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APPLICATION OF HOLDITCH THEOREM TO BEZIER AND QUADRATIC UNIFORM B-SPLINE CURVES

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Abstract-Parametric polynomial functions are one of the most common types of functions that are used in curve and surface design. Bezier and quadratic uniform B-Spline curves are some common examples of those functions. Crossectional areas between these curves have great importance in computer aided geometric design (CAGD) and computer aided geometric manufacturing (CAM). While calculating these areas, the kind of the curves and surfaces of which the objects are composed, is very important. After determining the types of the curves, we choose one of the most suitable methods to calculate the area. In this paper, we used Holditch Theorem to calculate the areas between two Bezier curves and two quadratic uniform B-Spline curves.

1.INTRODUCTION

The shapes of industrial products can be classified into two groups : first, expressible in terms of the combination of simple geometric curves and surfaces; second, can not be expressed, [1].

Most of the machines composed of planar, cylindrical or spherical parts are the elements of the first group, but the objects like the bodies of the cars, ships or electrical machines are the combinations of complex curves and surfaces. Geometric modeling is the process of constructing a complete mathematical description of the shape of a physical object so that everybody can understand. So, we get not a subjective but an objective definition of the object. Such a definition can be manufactured by 3-dimensional numerical control machines, [1] . It's obvious that only a mathematical description is not sufficient to understand that object. So a visual display and manipulation of this model is done through computer graphics techniques. An understanding of both is utmost importance to engineers. We use mathematical transformations like translation, rotation, scaling and shearing for this purpose, [4].

2.SOME PRINCIPLES OF PRODUCING CURVES

One of the most appropriate ways of defining curves and surfaces is the parametric representation. The first attempt for this was made by De Casteljau. Then, this algorithm has been generalized to Bezier curves and surfaces. We can produce plane curves by De Casteljau algorithm in the following way,

Let b_0, b_1 and b_2 be three points in E^3.

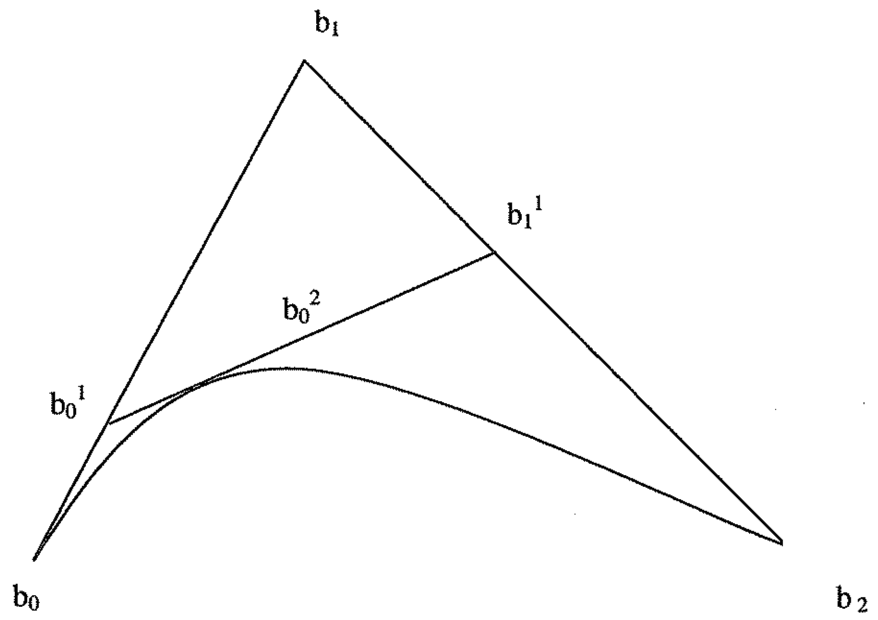

Fig. 1

According to figure1,

$$b_0^2(t)=(1-t)^2b_0+2t(1-t)b_1+t^2b_2 \quad (2.1)$$

is obtained, [2].(b_0^2 shows the degree of the curve, i.e. 2^{nd} degree)

By generalizing this curve representation, Bernstein-Bezier curves are produced. These curves have the following general definition,

$$b_i^n = \sum_{i=0}^{n} C(n;i)(1-t)^{n-i}t^i b_i \quad (2.2)$$

In figure2 the polygon formed by P_0,P_1 and P_2 is called convex polygon(hull) of the curve. The curve passes through points P_0, P_2 and the tangents of the curve at P_0 and P_2 are in the directions of P_0P_1 and P_1P_2, respectively, [4].

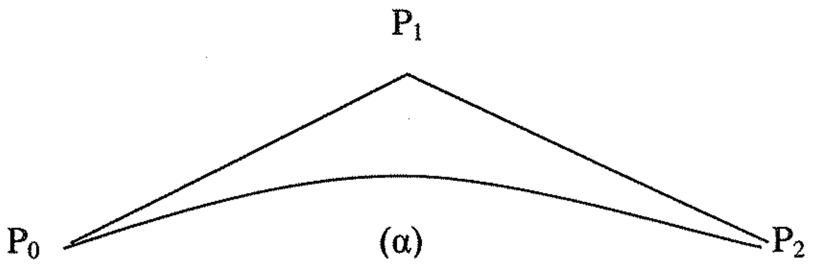

Fig. 2

In equation (2.1), the functions, $(1-t)^2 = B_{0,0}$; $2t(1-t) = B_{0,1}$; $t^2 = B_{0,2}$ are called Blending Functions,[3].

Here, $\Sigma B_{i,j}=1$. So $\alpha(t)$ is a *Baricentric Combination* of the points P_0 , P_1 and P_2.

Then, all the points of the curve lies in the convex polygon formed by the points P_0 ,P_1 and P_2, [2].

3.FORMING CLOSED BEZIER CURVES

To form a closed Bezier curve, let's draw an n-sided equilateral convex polygon with the vertices A_i (fig3). On the sides of this polygon, let's take points P_i which divides each side into two parts of length a and b units. Now, we begin to form our closed Bezier curve.

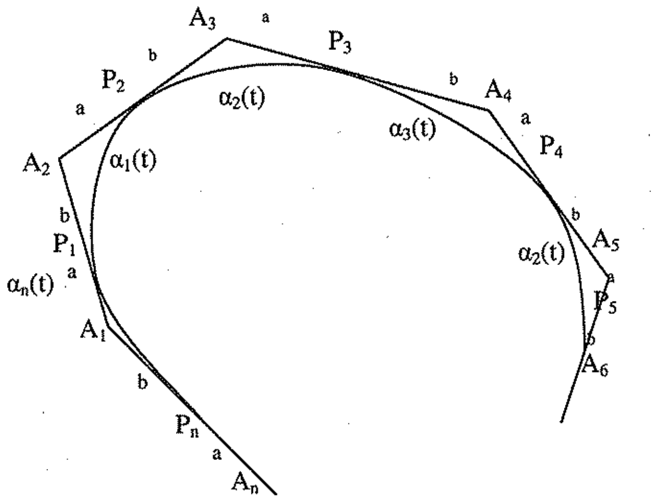

Fig. 3

Curves $\alpha_i(t)$ defined in each interval have the following equations,

$$\alpha_1(t)=(1-t)^2 P_1+2t(1-t)A_2 +t^2P_2 \quad (3.1)$$

$$\alpha_2(t)= (1-t)^2 P_2+2t(1-t)A_3 +t^2P_3 \quad (3.2)$$

.

.

$$\alpha_n(t)=(1-t)^2 P_n+2t(1-t)A_1 +t^2P_1 , \qquad t\in[0,1] \quad (3.n)$$

Let's define, $(\beta) = (\alpha_1) \cup (\alpha_2) \cup ... \cup (\alpha_n)$. Here, $\beta(t)$ is a closed Bezier curve and α_i have position and tangent continuity at the junction points,since

$\alpha_i(1) = \alpha_{i+1}(0)$ and $\alpha_i(1) = k_1\, \alpha_{i+1}(0)$ [4] .

Now our figure becomes,

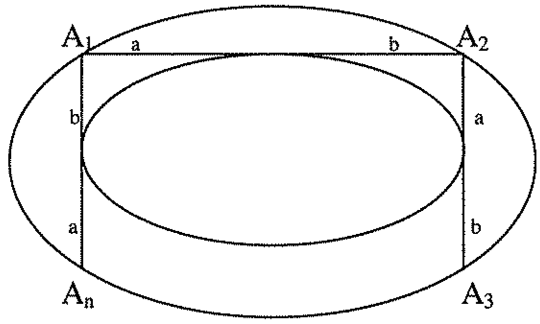

Fig. 4

4.CALCULATING CROSSECTIONAL AREAS BETWEEN BEZIER CURVES

We had pointed out that in some designs crossectional areas are very important. Now, let's try to calculate the area between two curves given in figure4. Problem turns to figure5,

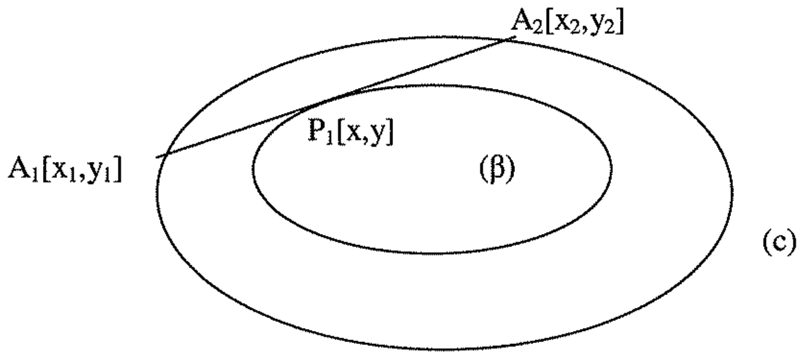

Fig. 5

A curve (c), a chord A_1A_2 taken on that curve and a point P_1 which divides that chord in the ratio a/b is the general aspect of the figure. While the chord A_1A_2 moving on the curve (c), point P_1 will also draw a curve (β). Let $A_1 = (x_1,y_1)$, $A_2=(x_2,y_2)$ and $P_1=(x,y)$.

Here, since both of the end points of the chord A_1A_2 draws the same curve (c), both end points will satisfy the equation of the curve (c). The point $P_1(x,y)$ which divides the chord A_1A_2 in the ratio $\pm$ a/b will draw a curve (β), [5]. Here,

$$x=(ax_1+bx_2)/(a+b) \quad \text{and} \quad y=(ay_1+by_2)/(a+b) \tag{4.1}$$

If we accept the area S_1 of the closed curve (c), as the area of the curve drawn by A_1 first and then the area of the curve drawn by A_2, this area S_1 will be [5],

$$S_1 = \int_{t_1}^{t_2} y_1 dx_1 = \int_{t_1}^{t_2} y_2 dx_2 \tag{4.2}$$

Let the area of the curve (β) which the point P_1 draws be S_2 [5],

$$S_2 = \int_{t_1}^{t_2} y dx \tag{4.3}$$

To find the crossectional area $S= S_1-S_2$, let's multiply S_1 by a and by b, then we get [5],

$$a S_1 = \int_{t_1}^{t_2} a y_1 dx_1 \tag{4.4}$$

$$b S_1 = \int_{t_1}^{t_2} b y_2 dx_2 \tag{4.5}$$

After simplifying we get [5],

$$S = \frac{ab}{(a+b)^2} \int_{t_1}^{t_2} (y_2 - y_1)(dx_2 - dx_1) \tag{4.6}$$

If we draw a parallel line from the origin to the chord, the coordinates of the end points of the line segment formed will be,

$$(r,k)=(x_2-x_1 , y_2- y_1) \ . \tag{4.7}$$

While the points A_1, A_2 are drawing a closed curve, the end point (r,k) will draw a circle with the radius a+b. If we calculate the area of the circle and make necessary simplifications, we get the crossectional area S as [5],

$$S=\pi ab. \tag{4.8}$$

5.UNIFORM QUADRATIC B-SPLINES

We can show a quadratic spline in matrix form as follows [1],

$$\mathbf{P}_i(t) = \frac{1}{2} \cdot \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \end{bmatrix} \tag{5.1}$$

In this case, only 3 control vertices are used.

Uniform B-Splines are convenient to represent closed curves. The only thing needed is a change in the number of the curve segments. Figure6 shows a closed quadratic B-Spline curve produced by 6 control points. B-Spline has 6 segments produced in following form,

Segment	Control Vertices
1	V_0 V_1 V_2
2	V_1 V_2 V_3
3	V_2 V_3 V_4
4	V_3 V_4 V_5
5	V_4 V_5 V_0
6	V_5 V_0 V_1

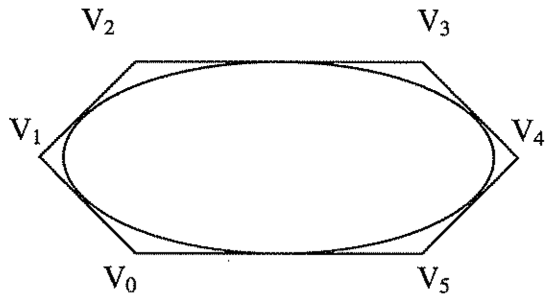

Fig.6 Closed quadratic uniform B-Spline curve

For a quadratic uniform B-Spline curve segment, the starting and ending locations can be found by the following way.

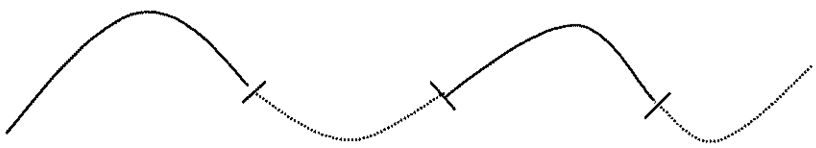

Fig. 7 Quadratic uniform B-Spline curve

From (5.1), for t=0 and t=1, we find,

$$\mathbf{P}_i(0) = \tfrac{1}{2}(V_{i-1} + V_i)$$

$$\mathbf{P}_i(1) = \tfrac{1}{2}(V_i + V_{i+1}) \tag{5.2}$$

respectively. Then, in quadratic uniform B-Splines, the joints between segments are located halfway between control vertices, [1].

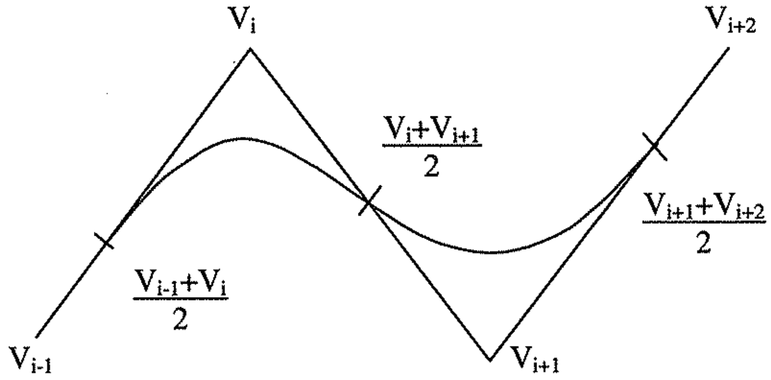

Fig. 8 Location of joints between uniform quadratic B-Spline segments

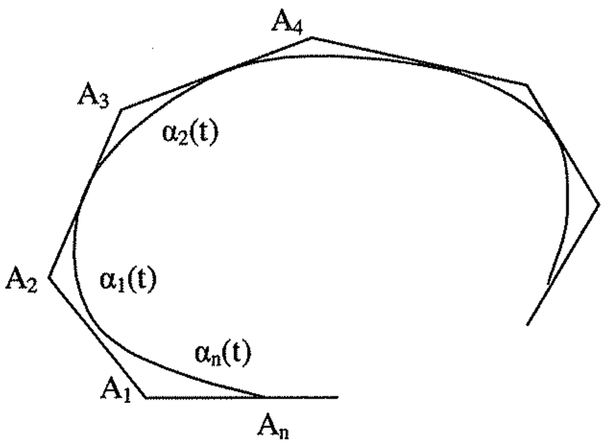

Fig. 9 Closed quadratic uniform B-Spline curve with n control points

If we take the vertices of this polygon as control points, quadratic uniform B-Spline curve will pass from midpoints of sides. Equations of the curve segments will be,

$$\alpha_1(t) = \frac{1}{2}\cdot\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$\alpha_2(t) = \frac{1}{2}\cdot\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

.

.

$$\alpha_n(t) = \frac{1}{2}\cdot\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} A_n \\ A_1 \\ A_2 \end{bmatrix} \tag{5.3}$$

$$(\beta) : (\alpha_1) \cup (\alpha_2) \cup \ldots \cup (\alpha_n) = \beta(t) \tag{5.4}$$

will give us a closed curve. As in the Bezier case, we can find the area between these closed curves by the help of Holditch Theorem as πa^2, (see fig.10).

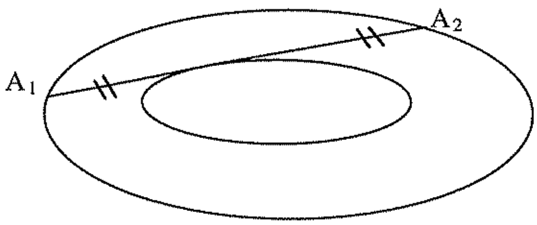

Fig. 10 Two closed quadratic uniform B-Spline curves

6.CONCLUSION

Bezier and uniform B-Spline curves have great importance in curve and surface design. In some applications, we need crossectional areas between two curves. Holditch theorem can be applied to find the area between two Bezier or two quadratic uniform B-Spline curves.

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