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FREDHOLM-VOLTERRA INTEGRAL EQUATION WITH POTENTIAL KERNEL

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Abstract-A method is used to solve the Fredholm-Volterra integral equation of the first kind in the space $L_2(\Omega) \times C(0, T)$, $\Omega = \{(x, y) \in \Omega : \sqrt{x^2 + y^2} \le a \, , \, z = 0\}$ and $T < \infty$. The kernel of the Fredholm integral term is considered in the generalized potential form belongs to the class $C([\Omega] \times [\Omega])$, while the kernel of Volterra integral term is a positive and continuous function belongs to the class $C[0,T)$. Also in this work the solution of Fredholm integral equation of the second and first kind with a potential kernel is discussed. Many interesting cases are derived and established from the wok.

Keywords-Fredholm-Volterra integral equations - generalized potential kernel - logarithmic kernel - Carleman kernel - Jacobi polynomials.

1.INTRODUCTION

Many problems of mathematical physics, theory of elasticity and mixed problems of mechanics of continuous media reduce to an integral equation with a kernel that have either of the following form

$$K^{\alpha,\gamma}_{n,m}(x, y) = \frac{x^{\alpha}}{y^{\varepsilon+\gamma-1}} W^{\alpha}_{n,m}(x, y)$$

$$W^{\alpha}_{n,m}(x, y) = \int_0^{\infty} \lambda^{\alpha} J_n(x\lambda) J_m(y\lambda)\, d\lambda \tag{1.1}$$

where $J_n(x)$ is a Bessel function of the first kind of order n. Arutyunyan [1] has shown that, the plane contact problem of the nonlinear theory of plasticity, in its first approximation, can be reduced to Fredholm integral equation of the first kind with Carleman kernel

$$K^{\alpha,\frac{1}{2}}_{\pm\frac{1}{2},\pm\frac{1}{2}}(x, y) = |x-y|^{-\alpha} = \sqrt{xy} \int_0^{\infty} \lambda^{\alpha} J_{\pm\frac{1}{2}}(x\lambda) J_{\pm\frac{1}{2}}(y\lambda) d\lambda \quad , \quad (\varepsilon = 0 \; , \; 0 \le \alpha < 1) \tag{1.2}$$

(for the symmetric and skew symmetric cases respectively)

In [2,3] Mkhitaryan and Abdou obtained the general formulas, even and odd, of the potential analytic function, using Krein's method [4], for the Fredholm integral equation of the first kind with Carleman kernel [2] and logarithmic kernel [3].

$$K^{0,\frac{1}{2}}_{\pm\frac{1}{2},\pm\frac{1}{2}}(x, y) = -\ln|x-y| = \sqrt{xy} \int_0^{\infty} J_{\pm\frac{1}{2}}(x\lambda) J_{\pm\frac{1}{2}}(y\lambda) d\lambda \quad , \quad (\varepsilon = 0) \tag{1.3}$$

(for symmetric and skew symmetric, respectively)

Kovalenko [5] developed the Fredholm integral equation of the first kind for the mechanics mixed problems of continuous media and obtained an approximate solution for the Fredholm integral equation of the first kind with an elliptic kernel

$$K_{0,0}^{0,1}(x,y) = \frac{2\sqrt{xy}}{\pi(x+y)} K\left(\frac{\sqrt{2xy}}{x+y}\right) = \int_0^\infty J_0(x\lambda)J_0(y\lambda)d\lambda \quad , \quad (\varepsilon = 0) \tag{1.4}$$

Abdou in [6] obtained the solution of Fredholm integral equation of the second kind with potential function kernel, $\left(K(x-\xi, y-\eta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \right)$,

$$K_{m,m}^{0,\frac{1}{2}}(x,y) = \sqrt{xy} \int_0^\infty J_m(\lambda x)J_m(\lambda y)d\lambda \quad , \quad (\varepsilon = 0) \tag{1.5}$$

Also, in [7], the structure resolvent for the Fredholm integral equation of the second kind with potential function kernel is obtained by Abdou. The potential theory method is used in [8,9] to obtain the eigenvalues and eigenfunction for a system of Fredholm integral equation of the first kind with Carleman kernel in [8] and logarithmic kernel in [9]. Abel's theorem is used in [10] to obtain the general solution of Fredholm integral equation of the first kind with a kernel in the form of Gauss hypergeometric function

$$K(x,y) = \frac{1}{(x^2+y^2)^{2n}} F\left(n, n+\frac{1}{2}, m, \left(\frac{2xy}{x^2+y^2}\right)^2\right) \tag{1.6}$$

The solution in Matheiu function form is obtained in [9], where the potential theory method is used for the contact problem, where the domain of integration Ω is represented as $\Omega : (x,y,z) \in \Omega : -\infty < x,y < \infty,\ z > 0$, of mechanics of continuous media between a finite system of stamps varying width and an elastic half-space in a three dimensional formulation.

In this paper, the solution of Fredholm-Volterra integral equation of first kind is obtained in $L_2(\Omega) \times C(0, T)$ where $\Omega = \left\{(x,y) \in \Omega : \sqrt{x^2 + y^2} = r \le a,\ z = 0\right\}$ and the time $t \in [0,T]$, $T < \infty$. The problem is investigated from the three dimensional semi-symmetric contact problem in the theory of elasticity of frictionless impression of a rigid surface (G,ν) having an elastic material occupying the domain Ω, where the external forces are neglected. Assume a function $f(x,y) \in L_2(\Omega)$ which describing the surface of stamp, such that, this stamp is impressed into the elastic layer surface (plane) by a variable force $M(t)$, whose eccentricity of application $e(t)$, that cases a rigid displacement $\delta(t) \in C(0,T)$. The integral equation, in this case, becomes [6]

$$\iint_\Omega \frac{P(\xi,\eta,t)\,d\xi\,d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\frac{1}{2}}} + \int_0^t F(\tau)P(x,y,\tau)d\tau = \pi\theta[\delta(t) - f(x,y)] = f(x,y,t)$$

$$(\theta = G(1-\nu)^{-1}) \tag{1.7}$$

under the consdition

$$\iint_\Omega P(x,y,t)\,dx\,dy = M(t) \quad , \quad 0 \le t \le T < \infty. \tag{1.8}$$

Here $F(t)$ is a positive continuous function belongs to the class $C(0,T)$ and represents the characterized resistance of the elastic layer, $P(x,y,t)$ is the unknown potential normal stress function between the surface of stamp and the elastic layer, G is the displacement magnitude and ν is poisson's coefficient.

In this work, the Fredholm integral equations of the first and second kind with a generalized potential kernel are established and their solutions are discussed the kernel is represented in the Weber-Sonin integral formula. Many interesting spectral relationships are derived from the problem. Finally a numerical example is considered for the solution of Fredholm integral equation of the second kind.

2.BASIC EQUATIONS

Here, in this section, a method is used to obtain a finite system of integral equation in three dimensional, then by using the method of separation of variables we represent the integral equation to a system of Fredholm integral equation of the second kind in one dimensional. Also the kernel of Fredholm integral equation is represented in the Weber-Sonin integral formula.

So, we divide the interval $[0,T]$, $0 \le t \le T < \infty$ as $0 = t_0 < t_1 < t_2 \ldots < t_N = T$, where $t = t_k \in [0,T]$, $k = 0, 1, \ldots N$; then by using the quadratic formula [11] u_j, $j = 0, 1, \ldots k$, in the Volterra integral term of (1.1), we have

$$\int_0^{t_k} F(\tau)\, P(x,y,\tau)\, d\tau = \sum_{j=0}^{k} u_j F_j P_j(x,y) + 0(\hbar^{p+1}) \qquad (h_k \to 0\,,\ p>0) \tag{2.1}$$

where $\hbar_k = \max\limits_{0\le k\le N} h_k$, $h_j = t_{j+1} - t_j$, $P(x,y,t_k) = P_k(x,y)$, $F(t_j) = F_j$, and

$$u_j = \begin{cases} \dfrac{h}{2} & j = 0,k \\ h & j \ne 0,k \end{cases}$$

The number values of u_j and p, $p \simeq k$ depend on the number of derivatives of F(t) (see [11]).

Using (2.1) in (1.1), we have

$$u_k F_k P_k(x,y) + \iint_\Omega \frac{P_k(\xi,\eta)\, d\xi\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\frac{1}{2}}} + \sum_{j=0}^{k-1} u_j F_j P_j(x,y)$$

$$= \pi\theta\left[\delta_k - f(x,y)\right] = f_k(x,y) \tag{2.2}$$

$$(\delta_k = \delta(t_k) \quad , \quad k\ 0,1,\ldots N)$$

Also the condition (1.2) becomes

$$\iint_\Omega P_k(x,y)\, dx\, dy = M_k \qquad (M(t_k) = M_k)\,. \tag{2.3}$$

The solution of the integral equation (2.2) depends on the kernel and the values of F_k at the two points t_0 and t_N, for example if $F(t_0) = F_0 = 0$ the first equation of the linear integral system of (2.2) represents an integral equation of the first kind, then for all values of $k > 1$ we have a linear system of integral equation of the second kind, while for $t_N = 0$, the formula (2.2), for $0 \le k \le N-1$, represents a linear system of integral equation of the second kind and the final equation, at $k = N$, represents an integral equation of the first kind.

To separate the variables, one assume

$$P_k(x,y) = P_{km}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases} \quad , \qquad f(x,y) = f_{km}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases} \tag{2.4}$$

Using (2.4) in (2.2) and (2.3), we have

$$u_k F_k P_{km}(r) + \int_0^a \rho\, W_m^{\frac{1}{2}}(r,\rho) P_{km}(\rho)\rho\, d\rho + \sum_{j=0}^{k-1} u_j F_j P_{jm}(r) = f_{km}(r) \tag{2.5}$$

and

$$\int_0^a \rho P_{km}(\rho)\,d\rho = \begin{cases} \dfrac{M_k}{2\pi} & m=0 \\ 0 & m\geq 1 \end{cases} \tag{2.6}$$

where

$$W_m^\alpha(r,\rho) = \int_{-\pi}^{\pi} \frac{\cos m\phi\, d\phi}{[r^2+\rho^2-2r\rho\cos\phi]^\alpha} \quad , \quad (\alpha = \frac{1}{2}+\ell \;,\; \ell < \frac{1}{2}) \tag{2.7}$$

For writing the integral (2.7) in the Bessel function form, firstly we use the following relations [12, pp. 81]

$$\int_0^{2\pi} \frac{\cos m\phi\, d\phi}{[1-2z\cos\phi+z^2)^\alpha} = \frac{2\pi(\alpha)_m z^m}{m!} F(\alpha, m+\alpha, m+1, z^2) \tag{2.8}$$

and

$$F(\gamma\,,\,\gamma+\frac{1}{2}-\beta\,,\,\beta+\frac{1}{2}\,,\,z^2) = (1+z)^{-2\gamma}\, F\left(\gamma\,,\,\beta\,;\,2\beta\,;\frac{4z}{1+z^2}\right)$$

$$\left(|z|<1\,,\,\mathrm{Re}\gamma>0\,,\,(\gamma)_m = \frac{\Gamma(m+\gamma)}{\Gamma(\gamma)}\right) \tag{2.9}$$

Hence Eq. (2.7) takes the form

$$W_m^\alpha(r,\rho) = \frac{2\pi\Gamma(m+\alpha)}{m\Gamma(\alpha)} \frac{(r\rho)^m}{(r+\rho)^{2m+1}} F\left(m+\alpha\,,\,m+\frac{1}{2}\,,\,2m+1\,,\,\frac{4r\rho}{(r+\rho)^2}\right) \tag{2.10}$$

where F(a,b,c;z) is the Gauss hypergeometric function, and $\Gamma(x)$ is the Gamma function. Formula (2.10) is symmetric and does not depend on the relation between ρ and r.

Secondly, using the relation [13]

$$\int_0^\infty J_\alpha(ax) J_\alpha(bx) x^{-\beta}\,dx =$$

$$= \frac{2^{-\beta} a^\alpha b^\alpha \Gamma\left(\alpha+\frac{1-\beta}{2}\right)}{(a+b)^{2\alpha-\beta+1}\Gamma(1+\alpha)\,\Gamma\left(\frac{1+\beta}{2}\right)} F\left(\alpha+\frac{1-\beta}{2}\,,\,\alpha+\frac{1}{2}\,,\,2\alpha+1\,,\,\frac{4ab}{(a+b)^2}\right),$$

($J_\alpha(x)$ is the Bessel fnction) (2.11)

equation (2.10) takes the form

$$W_m^{\frac{1}{2}}(r,\rho) = 2\pi \int_0^\infty \lambda_1^{2\ell} J_m(\lambda_1\rho) J_m(\lambda_1 r)\,d\lambda_1 \tag{2.12}$$

Using (2.12), and the following notations

$$u = \frac{r}{a}\,,\; v = \frac{\rho}{a}\,,\; \Phi_{km}(u) = \frac{P_{km}(au)}{\sqrt{au}}\,,\; \lambda = a\lambda_1\,,$$

$$c^* = a^{1+2\ell}\,2\pi\,,\; g_{km}(u) = \frac{f_{km}(au)}{\sqrt{au}} = \frac{2\pi\theta}{\sqrt{au}}[\delta_k - f_m(au)]\,,\; Q_k = \frac{M_k}{2\pi a}\,,$$

$$(k = 0, 1, \ldots N \quad ; \quad m \geq 0) \tag{2.13}$$

the integral equation (2.5) and the condition (2.6) become

$$\mu_k \Phi_{km}(u) + \int_0^1 K_m^{\frac{1}{2}}(u,v)\Phi_{km}(v)dv + \sum_{j=0}^{k-1}\mu_j \Phi_{jm}(u) = g_{km}(u) \tag{2.14}$$

and

$$\int_0^1 \sqrt{v}\ \Phi_{km}(v)dv = \begin{cases} Q_k & m=0 \\ 0 & m \ge 1 \end{cases} \tag{2.15}$$

where

$$K_m^{\alpha}(u,v) = 2\pi a \sqrt{uv}\int_0^{\infty} \lambda^{2\ell} J_m(\lambda u) J_m(\lambda v) d\lambda \tag{2.16}$$

which represents a Weber-Sonin integral formula.

It is easy to prove the following relation

$$\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) K_m^{\frac{1}{2}}(u,v) = \left((h(u) - h(v)\right) K_m^{\frac{1}{2}}(u,v) \tag{2.17}$$

where

$$h(x) = \left(m^2 - \frac{1}{4}\right)x^{-2} \qquad (m \ne \pm\frac{1}{2}) \tag{2.18}$$

The integral equation (2.14) represents a linear system of Fredholm integral equation of the first or second kind depending on the values of μ_k , $k \in [0,N]$. The general solution of (2.14) can be obtained using the recurrence relations for values of k and the mathematical induction. For this aim, let $k = 0$ in (2.14) and (2.15), we obtain

$$\mu_0 \Phi_{0m}(u) + \int_0^1 K_m^{\frac{1}{2}}(u,v)\Phi_{0m}(v)dv = g_{0m}(u) \tag{2.19}$$

under the condition

$$\int_0^1 \sqrt{v}\,\Phi_{0m}(v)dv = \begin{cases} Q_0 & m=0 \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

The kind and solution of (2.19) depends on the values of μ_0, for this, we go to obtain the solution of (2.19), firstly when $\mu_0 \to 0$ and secondly when μ_0 satisfies the relation

$$\mu_0 > \int_0^1\int_0^1 K_m^{\frac{1}{2}}(u,v)\,du\,dv \quad , \quad m = 0, \pm\frac{1}{2}, \pm 1, \ldots\ldots. \tag{2.21}$$

3.FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

In this section, we will obtain the general solution of Fredholm integral equation of the first kind when the kernel takes a Weber-Sonin integral formula and for any continuous values of $g_{0m}(u)$. Also many spectral relationships are established here.

When $F_0 = 0$, we have $\mu_0 = 0$ and Eq. (2.19) becomes

$$\int_0^1 K_m^{\frac{1}{2}}(u,v)\Phi_{0m}(v)dv = g_{0m}(u) \tag{3.1}$$

Abdou in [8] used potential theory method [14], to solve a linear system of Fredholm integral equation in a form of (3.1) under the condition (2.20) where the given function is represented in the Jacobi polynomials form. Here, we go to obtain the solution of (3.1) under (2.20) for given continuous function $g_{0m}(u)$. For this, rewrite (3.1) and (2.20) as an integral equation of the Wiener-Hopptype [15,16]. For setting

$u = e^{-\xi}$, $v = e^{-\eta}$, $e^{-\xi}\,\Phi_{0m}(e^{-\xi}) = \Psi_m(\xi)$, and $g_{0m}(e^{-\xi})\,e^{-\gamma\xi} = h_m(\xi)$ in (3.1) and (2.20), we have

$$\int_0^\infty M(\xi-\eta)\,\Psi_m(\eta)d\eta = h_m(\xi) \qquad 0 \le \xi < \infty \tag{3.2}$$

and

$$\int_0^\infty e^{-\eta/2}\Psi_m(\eta)\,d\eta = \begin{cases} Q_0 & m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

where

$$M(\xi-\eta) = e^{-\gamma(\xi-\eta)}K_m^{\gamma}(e^{-\xi}, e^{-\eta}) \tag{3.4}$$

Popov [16] stated that in order to obtain the solution of (3.2) under the condition (3.3), it suffices to obtain the most simple equation

$$\int_0^\infty M(\xi-\eta)\,\psi_{zm}(\eta)d\eta = e^{iz\xi} \quad , \xi, \operatorname{Im} z \ge 0 \tag{3.5}$$

Now, making use of the formulae

$$\Psi_m(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(-z)\,\psi_{zm}(\xi)\,dz \,,$$

$$G(z) = \int_0^\infty h_m(\xi)e^{i\xi z}\,d\xi \tag{3.6}$$

The solution of (3.5) (see [1, 16]) is given by

$$\psi_{zm}(u) = \frac{1}{u}\psi_{zm}(\ell n\frac{1}{u}) = \frac{\psi_m^-(-z)}{\Gamma(\frac{3}{4})}\{(1-u^2)^{-\frac{1}{4}} + (m+1+iz)\int_u^1 \frac{t^{-m-2-iz}}{(t^2-u^2)^{\frac{1}{4}}}dt\} \tag{3.7}$$

where

$$\psi_m^-(-z) = \sqrt{2}\;\Gamma(\frac{1}{2}(m+\frac{5}{4}-iz))\;(\Gamma(m+\frac{1}{2}-iz))^{-1} \tag{3.8}$$

After obtaining the solution of (3.7), we can derive the general solution of Eq. (3.2). It is easy to see that the function $\sqrt{u}\;\psi_{zm}(u)$ is a solution of Eq. (3.1) when $g_{om}(u) = u^{-1-iz}$. Therefore the general solution of the integral equation

$$\int_0^1 K_m^{\frac{1}{2}}(u,v)\,q_m^{\frac{1}{2}}(v,1)\,dv = 1 \qquad 0 \le u < 1 \tag{3.9}$$

is given by

$$q_m^{\frac{1}{2}}(u,1) = \sqrt{u}\,[\psi_{zm}(u)]_{z=i} \tag{3.10}$$

By using the principal of Krein's [4], with the aid of (3.10), the general solution of (3.1) takes the form

$$\Psi_{0m}(u) = \frac{\sqrt{2}\,u^{m+1/2}}{\Gamma(w^{\frac{3}{4}})\Gamma(w^{\frac{1}{4}})}\left\{\frac{X(1)}{(1-u^2)^{\frac{1}{4}}} - \int_u^1 \frac{X^1(v)\,dv}{(v^2-u^2)^{\frac{1}{4}}}\right\}$$

$$X(u) = \frac{u^{-2m-\frac{1}{2}}}{c^*}\frac{d}{du}\int_0^u \frac{s^{m+1/2}\,g_{0m}(s)\,ds}{(u^2-s^2)^{\frac{1}{4}}} \quad , \qquad (c^* = 2\pi a) \tag{3.11}$$

Now, we can obtain many interesting cases, for example:

Replacing $g_{0m}(u)$ in (3.11) by a Jacobi polynomial i.e. let $g_{0m}(u) = P_m^{(m,-\frac{1}{4})}(1-2u^2)$, then Eq. (3.1) is transformed to become

$$\int_0^1 \frac{u^{1+m} K_m^{\frac{1}{2}}(u,v) P_m^{(m,-\frac{1}{4})}(1-2u^2)\,du}{(1-u^2)^{\frac{1}{4}}} = \lambda_m v^m P_m^{(m,-\frac{1}{4})}(1-2v^2),$$

$$\lambda_m = 2^{\frac{3}{2}} \Gamma(m+\frac{3}{4}) \Gamma(2m+\frac{3}{4}) [m! \Gamma(1+2m)]^{-1} \tag{3.12}$$

In terms of Gauss hypergeometric function of formula 8 of [3] pp. 715, we can obtain the following important property

$$K_m^{\alpha}(u^{-1}, v^{-1}) = (uv)^{\alpha} K_m^{\alpha}(u,v) \tag{3.13}$$

Using in (3.12) the substitution $u = x^{-1}$, $v = y^{-1}$, and making use of property (3.13), we obtain spectral relations of the semi-infinite interval

$$\int_1^{\infty} \frac{K_m^{\frac{1}{2}}(x,y) P_m^{(m,-\frac{1}{4})}(1-2y^{-2})\,dy}{y^z (y^2-1)^{\frac{1}{4}}} = \frac{\lambda_m P_m^{(m,-\frac{1}{4})}(1-2x^{-2})}{x^{\frac{3}{2}+m}} \tag{3.14}$$

$$(z = \frac{5}{4} + m, \; 1 \le x < \infty)$$

4.FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

In this section, the general solution of Fredholm integral equation of the second kind is obtained. Also the mathematical induction is used to obtain the general solution of Eq. (2.14) under the condition (2.15).

Now, our attention comes to obtain the solution of Fredholm integral equation of the second kind (2.19) under the condition (2.20), where its solution depends on the kernel (2.16) and the surface $f_m(r)$. When the initial and the tangent points of the surface are in contact with the origin 0, we can expand $f_m(u)$ in Macklorien expansion near $u = 0$

$$f_m(u) \cong \frac{f_m''(0)}{2!} u^2 + \frac{f_m'''(0)}{3!} u^3 + \dots\dots\dots\dots + \frac{f_m^{(n)}(0)}{n!} u^n + \dots\dots \tag{4.1}$$

The last equation gives the degree of displacement of the surface for any degree. For example, if the displacement is very small and $\frac{f_m''(0)}{2!} = A_2 \neq 0$, we obtain $f_m(u) = A_2 u^2$

In general, we write

$$f_m(u) = A_{2m} u^{2m} \quad , \quad A_{2m} = \frac{f_{(0)}^{(2m)}}{(2m)!} \quad (m \ge 0) \tag{4.2}$$

where m is the order harmonic of the contact problem.

Hence, the function $g_{0m}(u)$ takes the form

$$g_{om}(u) = (\Delta_0 - \beta A_{2m} u^{2m}) \sqrt{u} \qquad (\Delta_0 = \beta \delta_0 \quad , \quad \beta = \pi\theta) \tag{4.3}$$

Equation (4.3) represents a polynomial of degree $2m+\frac{1}{2}$ and the solution of Eq. (2.19) under the condition (2.20) depends on the kernel (2.16) and the function (4.3). So, rewrite (2.19) and (2.20) to take the following forms

$$\mu_0 Z_m(u)+\int_0^1 K(u,v)Z_m(v)\,dv = u^{2m+\frac{1}{2}} \tag{4.4}$$

and

$$\Delta_0\int_0^1 \sqrt{u}\,Z_0(u)\,du - A_{2m}\int_0^1 \sqrt{u}\,Z_m(u)\,du = Q_0 \tag{4.5}$$

where

$$\Phi_{om}(u) = \Delta_0 Z_0(u) - A_{2m} Z_m(u) \qquad (m \ge 1) \tag{4.6}$$

To solve Eq. (4.4), we use the formula (7.3911) of [13] and with the aid of [8,10], we can write the kernel (2.16) in the form

$$K_m^{\frac{1}{2}}(u,v) = c^*\sqrt{2}\,(u\,v)^{m+\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\Gamma^2(j+m+\frac{3}{4})P_j^m(u)P_j^m(v)}{\Gamma^2(j+1+m)(2j+m+\frac{3}{4})^{-1}} \tag{4.7}$$

where

$$P_j^m(u) = P_j^{(m,-\frac{1}{4})}(1-2u^2) \tag{4.8}$$

Here $P_j^{(m,-\frac{1}{4})}(x)$ is the Jacobi polynomial.

Hence, the solution of (4.4) with the kernel of (4.7) is equivalent to the solution of the linear system

$$\mu_0 X_i + c^*\sum_{j=0}^{\infty} A_j B_{ij} X_j = f_i \tag{4.9}$$

where

$$f_j = (2j+m+\frac{3}{4})^{\frac{1}{4}}\int_0^1 f_m(u)\,u^{m+1}\,P_j^m(u)\,du$$

$$A_j = \frac{1}{\sqrt{2}}\,\frac{\Gamma^2(j+m+\frac{3}{4})(2j+m+\frac{3}{4})^{\frac{1}{4}}}{\Gamma^2(j+m+1)},$$

and

$$B_{ij} = (2j+m+\frac{3}{4})(2i+m+\frac{3}{4})\int_0^1 u^{2m+1}P_i^m(u)P_j^m(u)\,du \tag{4.10}$$

The infinite linear system of (4.9) is solvable under the condition

$$\sum_{j=0}^{\infty}\left|c^* A_j B_{ij}\right| < \mu_0 \qquad , \qquad (c^* = 2\pi a) \tag{4.11}$$

Using the orthogonality of Jacobi polynomial, the general solution of (4.4) takes the form

$$\mu_0 Z_m(u) = u^{2m+\frac{1}{2}} - c^*\sum_{j=0}^{\infty}\frac{\sqrt{2}\,\Gamma^2(j+m+\frac{3}{4})u^m P_j^m(u)X_j^m}{(j+m+1)(2j+m+\frac{3}{4})^{-\frac{3}{4}}} \tag{4.12}$$

Hence by the mathematical induction, the solution of Eq. (2.14) can be obtained.

5.NUMERICAL COMPUTATIONS

In Table 1 for $j = 2$, $m = 3$, $\mu_0 = c^* = 1$, we have the following results

u	u^{2m+1}	Z(u)
0.1	0.00000	0.00000
0.2	0.0007	0.0009
0.3	0.0044	0.0045
0.4	0.0162	0.0156
0.5	0.1442	0.0415
0.6	0.1004	0.0937
0.7	0.2001	0.1883
0.8	0.3270	0.3096
0.9	0.6224	0.6070
1.0	1.0	1.03

6.CONCLUSIONS

From the above results and discussions, the following may be concluded:

(1) The three-dimensional semi-symmetric contact problem for a stamp impressed into a layer surface, which made of material according to the power law $\sigma_j = K_0\varepsilon_j$, $j = 1,2,3$, by a variable force N(t) represents a Fredholm-Volterra integral equation of the first kind.

(2) The generalized potential kernel represents a Weber-Sonin integral formula

$$K(u,v) = \sqrt{u\,v}\int_0^\infty J_m(tu)\,J_m(tv)\,dt$$

which represents a non homogeneous wave equations and the kernel can be written in the Legender polynomial form as follows

$$K_m^{\frac{1}{2}}(u,v) = \frac{1}{\sqrt{2}}(u\,v)^{m+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+m+\frac{3}{4})P_n^m(u)\,P_n^m(v)}{\Gamma^2(n+m+1)(2n+m+\frac{3}{4})^{-1}}$$

where $P_n^m(u)$ is Legendre polynomial.

(3) The Fredholm-Volterra integral equation of the first kind can be reduced to a finite linear system of Fredholm integral equation of the second kind.

(4) This paper is considered as a generalization of the worker of the contact problems in continuous media for the Fredholm integral equation of the first and second kind when the kernel takes the following forms: Logarithmic kernel, Carleman kernel, elliptic integral kernel, and potential kernel. Moreover the contact problems which leads us to the integro-differential equation with Cauchy kernel is contained also as a special case of Eq. (2.19). Also in this work the contact problems of higher-order ($m \geq 1$) harmonic are included as special cases.

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