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A TOPOLOGICAL APPLICATION OF THE MONODROMY GROUPOID ON PRINCIPAL BUNDLES

Osman Mucuk
Erciyes University, Faculty of Science and Art, Department of Mathematics
Kayseri 38039, Turkey

Abstract-The idea of monodromy groupoid we deal here is due to Pradines [1] . An application of the monodromy groupoid on principal bundles was earlier given in [2]. In this paper, a topological version of this application is given.

Keywords-Monodromy groupoid, principal bundle, groupoid action

1. INTRODUCTION

A *groupoid* is a small category such that every morphism has an inverse. A *topological groupoid* is a groupoid in which both sets of objects and morphisms have topologies such that all maps of the groupoid structure are continuous.

Let G be a topological groupoid and W an open neighbourhood of the identities in G. Then we have a groupoid $M(G,W)$ called *monodromy groupoid* as given in Definition 3.10.

In the case where G is a topological group, which can be considered as a topological groupoid, this problem was studied in [3]. The monodromy groupoid of a locally trivial topological groupoid was also studied by Mackenzie in [4]. He constructed the monodromy groupoid directly from the universal covers of the stars G_x's.

As an example if X is a topological space then $G = X \times X$ becomes a topological groupoid. Further if X is semilocally simply connected then we can choose a suitable open subset W of G such that the monodromy groupoid $M(G,W)$ of the pair (G,W) is the fundamental groupoid $\pi_1 X$. If G is a topological group which is semilocally simply connected then the monodromy groupoid $M(G,W)$ of (G,W) for a suitable neighbourhood W of the identity is just the universal cover of G.

In this paper we prove that if $p: E \to X$ is a principal bundle in the sense of Definition 3.1, then the topological monodromy groupoid $M(G,W)$ with $G = X \times X$ acts topologically on the topological space E.

2. GROUPOIDS AND TOPOLOGICAL GROUPOIDS

Definition 2.1 A *groupoid* consists of two sets O_G and G called respectively the set of objects and the set of elements or morphisms of the groupoid together with two maps $\alpha, \beta : G \to O_G$, called respectively the source and target maps, a map $\varepsilon : O_G \to G, x \mapsto \varepsilon(x) = 1_x$ called the object inclusion map, where 1_x acts as identity at x, and a partial multiplication $G_\alpha \times_\beta G \to G$, $(b,a) \mapsto ba$ defined on

$$G_\alpha \times_\beta G = \{(b,a) \in G \times G : \alpha(b) = \beta(a)\}.$$

These maps are subject to the following conditions:
(i) $\alpha(ba) = \alpha(a)$ and $\beta(ba) = \beta(b)$ for each $(b,a) \in G_\alpha \times_\beta G$;
(ii) $c(ba) = (cb)a$ for all $c, b, a \in G$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$;
(iii) $\alpha(1_x) = \beta(1_x) = x$ for each $x \in O_G$, where 1_x is the identity at x;
(iv) $a1_{\alpha(a)} = a$ and $1_{\beta(a)}a = a$ for all $x \in O_G$ and;
(v) each $a \in G$ has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $a^{-1}a = 1_{\alpha(a)}$, $aa^{-1} = 1_{\beta(a)}$.

If (G, O_G) is a groupoid we say G is a groupoid on O_G. For a groupoid G, we write G_x for $\alpha^{-1}(x)$ and $G(x,y)$ for $\alpha^{-1}(x) \cap \alpha^{-1}(y)$, where $x, y \in O_G$.

In a groupoid G, the set O_G is mapped bijectively to the set of identities by $\varepsilon : O_G \to G$. So we sometimes write O_G for the set of identities.

Example 2.2 Let $p : E \to X$ be a continuous map. Let S_p denote the set of all bijections $f : E_x \to E_y$ for $x, y \in X$, where $E_x = p^{-1}(x)$. Then S_p becomes a groupoid on X with respect to the following structure: For a bijection $f : E_x \to E_y$ the source and target of f are defined by $\alpha(f) = x$, $\beta(f) = y$. The identity at $x \in X$ is the identity map 1_{E_x} and the partial multiplication is the composition of the maps. The inverse of $f \in S_p$ is just inverse map. This groupoid S_p is called *symmetry groupoid* of $p : E \to X$

Definition 2.3 Let G and H be groupoids. A *local morphism* from G to H is a map $f : W \to H$ from a subset W of G containing all the identities in G such that for $a \in W$, $\alpha_H(fa) = f(\alpha_G a)$, $\beta_H(fa) = f(\beta_G a)$, and $f(ba) = f(b)f(a)$ whenever $b, a \in W$, ba is defined and belongs to W.

A *morphism* from G to H is a pair of maps $f : G \to H$ and $O_f : O_G \to O_H$ such that $\alpha_H \circ f = O_f \circ \alpha_G$, $\beta_H \circ f = O_f \circ \beta_G$ and $f(ba) = f(b)f(a)$ for all $(b,a) \in G_\alpha \times_\beta G$.

For such a morphism we simply write $f : G \to H$.

The following notions of subgroupoid, normal subgroupoid, and quotient groupoid are from [5] and [6].

Definition 2.4 Let G be a groupoid. A subgroupoid of H is a pair of subsets $H \subseteq G$ and $O_H \subseteq O_G$ such that $\alpha(H) \subseteq O_H$, $\beta(H) \subseteq O_H$, $1_x \in H$ for all $x \in O_H$ and H is closed under the partial multiplication and the inversion in G.

A *normal subgroupoid* of G is a subgroupoid N of G such that $O_N = O_G$ and for each $x, y \in O_G$ and $a \in G(x,y)$ we have $aN(x) = N(y)a$. Let G be a groupoid and N a normal subgroupoid of G such that $N(x,y) = \varnothing$ if $x \neq y$. Define a groupoid G/N on O_G by $G/N(x,y) = \{aN(x) : a \in G(x,y)\}$ for any $x, y \in O_G$ with the multiplication that if

$a \in G(x,y)$ and $b \in G(y,z)$ then $bN(y)aN(x) = baN(x)N(x) = baN(x)$. This groupoid is called quotient groupoid of G by N.

The construction of the free groupoid is as follows ([6]): Let W be a directed graph. Let $p = (a_n,...,a_1)$ be a sequence of the edges such that the target of a_i is equal to the source of a_{i+1}. Such a p is called *directed path.* Write $(\ )_x$ for the empty path associated to x. The composition of two paths $p = (a_n,...,a_1)$ and $q = (b_m,...,b_1)$ is defined by $qp = (b_m,...,b_1,a_n,...,a_1)$ if the target of a_n is the source of b_1. Then we have a category $P(W)$. Let $\hat{a}$ denote the converse path of a in W. Define an equivalence relation on $P(W)$ as follows: Two directed paths p, q are equivalent if we can obtain one from the other by adding or deleting a number of $a\hat{a}$ or $\hat{a}a$. This is an equivalence relation. The set of equivalence classes $[p]$ is denoted by $F(W)$. A groupoid multiplication on $F(W)$ is defined by $[q][p]=[qp]$. So $F(W)$ becomes a groupoid which is called *free groupoid* on the graph W.

Let G be a groupoid and R a subgroupoid of G. By the normal subgroupoid $N(R)$ generated by R we mean the smallest normal groupoid including R. A direct construction of $N(R)$ is given in [5] as follows: Let G be a groupoid and R a subset of G such that $O_R = O_G$ and $R(x,y) = \varnothing$ if $x \neq y$. Let $N(x)$ be the set of all elements $r = a_n^{-1} r_n a_n ... a_1^{-1} r_1 a_1$ for $a_i \in G(x,x_i)$ and r_i or r_i^{-1} an element of $R(x,x_i)$. Let $N(x)$ be the family of $N(x)$ for all $x \in O_G$. Then $N(R)$ is a normal subgroupoid of G such that $O_{N(R)} = O_G$ and $NR(x,y) = \varnothing$ if $x \neq y$. $N(R)$ is called the *normal subgroupoid generated* by R.

Definition 2.5 ([4]) A *topological groupoid* is a groupoid G in which the both sets of objects and morphisms have topologies such that the following maps are continuous:

(i) the partial multiplication $G_\alpha \times_\beta G \to G$, $(b,a) \mapsto ba$,where $G_\alpha \times_\beta G$ has the relative topology;

(ii) the inverse map $G \to G$, $a \mapsto a^{-1}$;

(iii) the source and target maps $\alpha, \beta : G \to O_G$;

(iv) the object inclusion map $\varepsilon : O_G \to G, x \mapsto 1_x$.

For example a topological group can be thought as a topological groupoid with only one object. An other example is that if X is a topological space then $X \times X$ is a topological groupoid on X in which each pair (y,x) is a morphism from x to y and the groupoid multiplication is defined by $(z,y) \circ (y,x) = (z,x)$. The inverse of (y,x) is (x,y) and the identity 1_x at $x \in X$ is (x,x). This groupoid $G = X \times X$ is called *trivial groupoid.*

3. TOPOLOGICAL ACTION AND PRINCIPAL BUNDLE

The following definition comes from [7].

Definition 3.1 Let E and X be topological spaces and let $p : E \to X$ be a continuous surjective map and G a topological group acting effectively on E by

$G \times E \to E$, $(a,e) \mapsto ae$. By effectively we mean that if $b, a \in G$, and for all $e \in E$, *ae=be* then *a=b*. Then the triple (E, X, p) is called a *principal G-bundle* if the following conditions are satisfied.

(i) The fibres of p are equal to the orbits of G, that is, for $e, e' \in E$ the statement $p(e) = p(e')$ is equivalent to that there is an element $a \in G$ such that $e' = ae$.

(ii) The map $\delta : E \times_p E \to G$, $(e, ae) \mapsto a$ is continuous, where

$$E \times_p E = \{(e,e') \in E \times E : p(e) = p(e')\}.$$

(iii) There is an open cover $\{U_i : i \in I\}$ of X and for each $x \in X$ there is a continuous map $s_i : U_i \to E$ such that ps_i is the identity at U_i. Such a map $s_i : U_i \to E$ is called *local section.*

We call the set of these local sections $s_i : U_i \to E$ *atlas of sections* and the maps $s_{ij} : U_i \cap U_j \to G$ defined by $s_{ij}(x) s_i(x) = s_j(x)$ are called *transition functions.*

Example 3.2 ([4]) Let $p : E \to X$ be a principal G-bundle. Then G acts on $E \times E$ by the action $G \times (E \times E) \to E \times E$ with $a(e_2, e_1) = (ae_2, ae_1)$ induced by the action of G on E. Denote the orbit of (e_2, e_1) by $[e_2, e_1]$ and the set of orbits by $\frac{E \times E}{G}$. Then $\frac{E \times E}{G}$ is a groupoid on X with respect to the following structure: The source and target maps are defined by $\alpha[e_2, e_1] = p(e_1)$, $\alpha[e_2, e_1] = p(e_2)$, the object inclusion map is defined by $\varepsilon : X \to \frac{E \times E}{G}$, $x \mapsto 1_x = [e,e]$, where e is any element of $p^{-1}(x)$ and the partial multiplication is defined by

$$\left[e_3, e_2'\right] \circ [e_2, e_1] = \left[e_3, e_1 \delta\left(e_2', e_2\right)\right].$$

Here $\delta : E \times_p E \to G$ is the map defined by $(e, ae) \mapsto a$. The condition $\alpha\left(\left[e_3, e_2'\right]\right) = \beta([e_2, e_1])$ ensures that $\left(e_2', e_2\right) \in E \times_p E$. Note that one can always choose representatives so that $e_2' = e_2$ and the multiplication is then simply $[e_3, e_2] \circ [e_2, e_1] = [e_3, e_1]$. The inverse of $[e_2, e_1]$ is $[e_1, e_2]$. The groupoid $\frac{E \times E}{G}$ is called the groupoid associated to the principal G-bundle $p : E \to X$. Then $\frac{E \times E}{G}$, with the identification topology from $E \times E \to \frac{E \times E}{G}$ $(e_2, e_1) \mapsto [e_2, e_1]$ becomes a topological groupoid.

Example 3.3 Let $p : E \to X$ be a principal G-bundle and for $x \in X$, $E_x = p^{-1}(x)$. Let us consider the symmetry groupoid S_p of $p : E \to X$. Then there is an isomorphism

$\phi : \frac{E\times E}{G} \to S_p$ of groupoids defined by $(\phi[e,e'])(e'') = e''(e'e^{-1})$, where $e'e^{-1}$ represents the element $a \in G$ such that $ae = e'$. So S_p becomes a topological groupoid.

The following concept of locally trivial topological groupoid is due to Ehresmann [7].

Definition 3.4 Let G be a topological groupoid on $X = O_G$. Then G is called *locally trivial* if there exists a point x and an open cover $\{U_i : i \in I\}$ of X and for each $i \in I$ there is a continuous map $s_i : U_i \to G_x$ such that $\beta(s_i(y)) = y$, for all $y \in U_i$.

Example 3.5 ([4]) Let $p : E \to X$ be a principal G-bundle. Then the topological groupoid $\frac{E\times E}{G}$ associated to the bundle $p : E \to X$ is locally trivial. For if $s : U \to E$ is a local section of the principal G-bundle $p : E \to X$, then

$$U \to \left(\frac{E\times E}{G}\right)_{p(e_0)}, x \mapsto [s(x), e_0]$$

is a local section of $\frac{E\times E}{G}$. So the symmetry groupoid S_p of $p : E \to X$, which is isomorphic to $\frac{E\times E}{G}$, becomes a locally trivial topological groupoid.

Definition 3.6 Let G and H be topological groupoids. A *local morphism* of topological groupoids is a continuous map $f : W \to H$ defined on an open neighbourhood of the identities in G such that
for $a \in W$, $\alpha_H(fa) = f(\alpha_G a)$, $\beta_H(fa) = f(\beta_G a)$, and $f(ba) = f(b)f(a)$ whenever $b, a \in W$, ba is defined and belongs to W.

A *morphism* from G to H is a pair of continuous maps $f : G \to H$ and $O_f : O_G \to O_H$ such that $\alpha_H \circ f = O_f \circ \alpha_G$, $\beta_H \circ f = O_f \circ \beta_G$ and $f(ba) = f(b)f(a)$ for all $(b,a) \in G_\alpha \times_\beta G$.

Example 3.7 Let G be a locally trivial topological groupoid with an atlas of sections $\{s_i : U_i \to G_x\}_{i\in I}$ which has the property that each transition map $s_{ij} : U_i \cap U_j \to G$ defined by $s_{ij}(x)s_i(x) = s_j(x)$ is constant. Let

$$W = \bigcup_{i\in I}(U_i \times U_i)$$

and define $f : W \to G$ by $f(y,x) = s_i(y)(s_i(x))^{-1}$ whenever $(y,x) \in U_i \times U_i$. Since the transition functions are constant, f is well defined and so is a local morphism of topological groupoids (see [4] for details).

Definition 3.8 ([6]) Let G be a topological groupoid with $O_G = X$, E a topological space and $p : E \to X$ a continuous function. Let $G_\alpha \times_p E$ denote the subset

$$\{(a,e) \in G \times E : \alpha(a) = p(e)\}$$

of $G \times E$. A *topological action* on E via p is a continuous function $G_{\alpha} \times_p E \to E$, $(a,e) \mapsto ae$ such that

(i) $p(ae) = \beta(a)$ for all $(a,e) \in G_{\alpha} \times_p G$;

(ii) $b(ae) = (ba)e$ for all $(b,a) \in G_{\alpha} \times_{\beta} G$;

(iii) $(1_{pe})e = e$ for all $e \in E$.

We recall the following definition which is due to Ehresmann [7].

Definition 3.9 Let G be a groupoid and $O_G = X$ a topological space. An *admissible local section* of G is a function $s : U \to G$ from an open subset of X such that

(i) $\alpha s(x) = x$ for all $x \in U$;

(ii) $\beta s(U)$ is open in X; and

(iii) βs maps U homeomorphically to $\beta s(U)$.

Let G be a groupoid, W a subset of G such that $O_G = X \subseteq W$ and let W have the structure of topological space. We give X the subspace topology. We say that (α, β, W) *has enough continuous admissible local sections* if for each $a \in W$ there is an admissible local section $s : U \to G$ of G such that

(i) $s\alpha(a) = a$;

(ii) $s(U) \subseteq W$;

(iii) s is continuous from U to W.

For the concepts of free groupoid, normal subgroupoid and quotient groupoid we refer to [5] (see also [6]).

Let G be a topological groupoid and W an open neighbourhood of the identities in G. So W has a directed graph structure inherited from the groupoid multiplication of G. Hence we have a free groupoid $F(W)$ on W. Let N be the normal subgroupoid of $F(W)$ generated by the elements of $F(W)$ in the form $[ba]^{-1}[b][a]$ for $a, b \in W$ such that ba is defined and $ba \in W$. Let $M(G,W)$ be the quotient groupoid $F(G,W)/N$ of $F(W)$ by N. So we have an inclusion $\tilde{\imath} : W \to M(G,W)$ and by the universal property of $F(W)$ we have a projection map $p : M(G,W) \to G$ induced by the inclusion map $i : W \to G$. The inclusion map $\tilde{\imath} : W \to M(G,W)$ has the universal property that if $f : W \to H$ is a local morphism of groupoids as defined in Definition 2.3, then we have a morphism of groupoids $\phi : M(G,W) \to H$ such that $\phi\tilde{\imath} = f$. This groupoid $M(G,W)$ is called *monodromy groupoid* of the pair $(G;W)$. In [8] the monodromy groupoid $M(G,W)$ was given a topological groupoid structure such that the projection map $p : M(G,W) \to G$ is reduced to a universal cover on each fibre $M(G,W)_x$. Lie groupoid version of this problem is given in [9] and [10]. We give this construction as a definition.

Definition 3.10 Let G be a topological groupoid and W an open subset of G such that $O_G \subseteq W$. Let $F(W)$ be the free groupoid on W and let N be the normal subgroupoid of $F(W)$ generated by the elements in the form $[ba]^{-1}[b][a]$ for $a, b \in W$ such that ba is defined and $ba \in W$. Then the quotient groupoid $F[G,W]/N$ is called *monodromy groupoid* of G for W and denoted by $M(G,W)$

Theorem 3.11 ([8]) *Let G be a topological groupoid and W an open subset of G such that*
(i) $O_G \subseteq W$, *that is, W contains all the identities;*
(ii) $W = W^{-1}$, *that is, if* $a \in W$, *then we have* $a^{-1} \in W$;
(iii) (α_W, β_W, W) *has enough continuous admissible local sections.*
Then the monodromy groupoid $M(G,W)$ *has a structure of topological groupoid such that* $\widetilde{W} = \tilde{\imath}(W)$ *is an open subset of* $M(G,W)$ *and for any continuous local morphism* $f : W \to H$ *there exists a morphism of topological groupoids* $\phi : M(G,W) \to H$ *such that* $f = \phi\tilde{\imath}$

We now give our main theorem.

Theorem 3.12 *Let* $p : E \to X$ *be a principal G-bundle such that X is a topological manifold and each transition map* $s_{ij} : U_i \cap U_j \to G$ *defined by* $s_{ij}(x)s_i(x) = s_j(x)$ *is constant. Then an open subset W of the topological groupoid* $G = X \times X$ *can be chosen such that the monodromy groupoid* $M(G,W)$ *is a topological groupoid and acts topologically on the topological space E via p.*

Proof: Let $G = X \times X$ be the trivial topological groupoid defined above and $\{s_i : U_i \to E\}_{i\in I}$ an atlas of sections of the principal bundle $p : E \to X$. Let

$$W = \bigcup_{i\in I}(U_i \times U_i).$$

We now prove that the pair (G,W) satisfies the conditions of Theorem 3.11:
(i) Since $(x,x) \in W$ for all $x \in O_G = X$, we have $X = O_G \subseteq W \subseteq G$.
(ii) If $(y,x) \in W$, then $(x,y) \in W$, that is, $W = W^{-1}$.
(iii) To show that the pair (α, β, W) has enough continuous admissible local sections let $(y,x) \in W$. Then $\alpha(y,x) = x$ and $\beta(y,x) = y$ and by the definition of W, we have $(y,x) \in W$ for an $i \in I$. Choose open neighbourhoods U_x and V_y of x and y respectively such that $h : U_x \to R^n$ and $k : V_y \to R^n$ are manifold charts. Let $U = U_x \cap U_i$ and $V = V_y \cap U_i$. Choose neighbourhoods U' and V' of $h(x)$ and $k(y)$ respectively in $h(U)$ and $k(V)$ such that $V' = U' - h(x) + k(y)$ and define a homeomorphism

$$f' : U' \to V',\ a \mapsto a - h(x) + k(y).$$

in which $h(x) \mapsto k(y)$. Since $h : U \to h(U)$ and $k : V \to k(V)$ are homeomorphisms the sets $h^{-1}(U')$ and $k^{-1}(V')$ are open neighbourhoods of x and y respectively in U and V. Then the map $f : h^{-1}(U') \to k^{-1}(V'), z \mapsto (k^{-1}f'h)(z)$
is a homeomorphism and $f(x) = y$. Now define a continuous admissible local section $s : U \to W,\ z \mapsto (f(z), z)$. So (α, β, W) has enough continuous admissible local sections.

So by Theorem 3.11 the monodromy groupoid $M(G,W)$ is a topological groupoid. On the other hand by Example 3.5 the symmetry groupoid S_p of $p:E \to X$ is a locally trivial topological groupoid on X and by Example 3.7 there exists a local morphism $f:W \to S_p$ of topological groupoids. Again by Theorem 3.11 for this local morphism $f:W \to S_p$ we have a morphism $\phi : M(G,W) \to S_p$ of topological groupoids. This gives a continuous map $M(G,W)_\alpha \times_p E \to E$, which means that the topological groupoid $M(G,W)$ acts topologically on E.

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