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SOME GENERATING FUNCTION RELATIONS OF MULTIINDEX HERMITE POLYNOMIALS

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Abstract : In the present paper some generating functions relations of Hermite polynomial also of two, three and in turn several index are derived. Some known and unknown particular cases have also been discussed.

1. INTRODUCTION

In the course of an attempt to unify several results in the theory of polynomials, also in hypergeometric functions of one or more variables, author has defined the multiindex Hermite polynomials [3]. The results are interpreted interms of triple hypergeometric function [5] and generalized Lauricella function of several variables defined by H.M. Srivastava[4].

The well known Hermite polynomials of single variable and single index $H_n(x)$ may be defined by generating function and series [1; eq.(1), (2), p.187] as follows:

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)\ t^n}{n!}, \tag{1.1}$$

valid for all finite x and t;

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!\ (2x)^{n-2k}}{(n-2k)!k!} \tag{1.2}$$

where n indicates its index and index represent the degree of the polynomials in variable x.

The Hermite polynomials of index two, three and inturn p-index interms of series and generating function are represented as follows [3].

$$H_{n,m}(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{m!n!(-1)^{r+s}(2x)^{n+m-2r-2s}(2r+2s)!}{(n-2r)!(m-2s)!(2r)!(2s)!(r+s)!} \tag{1.3}$$

$$= (2x)^{n+m} F^{1:2;2}_{0:1;1}\left(\begin{matrix} \left(\frac{1}{2};1,1\right), \left(-\frac{n}{2},1\right), \left(\frac{-n+1}{2},1\right); \left(-\frac{m}{2},1\right), \left(\frac{-m+1}{2},1\right); \\ :--; \left(\frac{1}{2},1\right), \quad -- \quad ; \left(\frac{1}{2},1\right), \quad -- \quad ; \end{matrix} \left(-\frac{1}{2x^2}\right), \left(-\frac{1}{2x^2}\right) \right) \tag{1.4}$$

$$H_{n,m,p}(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{u=0}^{\left[\frac{p}{2}\right]} \frac{n!m!p!(-1)^{r+s+u}(2x)^{n+m+p-2r-2s-2u}(2r+2s+2u)!}{(n-2r)!(m-2s)!(p-2u)!(2r)!(2s)!(2u)!(r+s+u)!} \tag{1.5}$$

$$= (2x)^{n+m+p} F^{1:2;2;2}_{0:1;1;1}\left(\begin{matrix} \left(-\frac{1}{2};1,1,1\right),\left(-\frac{n}{2},1\right),\left(\frac{-n+1}{2},1\right);\left(-\frac{m}{2},1\right),\left(\frac{-m+1}{2},1\right);\left(-\frac{p}{2},1\right),\left(\frac{-p+1}{2},1\right); \\ ,\left(-\frac{1}{2},1\right), \ ---- \ \left(-\frac{1}{2},1\right), \ --- \ ;\left(-\frac{1}{2},1\right), \ --- \end{matrix} \right.$$

$$\left.\left(-\frac{1}{2x^2}\right),\left(-\frac{1}{2x^2}\right),\left(-\frac{1}{2x^2}\right)\right) \tag{1.6}$$

$$H_{n_1,n_2,\ldots,n_p}(x) = \sum_{\substack{r_j=0 \\ (j=1,\ldots,p)}}^{\left[\frac{n_j}{2}\right]} \frac{\left(\sum_{j=1}^{p}(2r_j)!\right)\prod_{j=1}^{p} n_j!(-1)^{\sum_{j=1}^{p} r_j}(2x)^{\sum_{j=1}^{p} n_j - 2\sum_{j=1}^{p} r_j}}{\left(\sum_{j=1}^{p} r_j\right)!\prod_{j=1}^{p}(2r_j)!\prod_{j=1}^{p}(n_j-2r_j)!} \tag{1.7}$$

$$= (2x)^{n_1+n_2+\ldots+n_p} F^{1:2;\ldots;2}_{0:1;\ldots;1}\left(\begin{matrix} \left(\frac{1}{2};1,\ldots,1\right),\left(-\frac{n_1}{2},1\right),\left(\frac{-n_1+1}{2},1\right);\ldots;\left(-\frac{n_p}{2},1\right),\left(\frac{-n_p+1}{2},1\right): \\ ---- \ ;\left(\frac{1}{2},1\right), \ ---- \ ;\ldots;\left(-\frac{1}{2},1\right), \ ---- \ : \end{matrix} \left(-\frac{1}{2x^2}\right),\ldots,\left(-\frac{1}{2x^2}\right)\right) \tag{1.8}$$

$$\exp[2x(t+h)-(t+h)^2] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{H_{n,m}(x)t^n h^m}{n!\ m!} \tag{1.9}$$

$$\exp[2x(t+h+g)-(t+h+g)^2] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{H_{n,m,p}(x)t^n h^m g^p}{n!\ m!\ p!} \tag{1.10}$$

$$\exp[2x(t_1+\ldots+t_p)-(t_1+\ldots+t_p)^2] = \sum_{n_1=0}^{\infty}\ldots\sum_{n_p=0}^{\infty} \frac{H_{n_1,\ldots,n_p}(x)t_1^{n_1}\ldots t_p^{n_p}}{n_1!\ldots n_p!} \tag{1.11}$$

where

(i) F-function in (1.4), (1.6), (1.8) represents generalized Lauricella function of different variables [3,4].

(ii) It is obvious from equations (2.1) and (1.2)

$$H_{n,m}(x) \neq H_n(x).H_m(x);$$

$$H_{n,m}(x) \neq H_{n+m}(x) \text{ and so on.}$$

(iii) In (1.3), put $m=0$ (s-series vanishes only r-series exists)

or $n=0$ (r-series vanishes only s-series exists),

we get

$$H_{n,0}(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r n!(2x)^{n-2r}}{(n-2r)!r!} = H_n(x) \tag{1.12}$$

$$H_{0,m}(x) = \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^s m!(2x)^{m-2s}}{(m-2s)!s!} = H_m(x) \tag{1.13}$$

Similarly, we can write from (3.3) and (3.4)

$$\left.\begin{aligned} H_{n,m,0}(x) &= H_{n,m}(x) \\ H_{n,0,0}(x) &= H_n(x) \\ H_{n_1,\dots,n_{p-1},0}(x) &= H_{n_1,\dots,n_{p-1}}(x) \\ H_{n_1,0\dots,0}(x) &= H_{n_1}(x) \end{aligned}\right\} \tag{1.14}$$

2. GENERATING FUNCTIONS OF MULTIINDEX HERMITE POLYNOMIALS

In this section few generating function relations of Hermite polynomials of two, three and in turn several index are proved :

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(c)_n(d)_m H_{n,m}(x)t^n h^m}{n!\ m!}$$

$$= F^{3:0;0;0;0}_{0:0;0;1;1}\left(\begin{matrix}(c;1,0,2,0),(d;0,1,0,2),\left(\frac{1}{2};0,0,1,1\right): --\ ;\ --\ ;\ --\ ;\ --\ ; \\ \text{------------}\ \qquad ;--\ ;\ --\ ;\ \left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);\end{matrix}\ (2xt),(2xh),(-t^2),(-h^2)\right), \tag{2.1}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(c)_n(d)_m(e)_p H_{n,m,p}(x)t^n h^m g^p}{n!\ m!\ p!}$$

$$= F^{4:0;0;0;0;0;0}_{0:0;0;0;1;1;1}\left(\begin{matrix}(c;1,0,0,2,0,0),(d;0,1,0,0,2,0),(e;0,0,1,0,0,2),\left(\frac{1}{2};0,0,0,1,1,1\right): -\ ;\ -\ ;\ -\ ;\ -\ \ -\ ;\ -\ ; \\ \text{------} \qquad ;-\ ;\ -\ ;\ -\ ;\ \left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);\end{matrix}\ (2xt),(2xh),(2xg),(-t^2),(-h^2),(-g^2)\right), \tag{2.2}$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\cdots\sum_{n_p=0}^{\infty}\frac{(c_1)_{n_1}(c_2)_{n_2}\cdots\cdots(c_p)_{n_p}H_{n_1,n_2,\dots,n_p}(x)t_1^{n_1}t_2^{n_2}\cdots t_p^{n_p}}{n_1!\cdots\cdots n_p!}$$

$$= F^{2p:0,\dots;0,0,\dots;0;}_{0:0,\dots;0;1,\dots;1}\left(\begin{matrix}(c_1;1,0,\dots,0,2,0,\dots,0),\dots\dots,(c_p;0,1,0,0,2,0),\left(\frac{1}{2};0,\dots,0,1,\dots,1\right): -\ ;\dots;\ -\ ;\dots;\ -\ ; \\ \text{------} \qquad ;\ -\ ;\dots;\left(\frac{1}{2},1\right);\dots;\left(\frac{1}{2},1\right);\end{matrix}\ (2xt_1),\dots\dots,(2xt_p),(-t_1^2),\dots\dots,(-t_p^2)\right), \tag{2.3}$$

$$\sum_{n,m=0}^{\infty}\frac{H_{n+r,m+s}(x)t^n h^m}{n!\ m!}$$

$$= \exp[2x(t+h)-(t+h)^2].H_{r,s}(x-t-h), \tag{2.4}$$

$$\sum_{n,m,p=0}^{\infty}\frac{H_{n+r,m+s,p+u}(x)t^n h^m g^p}{n!\ m!\ p!}$$

$$= \exp[2x(t+h+g)-(t+h+g)^2].H_{r,s,u}(x-t-h-g), \tag{2.5}$$

$$\sum_{n_1,\dots,n_p=0}^{\infty}\frac{H_{n_1+r_1,\dots\dots,n_p+r_p}(x)t_1^{n_1}\cdots\cdots t_p^{n_p}}{n_1!\cdots\cdots n_p!}$$

$$= \exp[2x(t_1+.....+t_p)-(t_1+.....t_p)^2].H_{r_1,......,r_p}(x-t_1-......-t_p), \tag{2.6}$$

$$\sum_{n,m,r_1,r_2,s_1=0}^{\infty} \frac{H_{n+r_1+r_2,m+s_1+s_2}(x)t^n h^m u_1^{r_1} u_2^{r_2} v_1^{s_1}}{n!\,m!\,r_1!\,r_2!\,s_1!}$$

$$= \exp(A_1)\exp(A_2)\exp(A_3)H_{s_2}(X), \tag{2.7}$$

$$\sum_{n,m,r_1,r_2,s_1,s_2,k_1=0}^{\infty} \frac{H_{n+r_1+r_2,m+s_1+s_2,p+k_1+k_2}(x)t^n h^m g^p u_1^{r_1} u_2^{r_2} v_1^{s_1} v_2^{s_2} w_1^{k_1}}{n!\;m!\;p!\;r_1!\;r_2!\;s_1!\;s_2!\;k_1!}$$

$$= \exp(B_1)\exp(B_2)\exp(B_3)\exp(B_4)H_{k_2}(Y), \tag{2.8}$$

$$\sum_{n_1,\dots,n_p=0}^{\infty}\;\sum_{r_1,s_1}^{\infty}.....\sum_{r_{p-1},s_{p-1}}^{\infty}\;\sum_{r_p=0}^{\infty} \frac{H_{n_1+r_1+s_1,n_2+r_2+s_2,......,n_p+r_p+s_p}(x)t_1^{n_1}.....t_p^{n_p}u_1^{r_1}......u_p^{r_p}}{n_1!.....n_p!\;r_1!.....r_p!\;s_1!......\;s_{p-1}!}$$

$$= \exp(C_0)\exp(C_1).......\exp(C_p)H_{s_p}(Z), \tag{2.9}$$

where F-function on the RHS of (2.1), (2.2), (2.3) are generalized Lauricella functions of several different variables [4, 5], and

$A_1 = 2x(t+h)-(t+h)^2$, $A_2 = 2(x-t-h)(u_1+u_2)-(u_1+u_2)^2$, $A_3 = 2(x-t-h-u_1-u_2)v_1-v_1^2$;
$X = x-t-h-u_1-u_2-v_1$

$B_1 = 2x(t+h+g)-(t+h+g)^2$, $B_2 = 2(x-t-h-g)(u_1+u_2)-(u_1+u_2)^2$,

$B_3 = 2x(x-t-h-g-u_1-u_2)(v_1+v_2)-(v_1+v_2)^2$, $B_4 = 2x(x-t-h-g-u_1-u_2-v_1-v_2)w_1-w_1^2$;
$Y = x-t-h-g-u_1-u_2-v_1-v_2-w_1$

$C_0 = 2x(t_1+.....+t_p)-(t_1+.....+t_p)^2$, $C_1 = 2(x-t_1-.......-t_p)(u_1+v_1)-(u_1+v_1)^2$,

$C_p = 2x[x-(t_1+.....+t_p)-\{(u_1+v_1)+....+(u_{p-1}+v_{p-1})\}]u_p-u_p^2$,

$Z = n-(t_1+.....+t_p)-\{(u_1+v_1)+....+(u_{p-1}+v_{p-1})\}-u_p$,

Proof of (2.1) : Taking LHS and using the definition (1.3) of $H_{n,m}(x)$, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(C)_n(d)_m H_{n,m}(x)t^n h^m}{n!\;m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}\frac{(c)_n(d)_m(-1)^{r+s}(2r+2s)!(2x)^{n+m-2r-2s}t^n h^m}{(2r)!(2s)!(r+s)!(n-2r)!(m-2s)!}$$

$$= \sum_{n,=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(c)_{n+2r}(d)_{m+2s}(-1)^{r+s}(2r+2s)!(2x)^{n+m}t^{n+2r}h^{m+2s}}{n!\;m!\;(2r)!\;(2s)!\;(r+s)!}$$

$$= \sum_{n,m,r,s=0}^{\infty} \frac{(c)_{n+2r}(d)_{m+2s}(-1)^{r+s}(2x)^{n+m}t^{n+2r}h^{m+2s}\left(\frac{1}{2}\right)_{r+s}}{(n)!(m)!r!s!\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_s},$$

using the definition of generalized Lauricella function of several variable [4, eq. (4.1), p.454], we get the required result (2.1).

Proceeding on the same lines we obtain the results (2.2) and (2.3).

Proof of (2.4) : Taking

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{H_{n+r,m+s}(x)t^n h^m u^r v^s}{n!m!r!s!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{r=0}^{\infty}\frac{n!}{(n-r)!}t^{n-r}u^r\right)\left(\sum_{s=0}^{\infty}\frac{m!}{s!(m-s)!}h^{m-s}v^s\right)\frac{H_{n,m}(x)}{n!m!}$$

using Binomial expansion, $(a+x)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$, we obtain

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{H_{n,m}(x)(t+u)^n(h+v)^m}{n!m!}$$

using the definition (1.9), we obtain

$$= \exp\left[2x(t+h+u+v)-(t+h+u+v)^2\right]$$

$$= \exp\left[2x(t+h)-(t+h)^2\right].\exp\left[2(x-t-h)(u+v)-(u+v)^2\right]$$

using the definition once again (1.9), we arrive at

$$= \exp\left[2x(t+h)-(t+h)^2\right].\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{H_{r,s}(x)(x-t-h)u^r v^s}{r!s!}$$

On comparing the coefficient of $u^r v^s$, we get the required result (2.4)

Similarly, the results (2.5) and (2.6) can easily be obtained.

Proof of (2.7) : Taking

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{H_{n+r_1+r_2,m+s_1+s_2}(x)t^n h^m u_1^{r_1}u_2^{r_2}v_1^{s_1}v_2^{s_2}}{n!m!r_1!r_2!s_1!s_2!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{H_{n+r_2,m+s_2}(x)t^{n-r_1}h^{m-s_1}u_1^{r_1}u_2^{r_2}v_1^{s_1}v_2^{s_2}}{(n-r_1)!(m-s_1)!r_1!r_2!s_1!s_2!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{s_1=0}^{m}\sum_{s_2=0}^{m-s_1}\frac{H_{n,m}(x)t^{n-r_1-r_2}h^{m-s_1-s_2}u_1^{r_1}u_2^{r_2}v_1^{s_1}v_2^{s_2}}{(n-r_1-r_2)!(m-s_1-s_2)!r_1!r_2!s_1!s_2!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\frac{n!t^{n-r_1-r_2}u_1^{r_1}u_2^{r_2}}{(n-r_1-r_2)!r_1!r_2!}\right)\left(\sum_{s_1=0}^{m}\sum_{s_2=0}^{m-s_1}\frac{m!h^{m-s_1-s_2}v_1^{s_1}v_2^{s_2}}{(m-s_1-s_2)!s_1!s_2!}\right)\frac{H_{n,m}(x)}{n!m!}$$

using $(x+y+z)^n = \sum_{r=0}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}x^{n-r-s}y^r z^s$, we get

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{H_{n,m}(x)(t+u_1+u_2)^n(h+v_1+v_2)^m}{n!m!}$$

using the definition (1.9), we obtain

$$= \exp\left[2x(t+h+u_1+u_1+v_1+v_2)-(t+h+u_1+u_1+v_1+v_2)^2\right]$$

$$= \exp(A_1).\exp\left[2x(u_1+u_1+v_1+v_2)-(u_1+u_2+v_1+v_2)^2-2(t+h)(u_1+u_1+v_1+v_2)\right]$$

$$= \exp(A_1).\exp(A_2).\exp\left[2(x-t-h)(v_1+v_2)-(v_1+v_2)^2+2(u_1+u_2)(v_1+v_2)\right]$$

$$= \exp(A_1).\exp(A_2).\exp(A_3).\exp\left[2(x-t-h-u_1-u_2-v_1)v_2-v_2^{\,2}\right]$$

using the definition, we arrive at

$$= \exp(A_1).\exp(A_2).\exp(A_3).\sum_{s_2=0}^{\infty}\frac{H_{s_2}(x)v_2^{\,s_2}}{s_2!}$$

On comparing the coefficient of $v_2^{\,s_2}$, we get the required result (2.7) and proceeding on the same lines we can easily get the results (2.8) and (2.9).

3. PARTICULAR CASES

(i) In (2.1) taking $m=0$ (h will vanish)
or In (2.2) taking $m=p=0$ (h, g will vanish)
or In (2.3) taking $n_2=n_3=\ldots=n_p=0$ ($t_1,\ldots,t_p$ will vanish)
we get known relation [1, eq. (1), p. 190].

(ii) In (2.4) taking $m=s=0$
or In (2.5) taking $m=s=p=u=0$
or In (2.6) taking $n_2=r_2=\ldots=n_p=r_p=0$
we get a known relation [1, eq. (2), p. 198] after little manoeuvring.

(iii) Similarly specializing the index in the results proved we obtain many more new results.

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