Mathematical & Computational Applications, Vol. 7, No. 2, pp. 139-149, 2002
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MULTIINDEX MULTIVARIABLE HERMITE POLYNOMIALS

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Abstract-In the present paper multiindex multivariable Hermite polynomials in terms of series and generating function are defined. Their basic properties, differential and pure recurrence relations, differential equations, generating function relations and expansions have been established. Few deductions are also obtained.
Keywords-Multiindex multivariable Hermite polynomials , orthogonal polynomials, Lauricella function of several variables, hypergeometric function of several variables :

1. INTRODUCTION

Special functions play a vital role in both classical as well as quantum physics. There are many different Special functions, which one is most convenient depends on the particular problem at hand. Hermite polynomials play a central role in optics and certain parts of quantum mechanics. The aim of the present paper is to generalize the well-known Hermite polynomials in terms of both indices and variables. In section 2, we defined Hermite polynomials of 2-index 2-variable and in general several- index several- variable and they are also interpreted in terms of generalized Lauricella function of two and several variables respectively. The generalized Lauricella function of several variables was defined by Srivastava and Daoust [9, eq. (1), p.454; see also 10,11,12]. In subsequent sections, we discussed some basic properties, obtained – recurrence relations, differential equations, generating functions and expansions of multiindex multivariable Hermite polynomials. The relationships of the Hermite polynomials with other orthogonal polynomials such as Jacobi, Gegenbauer, Laguerre and Legendre have also been established.

The well-known Hermite polynomials of single index and single variable $H_n(x)$ may be defined by generating function and series [1, eq. (1), (2), p.187] as follows :

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty} H_n(x)\,t^n/n!\,, \tag{1.1}$$

$$H_n(x)=\sum_{k=0}^{[n/2]} (-1)^k\, n!\,(2x)^{n-2k}/(n-2k)!\;k!, \tag{1.2}$$

valid for all finite x and t, where n indicates its index and index represents the degree of the polynomials in x.

The relationship between Hermite polynomials and well-known classical orthogonal polynomials-Jacobi, Gegenbauer, Laguerre and Legendre are as follows :

$$2^{2n}n!(-1)^n \lim_{\beta\to\infty} P_n^{(1/2,\beta)}\left(1-\frac{2x^2}{\beta}\right)=H_{2n}(x) \tag{1.3}$$

$$2^{2n+1}n!\;x\,(-1)^n \lim_{\beta\to\infty} P_n^{(1/2,\beta)}\left(1-\frac{2x^2}{\beta}\right)H_{2n+1}(x) \tag{1.4}$$

$$\lim_{v\to\infty} n!\left(1/\sqrt{x^2+v}\right)^n \; C_n^v\left(x/\sqrt{x^2+v}\right) = H_n(x) \tag{1.5}$$

$$(-1)^n\, 2^{2n}\, n!\; L_n^{(-1/2)}(x^2) = H_{2n}(x) \tag{1.6}$$

$$(-1)^n\, 2^{2n+1}\, n!\; x\; L_n^{(1/2)}(x^2) = H_{2n+1}(x) \tag{1.7}$$

$$H_n(x) = 2^{n+1}\exp(x^2)\int_x^{\infty} \exp(-t)^2\, t^{n+1}\, P_n(x/t)\,dt \tag{1.8}$$

2. MULTIINDEX MULTIVARIABLE HERMITE POLYNOMIALS

In this section, we define 2-index 2-variable and in turn p-index p-variable Hermite polynomials as follows :

$$\exp\left[2x(t+h)-(y+1)(t+h)^2\right] = \sum_{n,m=0}^{\infty} H_{n,m}(x,y)\, t^n h^m / n!\, m!\,, \tag{2.1}$$

$$\exp\left[2x_1(t_1+\ldots+t_p)-(x_2+\ldots+x_p+1)(t_1+\ldots+t_p)^2\right]$$
$$= \sum_{n_1,\ldots,n_p=0}^{\infty} H_{n_1,\ldots,n_p}(x_1,\ldots,x_p)\, t^{n_1}\ldots t^{n_p} / n_1!\ldots n_p!\,, \tag{2.2}$$

$$H_{n,m}(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{n!\; m!\; (2r+2s)!\; (-1)^{r+s}(2x)^{n+m-2r-2s}(y+1)^{r+s}}{(2r)!\; (2s)!\; (r+s)!\; (n-2r)!\; (m-2s)!}\,, \tag{2.3}$$

$$= (2x)^{n+m} F^{1:2;2}_{0:1;1}\left(\begin{matrix} \left(\frac{1}{2}:1,1\right):\left(-\frac{n}{2},1\right),\left(\frac{-n+1}{2},1\right);\left(-\frac{m}{2},1\right),\left(\frac{-m+1}{2},1\right); \\ -:\ \left(\frac{1}{2},1\right);\quad -;\ \left(\frac{1}{2},1\right);\quad -; \end{matrix}\ \frac{-(y+1)}{x^2}, \frac{-(y+1)}{x^2}\right), \tag{2.4}$$

$$H_{n_1,\ldots,n_p}(x_1,\ldots,x_p)$$
$$= \sum_{r_1=0}^{\left[\frac{n_1}{2}\right]} \cdots\cdots \sum_{r_p=0}^{\left[\frac{n_p}{2}\right]} \frac{n_1!\ldots n_p!\; (2r_1+\ldots+2r_p)!\; (-1)^{r_1+\ldots+r_p}(2x_1)^{n_1+\ldots+n_p-2r_1\ldots-2r_p}(x_2+\ldots+x_p+1)^{r_1+\ldots+r_p}}{(2r_1)!\ldots(2r_p)!\; (r_1+\ldots+r_p)!\; (n_1-2r_1)!\ldots(n_p-2r_p)!}\,, \tag{2.5}$$

$$= (2x_1)^{n_1+\ldots+n_p} F^{1:2,\ldots,2}_{0:1;\ldots,1}\left(\begin{matrix} \left(\frac{1}{2}:1,\ldots,1\right):\left(-\frac{n_1}{2},1\right),\left(\frac{-n_1+1}{2},1\right);\ldots;\left(-\frac{n_p}{2},1\right),\left(\frac{-n_p+1}{2},1\right); \\ -:\ \left(\frac{1}{2},1\right);\ \ldots;\ \left(\frac{1}{2},1\right);\quad -; \end{matrix}\right.$$
$$\left.\frac{-(x_2+\ldots+x_p+1)}{x_1^2},\ldots,\frac{-(x_2+\ldots+x_p+1)}{x_1^2}\right), \tag{2.6}$$

where

(i) F-function in (2.4) and (2.6) represent generalized Lauricella functions of two and several variables [9] respectively.

(ii) In (2.5), put $n_2 = ... = n_p = 0$ ($r_2,...,r_p$ – series vanish only r_1 - series will exist) we get series of Hermite Polynomials of single index and several variables and its generating function can be obtained from (2.2) by taking $t_2 = ... = t_p = 0$ [3].

(iii) In (2.5) and (2.2), taking $x_2 = ... = x_p = 0$ and $t_2 = ... t_p = 0$, we get the series (1.2) and the generating function (1.1) of well-known Hermite polynomials of single index and single variable.

(iv) It is obvious from (2.1) to (2.6), that

$$\left.\begin{aligned} & H_{n,m}(x,y) \neq H_{n+m}(x,y) \\ & H_{n,m}(x,y) \neq H_n(x)\, H_m(x) \\ & H_{n_1,...,n_p}(x_1,...,x_p) \neq H_{n_1}(x_1)...H_{n_p}(x_p) \\ & H_{n_1,...,n_p}(x_1,...,x_p) \neq H_{n_1+...+n_p}(x_1,...,x_p) \end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned} & H_{n,m}(x,0) = H_{n,m}(x) \\ & H_{n,0}(x,y) = H_n(x,y) \\ & H_{n,0}(x,0) = H_n(x) \\ & H_{n_1,...,n_{p-1},0}(x_1,...,x_p) = H_{n_1,...,n_{p-1}}(x_1,...,x_p) \\ & H_{n_1,...,n_p}(x_1,...,x_{p-1},0) = H_{n_1,...,n_p}(x_1,...,x_{p-1}) \\ & H_{n_1,...,n_{p-1},0}(x_1,...,x_{p-1},0) = H_{n_1,...,n_{p-1}}(x_1,...,x_{p-1}) \end{aligned}\right\} \tag{2.8}$$

we shall use following known relations in subsequent sections :

$$\text{exponential series } \exp(x) = \sum_{n=0}^{\infty} x^n/n!\,, \tag{2.9}$$

$$\text{Binomial series } (a+x)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} x^r, |x/a| < 1\,, \tag{2.10}$$

Proof of (2.3) : Taking (2.1) and writing

$$\sum_{n=0}^{\infty} H_{n,m}(x,y)\, t^n h^m/n!\, m! = \exp[2x(t+h)] \exp[-(y+1)(t+h)^2],$$

$$= \sum_{n=0}^{\infty} (2x)^n (t+h)^n/n! \sum_{r=0}^{\infty} (1)^r (y+1)^r (t+h)^{2r}/r!\,,\ (\text{in view of (2.9)})$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{r=0}^{\infty} \sum_{s=0}^{2r} \frac{(-1)^r (2r)! (2x)^n (1+y)^r t^{n-m+2r-s} h^{m+s}}{r!\ s!\ m!\ (n-m)!\ (2r-s)!},\ (\text{in view of (2.10)})$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{r+s} (2r+2s)! (2x)^{n+m} (y+1)^{r+s} t^{n+2r} h^{m+2s}}{n!\ m!\ (2r)!\ (2s)!\ (r+s)!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^{r+s} (2r)! (2r+2s)! (2x)^{n+m-2r-2s} (y+1)^{r+s} t^n h^m}{(n-2r)!\ (m-2s)!\ (2r)!\ (2s)!\ (r+s)!}$$

On comparing the coefficient of $t^n h^m$ from both the sides, we get the required result (2.3).

Similarly (2.5) can easily be obtained.

3. BASIC PROPERTIES AND RELATIONSHIP BETWEEN $H_{n_1,\dots,n_p}(x_1,\dots,x_p)$ and $H_{n_1,\dots,n_p}(x_1)$

(i) Earlier it has been proved [1, 3, 4] that

$$\left.\begin{aligned} &H_n(-x) = (-1)^n H_n(x) \\ &H_{n,m}(-x) = (-1)^{n+m} H_{n,m}(x) \\ &H_{n_1,\dots,n_p}(-x) = (-1)^{n_1+\dots+n_p} H_{n_1,\dots,n_p}(x) \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} &H_n(-x,y) = (-1)^n H_n(x,y) \\ &H_n(x,-y,z) \neq (-1)^n H_n(x,y,z) \\ &H_n(x,y,-z) \neq (-1)^n H_n(x,y,z) \end{aligned}\right\} \tag{3.2}$$

From (2.5) one can easily derive :

$$H_{n_1,\dots,n_p}(-x_1,x_2,\dots,x_p) = (-1)^{n_1+\dots+n_p} H_{n_1,\dots,n_p}(x_1,\dots,x_p)$$

$$H_{n_1,\dots,n_p}(x_1,x_2,\dots,-x_i,\dots,x_p) \neq (-1)^{n_1+\dots+n_p} H_{n_1,\dots,n_p}(x_1,\dots,x_p), \quad (i=2,\dots,p)$$

(ii) In (2.1) putting y=0 and using the relation (2.8), we get generating function of two index and single variable Hermite polynomials i.e.,

$$\sum_{n,m=0}^{\infty} H_{n,m}(x)\, t^n h^m / n!\; m! = \exp\left[2x(t+h) - (t+h)^2\right],$$

replacing x by $\left(x/\sqrt{y+1}\right)t$ by $t\sqrt{y+1}$ and h by $h\sqrt{y+1}$, we arrive at

$$\sum_{n,m=0}^{\infty} \frac{H_{n,m}\left(x/\sqrt{y+1}\right)(y+1)^{\frac{n+m}{2}}\, t^n h^m}{n!\; m!} = \exp\left[2x(t+h) - (y+1)(t+h)^2\right]$$

$$= \sum_{n,m=0}^{\infty} H_{n,m}(x,y)\, t^n h^m / n!\; m!, \text{ with the help of (2.1) ;}$$

On comparing the coefficient of $t^n h^m$, we obtain a relation between 2-index 2-variable Hermite polynomials and 2-index 1-variable Hermite polynomials as follows :

$$H_{n,m}(x,y) = (y+1)^{\left(\frac{n+m}{2}\right)} H_{n,m}\left(x/\sqrt{y+1}\right), \tag{3.3}$$

Similarly,

$$H_{n_1,\dots,n_p}(x_1,\dots,x_p) = (x_2+\dots+x_p+1)^{\frac{(n_1+\dots+n_p)}{2}} H_{n_1,\dots,n_p}\left(x_1/\sqrt{(x_2+\dots,x_p+1)}\right), \tag{3.4}$$

(iii) In (3.3), taking x=0, we obtain the value of $H_{n,m}(x,y)$ and $\frac{\partial}{\partial x}H_{n,m}(x,y)$ at x=0 as follows :

$$\left.\begin{aligned} &H_{n,m}(0,y) = (y+1)^{\frac{(n+m)}{2}} H_{n,m}(0)\ , \\ &\text{when n, m = even} \\ &H_{2n,2m}(0,y) = (y+1)^{n+m} H_{2n,2m}(0) = (y+1)^{n+m}(-1)^{n+m}2^{2n+2m}\left(\frac{1}{2}\right)_{n+m} ;\text{by [4]} \\ &\text{when n, m = odd} \\ &H_{2n+1,2m+1}(0,y) = (y+1)^{n+m+1} H_{2n+1,2m+1}(0) = 0\, ;\text{by [4]} \end{aligned}\right\} \tag{3.5}$$

Differenting partially (3.3) w.r.t. "x" and then putting x=0, we get

$$\frac{\partial}{\partial x}H_{n,m}(0,y)=(y+1)^{\frac{(n+m)}{2}}\left(\frac{\partial}{\partial x}H_{n,m}\left(\frac{x}{\sqrt{y+1}}\right)\right)_{x=0}\quad(y+1)^{\frac{(n+m)}{2}}H'_{n,m}(0), \tag{3.6}$$

when n,m=odd

$$\frac{\partial}{\partial x}H_{2n+1,2m+1}(0,y)=(y+1)^{n+m+1}H'_{2n+1,2m+1}(0)$$

$$=\frac{(y+1)^{n+m+1}\,2\,(-1)^{\frac{(2n+2m+1)}{2}}\{(2n+1)+(2m+1)\}!}{\left(\frac{2n+2m+1}{2}\right)!}\;;\text{by}\,[4] \tag{3.7}$$

when n,m=even

$$\frac{\partial}{\partial x}H_{2n,2m}(0,y)=(y+1)^{n+m}H'_{2n,2m}(0)=0,\ \text{by}\,[4] \tag{3.8}$$

4. RECURRENCE RELATIONS AND DIFFERENTIAL EQUATIONS

In this section, the differential and pure recurrence relations, differential equations of 2-index 2-variable and in turn p-index p-variable of Hermite polynomials are derived.

(i) Differentiating partially (2.1) w.r.t. "x", we get

$$\sum_{n,m=0}^{\infty}\frac{\frac{\partial}{\partial x}H_{n,m}(x,y)t^n h^m}{n!\ m!} \tag{4.1}$$

$$=2(t+h)\exp\left[2x(t+h)-(y+1)(t+h)^2\right],$$

$$=2\left(\sum_{r=0}^{1}\frac{t^{1-r}h^r}{r!\,(1-r)!}\right)\sum_{n,m=0}^{\infty}\frac{H_{n,m}(x,y)\,t^n h^m}{n!\ \text{m}!}\ ,\ \text{by (2.10) and (2.1)}$$

$$=2\sum_{n,m=0}^{\infty}\sum_{r=0}^{1}\frac{H_{n-(1-r),m-r}(x,y)\,t^n h^m}{(n-(1-r))!\ (m-r)!\ (1-r)!}$$

On comparing the coefficients of $t^n h^m$, we obtain

$$\frac{\partial}{\partial x}H_{n,m}(x,y)=2n\,H_{n-1,m}(x,y)+2m\,H_{n,m-1}(x,y)\ , \tag{4.2}$$

differentiating (4.1) partially w.r.t. "x" and proceeding as above, we get

$$\frac{\partial^2}{\partial x^2}H_{n,m}(x,y)=2^2\left[n(n-1)\,H_{n-2,m}(x,y)+2\,nm\,H_{n-1,m-1}(x,y)+m(m-1)\,H_{n,m-2}(x,y)\right], \tag{4.3}$$

On iteration

$$\frac{\frac{\partial^s}{\partial x^s}H_{n,m}(x,y)}{n!\ m!}=2^s\sum_{r=0}^{s}\frac{s!\ H_{n-(s-r),m-r}(x,y)}{(n-(s-r))!\ (m-r)!\ r!\ (s-r)!}\ , \tag{4.4}$$

(ii) Differenting partially (2.1) w.r.t. "y", and proceeding as in (i), we get

$$\frac{\partial}{\partial y} H_{n,m}(x,y) = -[n(n-1)\, H_{n-2,m}(x,y) + 2nm\, H_{n-1,m-1}(x,y) + m(m-1)\, H_{n,m-2}(x,y)] \;, \tag{4.5}$$

$$\frac{\frac{\partial^2}{\partial y^2} H_{n,m}(x,y)}{n!\; m!} = (-1)^2 \sum_{r=0}^{4} \frac{4!\; H_{n-(4-r),m-r}(x,y)}{(n-(4-r))!\; (m-r)!\; r!\; (4-r)!} \;, \tag{4.6}$$

$$\frac{\frac{\partial^s}{\partial y^s} H_{n,m}(x,y)}{n!\; m!} = (-1)^s \sum_{r=0}^{2s} \frac{(2s)!\; H_{n-(2s-r),m-r}(x,y)}{(n-(2s-r))!\; (m-r)!\; r!\; (2s-r)!} \;, \tag{4.7}$$

In general

$$D_{x_1} H_{n_1,\dots,n_p}(x_1,\dots,x_p) = 2a \sum_{r_i=1}^{n_i} \left(d!(n_1-d)!b\right)^{-1} H_{n_1-d,c}(x_1,\dots,x_p) \;, \tag{4.8}$$

$$D_{x_1}^{s} H_{n_1,\dots,n_p}(x_1,\dots,x_p) = 2^s s! a \sum_{r_i=0}^{n_i} \left(b!\; e!\; (n_1-e)!\right)^{-1} H_{n_1-e,c}(x_1,\dots,x_p) \;, \tag{4.9}$$

$$D_{x_2} H_{n_1,\dots,n_p}(x_1,\dots,x_p) = -(2!) a \sum_{r_i=0}^{n_i} \left(b\; f!(n_1-f)!\right)^{-1} H_{n_1-f,c}(x_1,\dots,x_p) \;, \tag{4.10}$$

$$D_{x_2}^{s} H_{n_1,\dots,n_p}(x_1,\dots,x_p) = (-1)^s (2s)!\; a \sum_{r_i=0}^{n_i} \left(b\; g!(n_1-g)!\right)^{-1} H_{n_1-g,c}(x_1,\dots,x_p) \;, \tag{4.11}$$

where $a \equiv (n_1!,\dots,n_p!), b \equiv ((n_2-r_1)!\dots(n_p-r_{p-1})!), c \equiv ((n_2-r_1),\dots,(n_p-r_{p-1}))$,
$d \equiv (1-r_1-\dots-r_{p-1}), e = (s-r_1-\dots-r_{p-1}), f \equiv (2-r_1-\dots-r_{p-1}), g = (2s-r_1-\dots-r_{p-1})$,
$D_{x_i}^{s} \equiv \partial^s/\partial x_i^s \,, (i=1,\dots,p)$

NOTE:

(a) For the variables $x_3, x_4, \dots, x_p$ the results will be same as the results for x_2

(b) From above equations it is worth to note the following conversion relations :

$$\left.\begin{aligned} \frac{\partial^2}{\partial x^2} H_{n,m}(x,y) &= 2^{2.1}(-1)^1 \frac{\partial}{\partial y} H_{n,m}(x,y) \;, \\ \frac{\partial^4}{\partial x^4} H_{n,m}(x,y) &= 2^{2.2}(-1)^2 \frac{\partial^2}{\partial y^2} H_{n,m}(x,y) \;, \\ \frac{\partial^6}{\partial x^6} H_{n,m}(x,y) &= 2^{2.3}(-1)^3 \frac{\partial^3}{\partial y^3} H_{n,m}(x,y) \;, \end{aligned}\right\} \tag{4.12}$$

In general

$$\frac{\partial^{2s}}{\partial x^{2s}} H_{n,m}(x,y) = 2^{2s}(-1)^s \frac{\partial^s}{\partial y^s} H_{n,m}(x,y) \;, \tag{4.13}$$

Similar results can also be written for multiindex multivariable Hermite polynomials.

(iii) Taking (2.1)

$$\sum_{n,m=0}^{\infty} \frac{H_{n,m}(x,y)\, t^n h^m}{n!\; m!} = \exp[2x(t+h) - (y+1)(t+h)^2] \;,$$

Let

$$\sum_{n,m=0}^{\infty} g_{n,m}(x,y)\, t^n h^m = G[2x(t+h) - (y+1)(t+h)^2] \;, \tag{4.14}$$

also

$$F = G[2x(t+h) - (y+1)(t+h)^2] \text{, then} \tag{4.15}$$

$$\frac{\partial F}{\partial x} = 2(t+h)\, G' \qquad ; \; \frac{\partial F}{\partial y} = -(t+h)^2\, G'$$

$$\frac{\partial F}{\partial t} = [2x - 2(y+1)(t+h)]\, G', \frac{\partial F}{\partial h} = [2x - 2(y+1)(t+h)]\, G' \Rightarrow \frac{\partial F}{\partial t} \equiv \frac{\partial F}{\partial h}$$

From above partial derivatives, we observe that the F of (4.15) satisfies the following partial differential equation.

$$(x-t-h)\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} - t\frac{\partial F}{\partial t} - h\frac{\partial F}{\partial h} = 0$$

Now using (4.14) and comparing the coefficients of $t^n h^m$, we get

$$x\frac{\partial}{\partial x} g_{n,m}(x,y) - \frac{\partial}{\partial x} g_{n-1,m}(x,y) - \frac{\partial}{\partial x} g_{n,m-1}(x,y) + 2y\frac{\partial}{\partial y} g_{n,m}(x,y) - n g_{n,m}(x,y)$$
$$- m g_{n,m}(x,y) = 0 \;, \tag{4.16}$$

hence $H_{n,m}(x,y)$ must satisfy

$$x\frac{\partial}{\partial x} H_{n,m}(x,y) - n\frac{\partial}{\partial x} H_{n-1,m}(x,y) - m\frac{\partial}{\partial x} H_{n,m-1}(x,y) + 2y\frac{\partial}{\partial y} H_{n,m}(x,y)$$

$$-(n+m) H_{n,m}(x,y) = 0 \;, \tag{4.17}$$

It is a *differential recurrence relation* of $H_{n,m}(x,y)$.

In (4.17), substituting the value of $\frac{\partial}{\partial x} H_{n,m}(x,y)$ and $\frac{\partial}{\partial y} H_{n,m}(x,y)$ from (4.2) and (4.5) respectively, we get a another *differential recurrence relation* of $H_{n,m}(x,y)$

$$(n+m)\, H_{n,m}(x,y) = 2x[n\, H_{n-1,m}(x,y) + m\, H_{n,m-1}(x,y)] - 2y[n(n-1)\, H_{n-2,m}(x,y)$$
$$+ m(m-1)\, H_{n,m-2} + 2nm\, H_{n-1,m-1}(x,y)] - n[\frac{\partial}{\partial x} H_{n-1,m}(x,y) + m\frac{\partial}{\partial x} H_{n,m-1}(x,y)], \tag{4.18}$$

In (4.18), once again substituting the value of $\frac{\partial}{\partial x} H_{n,m}(x)$ from (4.2) or in (4.17) removing all the derivatives with the help of (4.2) and (4.5), we obtain a *pure recurrence relation of* $H_{n,m}(x,y)$:

$$(n+m)\,H_{n,m}(x,y)=2x[n\,H_{n-1,m}(x,y)+m\,H_{n,m-1}(x,y)]$$

$$-(y+1)\left[2n\,(n-1)\,H_{n-2,m}(x,y)+2m(m-1)\,H_{n,m-2}(x,y)+4nm\,H_{n-1,m-1}(x,y)\right], \qquad (4.19)$$

In (4.19), removing all the index less then n,m (i.e. n-1,n-2,m-1,m-2) with the help of (4.2), (4.3) and (4.5), we obtain the *differential equation* of $H_{n,m}(x,y)$ which is as follows :

$$(y+1)\frac{\partial^2}{\partial x^2}H_{n,m}(x,y)-2x\frac{\partial^2}{\partial x^2}H_{n,m}(x,y)+2(n+m)\,H_{n+m}(x,y)=0 \qquad (4.20)$$

In view of relations (4.12), the differential equations (4.20) can also be written as :

$$2(y+1)\frac{\partial}{\partial y}H_{n,m}(x,y)+x\frac{\partial}{\partial x}H_{n,m}(x,y)-(n+m)\,H_{n,m}(x,y)=0\ , \qquad (4.21)$$

Proceeding on the same lines as above *differential equation* of $H_{n_1,\dots,n_p}(x_1,\dots,x_p)$ can easily be written as follows :

$$(x_2+\dots+x_p+1)\frac{\partial^2}{\partial x_1^2}H_{n_1,\dots,n_p}(x_1,\dots,x_p)-2x_1\frac{\partial}{\partial x_1}H_{n_1,\dots,n_p}(x_1,\dots,x_p)$$

$$+2(n_1+\dots+n_p)\,H_{n_1,\dots,n_p}(x_1,\dots,x_p)=0\ , \qquad (4.22)$$

5. GENERATING FUNCTION RELATIONS

In this section, the generating function relation of 2-index 2-variable and in turn p-index p-variable are obtained in terms of generalized Lauricella functions of several variables :

$$\sum_{n,m=0}^{\infty}\frac{(c)_n(d)_m H_{n,m}(x,y)\,t^n h^m}{n!\ m!}$$

$$=F^{3:0;0;0;0}_{0:0;0;1;1}\left(\begin{matrix}\Delta_1,\Delta_2,(1/2:0,0,1,1):-;-; & -; & -; \\ -\ :-\ ;- & ;(1/2,1); & (1/2,1);\end{matrix}\quad X_1,X_2,T_1,T_2\right), \qquad (5.1)$$

$$\sum_{n_1,\dots,n_p=0}^{\infty}\frac{(c_1)_{n_1}\dots(c_p)_{n_p}H_{n_1,\dots,n_p}(x_1,\dots,x_p)\,t_1^{n_1}\dots t_p^{n_p}}{n_1!\dots n_p!}$$

$$=F^{p+1:0;\dots;0;0;\dots;0}_{0:0;\dots;0;1;\dots;1}\left(\begin{matrix}\Delta_1,\Delta_2,(1/2:0,\dots,0,1,\dots,1):-;\dots;-; & -; & -; \\ -\ :-;\dots;-; & (1/2,1); & (1/2,1);\end{matrix}\quad X,T\right), \qquad (5.2)$$

provided the results exist and where

$\Delta_1\equiv(cd:1,0,2,0),\Delta_2\equiv(d:0,1,0,2),\Delta\equiv(c_1:1,0,\dots,0,2,0,\dots,0),\dots,(c_p:0,\dots,0,1,0,\dots,0,2),$

$X_1\equiv 2xt, X_2\equiv 2xh,,T_1\equiv-(y+1)t^2,T_2\equiv-(y+1)h^2,$

$X\equiv(2x_1t_1),\dots,(2x_pt_p);T\equiv(-(x_2+\dots+x_p+1)t_1^2),\dots,(-(x_2+\dots+x_p+1)t_p^2)$

Proof of (5.1) : Taking

$$\sum_{n,m=0}^{\infty} \frac{(c)_n (d)_m H_{n,m}(x,y)\, t^n h^m}{n!\, m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^{r+s}(2r+2s)!(c)_n (d)_m (2x)^{n+m-2r-2s}(y+1)^{r+s} t^n h^m}{(n-2r)!(m-2s)!(2r)!(2s)!(r+s)!}$$

$$= \sum_{n,m,r,s=0}^{\infty} \frac{(-1)^{r+s}\left(\frac{1}{2}\right)_{r+s} (c)_{n+2r}(d)_{m+2s}(2x)^{n+m}(y+1)^{r+s} t^{n+2r} h^{m+2s}}{n!\, m!\, r!\, s! \left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_s}$$

expressing the series in terms of generalized Lauricella function of several variable [10], we get the required result (5.1). Proceeding on the same lines the result (5.2) can easily be established.

6.EXPANSIONS

Expansion of x^{n+m} in the series of Hermite polynomials of two-index two-variable, and in general an expansion of $x^{n_1+\ldots+n_p}$ in the series of Hermite polynomials of multiindex multivariable are established in this section :

$$(x)^{n+m} = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \binom{n}{2r}\binom{m}{2s} 2^{2r+2s-n-m} \left(\frac{1}{2}\right)_{r+s} (y+1)^{r+s} H_{n-2r,m-2s}(x,y) \,, \tag{6.1}$$

$$(x)^{n_1+\ldots+n_p} = \sum_{r_1=0}^{\left[\frac{n_1}{2}\right]} \ldots \sum_{r_p=0}^{\left[\frac{n_p}{2}\right]} \binom{n_1}{2r_1} \ldots \binom{n_p}{2r_p} 2^{2r_1+\ldots+2r_p-n_1\ldots n_p} \left(\frac{1}{2}\right)_{r_1+\ldots+r_p} (x_2+\ldots+x_p+1)^{r_1+\ldots+r_p}$$

$$\times H_{n_1-2r_1,\ldots,n_p-2r_p}(x_1,\ldots,x_p) \,, \tag{6.2}$$

Proof of (6.1) : Taking (2.1), i.e.

$$\exp\left[2x(t+h)-(y+1)(t+h)^2\right] = \sum_{n,m=0}^{\infty} \frac{H_{n,m}(x,y)\, t^n h^m}{n!\, m!}$$

$$\exp\left[2x(t+h)\right] = \left[\exp(y+1)(t+h)^2\right] \sum_{n,m=0}^{\infty} \frac{H_{n,m}(x,y)\, t^n h^m}{n!\, m!}$$

$$\sum_{n=0}^{\infty} \frac{(2x)^n (t+h)^n}{n!} = \sum_{r=0}^{\infty} \frac{(y+1)^r (t+h)^{2r}}{r!} \sum_{n,m=0}^{\infty} \frac{H_{n,m}(x,y)\, t^n h^m}{n!\, m!}, \text{ by (2.9)}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2x)^n t^{n-m}h^m}{m!\,(n-m)!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{2r}\frac{(2r)!(y+1)^r H_{n,m}(x,y)\,t^{n+2r-s}h^{m+s}}{n!\,m!\,r!\,s!\,(2r-s)!},\ \text{by (2.10)}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2r+2s)!(y+1)^{r+s} H_{n,m}(x,y)\,t^{n+2r}h^{m+2s}}{n!\,m!\,(r+s)!\,(2r)!\,(2s)!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}\frac{(2r+2s)!(y+1)^{r+s} H_{n-2r,m-2s}(x,y)\,t^{n}h^{m}}{(n-2r)!\,(m-2s)!\,(r+s)!\,(2r)!\,(2s)!}$$

On comparing the coefficients of $t^n h^m$, we get the required result (6.1). Similarly the result (6.2) can easily be established.

7. DEDUCTIONS

(i) The following relationship can easily be established and hence the classical polynomials-Jacobi, Gegenbauer, Laguerre and Legendre polynomials are also generalized in terms of both index and variable.

$$H_{2n_1,\dots,2n_p}(x_1,\dots,x_p)$$

$$=2^{2(n_1+\dots+n_p)}\prod_{j=1}^{p}(n_j)!(-1)^{n_1+\dots+n_p}\lim_{\substack{\beta_i\to\infty\\(i=1,2,\dots,p)}}P_{n_1,\dots,n_p}^{\left(-\frac{1}{2},\beta_1;\dots;-\frac{1}{2},\beta_p\right)}\left(1-\frac{2x_1^2}{\beta_1},\dots,1-\frac{2x_p^2}{\beta_p}\right), \tag{7.1}$$

$$H_{2n_1+1,\dots,2n_p+1}(x_1,\dots,x_p)$$

$$=2^{2(n_1+\dots+n_p)+p}(-1)^{n_1+\dots+n_p}\prod_{j=1}^{p}(n_j)!\,(x_j)\lim_{\substack{\beta_i\to\infty\\(i=1,2,\dots,p)}}P_{n_1,\dots,n_p}^{(1/2,\beta_1;\dots;1/2,\beta_p)}\left(1-\frac{2x_1^2}{\beta_1},\dots,1-\frac{2x_p^2}{\beta_p}\right), \tag{7.2}$$

$$H_{n_1,\dots,n_p}(x_1,\dots,x_p)$$

$$=\lim_{\substack{v_i\to\infty\\(i=1,2,\dots,p)}}\prod_{j=1}^{p}(n_j!)\left(\frac{1}{\sqrt{x_1^2+v_1}}\right)^{n_1}\cdots\left(\frac{1}{\sqrt{x_p^2+v_p}}\right)^{n_p}C_{n_1,\dots,n_p}^{v_1,\dots,v_p}\left(\frac{x_1}{\sqrt{x_1^2+v_1}},\dots,\frac{x_p}{\sqrt{x_p^2+v_p}}\right), \tag{7.3}$$

$$H_{2n_1,\dots,2n_p}(x_1,\dots,x_p)=(-1)^{n_1+\dots+n_p}\,2^{2(n_1+\dots+n_p)}\prod_{j=1}^{p}(n_j!)L_{n_1,\dots,n_p}^{(1/2,\dots 1/2,)}\left(x_1^2,\dots,x_p^2\right), \tag{7.4}$$

$$H_{2n_1+1,\dots,2n_p+1}(x_1,\dots,x_p)=(-1)^{n_1+\dots+n_p}\,2^{2(n_1+\dots+n_p)+p}\prod_{j=1}^{p}(n_j!)\,(x_j)\,L_{n_1,\dots,n_p}^{(1/2,\dots 1/2,)}\left(x_1^2,\dots,x_p^2\right), \tag{7.5}$$

$$H_{n_1,\dots,n_p}(x_1,\dots,x_p)$$

$$=2^{n_1+\dots+n_p+p}\exp(x_1^2+\dots+x_p^2)\int_{x_1}^{\infty}\cdots\int_{x_p}^{\infty}\exp(-t_1^2-\dots-t_p^2)t_1^{n_1+1}\dots t_p^{n_p+1}$$

$$\times P_{n_1,\dots,n_p}\left(\frac{x_1}{t_1},\dots,\frac{x_p}{t_p}\right)dt_1\dots dt_p\ , \tag{7.6}$$

(ii) In (2.1), (2.3); (2.2), (2.5) taking $m = 0; n_2 = ... = n_p = 0$ respectively and using (2.8), we obtained the generating functions and series of Hermite polynomials of 2-variable, p-variable and single index [3].
(iii) In (4.8), (4.9), setting $n_2 = ... = n_p = 0$ and using (2.8); we get the derivatives of Hermite polynomials of single index multivariable [3].
(iv) In (4.17), (4.18), (4.19), (4.20) taking m=0 and using (2.8), we get differential & pure recurrence relations, differential equation of Hermite polynomials of single index two variable [3].
(v) In (5.2) taking $n_2 = ... = n_p = 0$ and using (2.8); $x_2 = ... = x_p = 0$ and using (2.8), we get respectively the generating function relation of Hermite polynomials of single-index multivariable; multiindex single variable [5, 6].
(vi) In (6.2) taking as in (v), we get respectively the expansion x^n and $x^{n_1+...+n_p}$ in the series of Hermite polynomials of single-index multivariable; multiindex single variable [7,8].

Similarly specializing index and variable, one can easily obtain many more known and unknown relations.

Acknowledgement - The author is thankful to the UGC, CRO, Bhopal (India) for the providing financial support for this work.

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