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MANN AND ISHIKAWA TYPE PERTURBED ITERATIVE ALGORITHMS FOR GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS

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Abstract- In this paper, we consider the generalized nonlinear variational inclusions and develop Mann and Ishikawa type perturbed iterative algorithms for finding the approximate solution of this problem. By using the definition of multivalued relaxed Lipschitz operators, we discuss the convergence criteria for the perturbed algorithms.

Key words- Generalized nonlinear variational inclusions, Algorithms, Relaxed Lipschitz operators, Fixed-point.

1. INTRODUCTION

In 1994, Hassouni and Moudafi [3] have introduced a perturbed method for solving a new class of variational inequalities, known as variational inclusions. Very recently, this class of variational inclusions have been extended and generalized for multivalued maps by Huang [4] and Chang et al [2]. In this paper, we consider the generalized nonlinear variational inclusions and develop Mann and Ishikawa type perturbed iterative algorithms. By using the definition of multivalued relaxed Lipschitz operators, we prove that the approximate solution obtained by these algorithms converges to the exact solution of our variational inclusions.

2. PRELIMINARIES AND FORMULATION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. Let $\phi : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\partial\phi$ be the subdifferential of ϕ. Given a multivalued map $T: H \to 2^H$, where 2^H denotes the family of nonempty subsets of H, and $f,g:H \to H$ be single-valued maps with $Im(g) \cap dom(\partial\phi) \neq \phi$, then we consider the following generalized nonlinear variational inclusions problem (GNVIP):

(GNVIP): Find $x \in H$, $w \in T(x)$, such that $g(x) \cap dom(\partial\phi) \neq \phi$, and

$$\langle g(x) - f(w), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \forall y \in H. \tag{2.1}$$

Inequality (2.1) is called generalized nonlinear variational inclusion.

When $\phi \equiv \partial\phi$, the indicator function of closed convex set K in H, defined by

$$\phi(x) = 0, x \in K, \text{ and } +\infty, x \notin K,$$

then (GNVIP) reduces to the following generalized variational inequality problem (GVIP) considered by Verma [7]:

(GVIP): Find $x \in H$, $w \in T(x)$, such that $g(x) \in K$ and

$$\langle g(x) - f(w), y - g(x) \rangle \geq 0, \ \forall\, y \in H. \tag{2.2}$$

3. ITERATIVE ALGORITHM

In this section, we first establish the equivalence of the generalized nonlinear variational inclusion (2.1) to a nonlinear equation. Then we suggest an iterative algorithm for finding the approximate solution of (2.1).

Lemma 3.1 [1]: Elements $x \in H$ and $w \in T(x)$ are solutions of (GNVIP) if and only if x and w satisfy the following relation

$$g(x) = J^{\phi}_{\eta}\{g(x) - \eta\ (g(x) - f(w))\}. \tag{3.1}$$

For finding the approximate solution of (2.1), we can apply a successive approximation method to the problem of solving

$$x \in F(x) \tag{3.2}$$

where $F(x) = x - g(x) + J^{\phi}_{\eta}\{g(x) - \eta\ (g(x) - f(w))\}$.

On the basis of lemma 3.1, we suggest the following iterative algorithms.

Mann Type Perturbed Iterative Algorithm (MTPIA)

Let $T: H \to 2^H$ and $f, g: H \to H$. Given $x_0 \in H$, the iterative sequences $\{x_n\}$ and $\{w_n\}$ are defined by

$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g(x_n) + J^{\phi_n}_{\eta}\{g(x_n) - \eta(g(x_n) - f(w_n))\}] + e_n$,

for all $w_n \in T(x_n)$ and $n \geq 0$, where $\{\alpha_n\}$ is a real sequence satisfying $\alpha_0 = 1$, $0 \leq \alpha_n \leq 1$, for $n > 0$ and $\sum^{\infty}_{n=0} \alpha_n = \infty$, $e_n \in H$, for all n, is an error which is taken into account for a possible inexact computation of the proximal point; $\{\phi_n\}$ is the sequence approximating ϕ and $\eta > 0$ is a constant.

Ishikawa Type Perturbed Iterative Algorithm (ITPIA)

Let $T: H \to 2^H$ and $f, g: H \to H$. Given $x_0 \in H$, the iterative sequences $\{x_n\}$ and $\{w_n\}$ are defined by

$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [y_n - g(y_n) + J^{\phi_n}_{\eta}\{g(y_n) - \eta(g(y_n) - f(w^*_n))\}] + e_n$,

$y_n = (1 - \beta_n) x_n + \beta_n [x_n - g(x_n) + J^{\phi_n}_{\eta}\{g(x_n) - \eta(g(x_n) - f(w_n))\}] + \beta_n r_n$,

for $n \geq 0$ where e_n and r_n are the error terms which are taken into account the possible inexact computation of the proximal points; $x_n \in H$, $w_n \in T(x_n)$, $w^*_n \in T(y_n)$; $\eta > 0$ is a constant, and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in [0,1] satisfy the following conditions:

(i) $\alpha_0 = 1, \alpha_n \leq 1, \beta_n \leq 1$, for $n \geq 0$,

(ii) $\sum^{\infty}_{n=0} \alpha_n$ diverges, and

(iii) $\sum^{n}_{i=0} \prod_{j=i+1} (1 - (1 - c)\alpha_j)$ converges, where $0 \leq c < 1$.

4. CONVERGENCE THEORY

We need the following concepts and results to prove the main result of this paper.

Definition 4.1: A mapping $g:H\to H$ is said to be

(i) strongly monotone, if there exists $t > 0$, such that

$$\langle g(x_1) - g(x_2), x_1 - x_2\rangle \geq t\ ||x_1 - x_2||^2, \text{ for all } x_1, x_2 \in H;$$

(ii) Lipschitz continuous, if there exists $s > 0$ such that

$$|| g(x_1) - g(x_2) || \leq s\ || x_1 - x_2 ||, \text{ for all } x_1, x_2 \in H.$$

Definition 4.2 : Let $f: H\to H$ be a map. Then a multivalued map $T:H\to 2^H$ is said to be relaxed Lipschitz with respect to f, if for given $k \leq 0$,

$\langle f(w_1) - f(w_2), x_1 - x_2\rangle \leq k||x_1 - x_2||^2$, for all $w_1 \in T(x_1)$, $w_2 \in T(x_2)$ and for all $x_1, x_2 \in H$.

The multivalued map T is called Lipschitz continuous, if for $m \geq 1$,

$$|| w_1 - w_2 || \leq m\ || x_1 - x_2 ||, \text{ for all } w_1 \in T(x_1),\ w_2 \in T(x_2)$$

and for all $x_1, x_2 \in H$.

A single-valued map $f: H\to H$ is said to be Lipschitz continuous with constant $p > 0$, such that

$$|| f(w_1) - f(w_2) || \leq p\ || w_1 - w_2 ||, \text{ for all } w_1 \in T(x_1),\ w_2 \in T(x_2)$$

and for all $x_1, x_2 \in H$.

Lemma 4.1 [5]: Let ϕ be a proper convex lower semicontinuous function. Then $J^{\phi}_{\eta} = (I + \eta\, \partial\phi)$ is nonexpansive, i.e.,

$$|| J^{\phi}_{\eta}(x) - J^{\phi}_{\eta}(y) || \leq || x - y ||, \text{ for all } x, y \in H.$$

Now, we prove our main result of this paper.

Theorem 4.1: Let $g:H\to H$ be strongly monotone and Lipschitz continuous with corresponding constant $t > 0$ and $s > 0$ and $f : H\to H$ be Lipschitz continuous with constant $p > 0$. Let $T: H\to 2^H$ be relaxed Lipschitz continuous with respect to f and Lipschitz continuous with corresponding constants $k \leq 0$ and $m \geq 1$, if there exists a constant $\eta > 0$ such that

$$|\eta-(s(q-1)-k)(p^2m^2-s^2)^{-1}| < [(k+s(1-q))^2-(p^2m^2-s^2)q(2-q)]^{1\backslash 2}[p^2m^2-s^2]^{-1}$$

$$k > s(q-1)+[(p^2m^2-s^2)\ q\ (2-q)]^{1\backslash 2}$$

$$s(q-1) < pm \tag{4.1}$$

and $q = 2(1-2t+s^2)^{1\backslash 2} < 1$,

then (x, w) is a solution of GNVIP. Moreover, if

$$\lim_{n\to\infty} ||J^{\phi^n}_{\eta}(y) - J^{\phi}_{\eta}(y)|| = 0, \text{ for all } y \in H,$$

and $\{x_n\}$ and $\{w_n\}$ are defined by ITPIA with conditions

(a) $\lim_{n\to\infty} ||e_n|| = 0 = \lim_{n\to\infty} || r_n ||$ and

(b) $\sum^{n}_{i=0} \prod^{n}_{j=i+1} (1- (1-c)\alpha_j)$ converges, $0 \leq c < 1$.

Then $\{x_n\}$ and $\{w_n\}$ strongly converges to x and w, respectively.

Proof: First we prove that the GNVIP has a solution (x, w). By Lemma 3.1, it is enough to show that the mapping $F:H\to 2^H$ defined by (3.2) has a fixed point x. For any $x, y \in H$, $u \in F(x)$ and $v \in F(y)$, there exist $w_1 \in T(x)$ and

$w_2 \in T(y)$ such that

$$u = x - g(x) + J^{\phi}_{\eta}\{g(x) - \eta\,(g(x) - f(w_1))\} \text{ and}$$
$$v = y - g(y) + J^{\phi}_{\eta}\{g(y) - \eta(g(y) - f(w_2))\}$$

By Lemma 4.1, we have

$$\|u - v\| \le 2\,\|x - y - (g(x) - g(y))\| + \eta\,\|g(x) - g(y)\| + \|x - y + \eta(f(w_1) - f(w_2))\|. \qquad (4.2)$$

By using the strong monotonicity and Lipschitz continuity of g , we obtain

$$\begin{aligned}\|x - y - (g(x) - g(y))\|^2 &= \|x - y\|^2 - 2\langle g(x) - g(y), x - y\rangle + \|g(x) - g(y)\|^2 \\ &\le (1 - 2t + s^2)\,\|x - y\|^2, \qquad (4.3)\end{aligned}$$

and

$$\|g(x) - g(y)\| \le s\,\|x - y\|, \text{ for all } x, y \in H. \qquad (4.4)$$

Again

$$\begin{aligned}\|x - y + \eta\,(f(w_1) - f(w_2))\|^2 &= \|x - y\|^2 + 2\eta\,\langle f(w_1) - f(w_2), x - y\rangle + \eta^2\,\|f(w_1) - f(w_2)\|^2 \\ &\le (1 + 2\eta k + p^2 m^2 \eta^2)\,\|x - y\|^2. \qquad (4.5)\end{aligned}$$

Since T is Lipschitz continuous and relaxed Lipschitz continuous with respect to f, and f is Lipschitz continuous. Therefore, by (4.2), (4.3), (4.4) and (4.5), it follows that

$$\begin{aligned}\partial\,(F(x), F(y)) &\le \{2\,(1 - 2t + s^2)^{1\backslash 2} + (1 + 2\eta k + p^2 m^2 \eta^2)^{1\backslash 2} + \eta s\}\times \|x - y\| \\ &= \{q + \mu(\eta) + \eta s\}\,\|x - y\| \\ &= \theta\,\|x - y\|, \qquad (4.6)\end{aligned}$$

where $q = 2(1 - 2t + s^2)^{1\backslash 2}$, $\mu(\eta) = (1 + 2\eta k + p^2 m^2 \eta^2)^{1\backslash 2}$ and $\theta = q + \mu(\eta) + \eta s$. By condition (4.1), we see that $0 < \theta < 1$. It follows from (4.6). Thus F has a fixed point $x \in H$.

Let $x \in H$, $w \in T(x)$ be a solution of GNVIP. Next, we prove that the iterative sequences $\{x_n\}$ and $\{w_n\}$ define by ITPIA strongly converges to x and w, respectively. Since GNVIP has a solution (x , w) then by Lemma 3.1, we have

$$x = x - g(x) + J^{\phi}_{\eta}\{g(x) - \eta\,(g(x) - f(w))\}.$$

By making use of the same arguments used for obtaining (4.3) and (4.5), we get

$$\|x_n - x - (g(x_n) - g(x))\| \le (1 - 2t + s^2)^{1\backslash 2}\,\|x_n - x\|,$$
$$\|x_n - x + \eta\,(f(w_n) - f(w))\| \le (1 + 2\eta k + p^2 m^2 \eta^2)^{1\backslash 2}\,\|x_n - x\|$$
$$\|y_n - x - (g(y_n) - g(x))\| \le (1 - 2t + s^2)^{1\backslash 2}\,\|y_n - x\|,$$
$$\|y_n - x + \eta\,(f(w_n) - f(w))\| \le (1 + 2\eta k + p^2 m^2 \eta^2)^{1\backslash 2}\,\|y_n - x\|.$$

By setting

$$h(x) = g(x) - \eta\,(g(x) - f(w)).$$

and

$$h(y_n) = g(y_n) - \eta(g(y_n) - f(w_n)), \text{ we have}$$

$$\begin{aligned}||x_{n+1} - x|| &= ||(1-\alpha_n)x_n + \alpha_n[y_n - g(y_n) + J^{\phi_n}_\eta(h(y_n))] + e_n \\ &\quad - (1-\alpha_n)x - \alpha_n[x - g(x) + J^{\phi}_\eta(h(x))]|| \\ &\le (1-\alpha_n)||x_n - x|| + \alpha_n||y_n - x - (g(y_n) - g(x))|| \\ &\quad + \alpha_n||J^{\phi_n}_\eta(h(y_n)) - J^{\phi}_\eta(h(x))|| + ||e_n||. \qquad (4.7)\end{aligned}$$

Now, since $J^{\phi_n}_\eta$ is nonexpansive, we have

$$\begin{aligned}||J^{\phi_n}_\eta(h(y_n)) - J^{\phi}_\eta(h(x))|| &\le ||h(y_n) - h(x)|| \\ &\quad + ||J^{\phi_n}_\eta(h(x)) - J^{\phi}_\eta(h(x))|| \\ &\le ||y_n - x - (g(y_n) - g(x))|| + ||y_n - x + \eta(f(w_n) - f(w))|| \\ &\quad + \eta||g(y_n) - g(x)|| + ||J^{\phi_n}_\eta(h(x)) - J^{\phi}_\eta(h(x))|| \\ &\le (1 - 2t + s^2)^{1\backslash 2}||y_n - x|| + (1 + 2\eta k + p^2m^2\eta^2)^{1\backslash 2}||y_n - x|| \\ &\quad + s\eta||y_n - x|| + ||J^{\phi_n}_\eta(h(x)) - J^{\phi}_\eta(h(x))||. \qquad (4.8)\end{aligned}$$

By combining (4.7) and (4.8), we obtain

$$\begin{aligned}||x_{n+1} - x|| &\le (1-\alpha_n)||x_n - x|| + \alpha_n[2(1 - 2t + s^2)^{1\backslash 2} + (1 + 2\eta k \\ &\quad + p^2m^2\eta^2)^{1\backslash 2} + \eta s]||y_n - x|| + \alpha_n||J^{\phi_n}_\eta(h(x)) - J^{\phi}_\eta(h(x))|| + ||e_n|| \\ &= (1-\alpha_n)||x_n - x|| + \alpha_n\theta||y_n - x|| + \alpha_n\varepsilon_n + ||e_n||, \qquad (4.9)\end{aligned}$$

where $\theta = 2(1 - 2t + s^2)^{1\backslash 2} + (1 + 2\eta k + p^2m^2\eta^2)^{1\backslash 2} + \eta s$ and

$$\varepsilon_n = ||J^{\phi_n}_\eta(h(x)) - J^{\phi}_\eta(h(x))||.$$

Next

$$\begin{aligned}||y_n - x|| &= ||(1-\beta_n)x_n + \beta_n[x_n - g(x_n) + J^{\phi_n}_\eta(h(x_n))] \\ &\quad + \beta_n r_n - (1-\beta_n)x - \beta_n[x - g(x) + J^{\phi}_\eta(h(x))]|| \\ &\le (1-\beta_n)||x_n - x|| + \beta_n||x_n - x - (g(x_n) - g(x))|| \\ &\quad + \beta_n||J^{\phi_n}_\eta(h(x_n)) - J^{\phi}_\eta(h(x))|| + \beta_n||r_n||. \qquad (4.10)\end{aligned}$$

By making use of the same arguments used for obtaining (4.8), we get

$$\begin{aligned}||J^{\phi_n}_\eta(h(x_n)) - J^{\phi}_\eta(h(x))|| &\le \{2(1 - 2t + s^2)^{1\backslash 2} \\ &\quad + (1 + 2\eta k + p^2m^2\eta^2)^{1\backslash 2} + \eta s\}||x_n - x|| + \varepsilon_n. \qquad (4.11)\end{aligned}$$

On combining (4.10) and (4.11), we get

$$\begin{aligned}||y_n - x|| &\le (1-\beta_n)||x_n - x|| + \beta_n\theta||x_n - x|| + \beta_n\varepsilon_n + \beta_n||r_n|| \\ &\le (1 - \beta_n(1-\theta))||x_n - x|| + \beta_n(\varepsilon_n + ||r_n||).\end{aligned}$$

Since $(1 - \beta_n(1-\theta)) \le 1$.

On combining (4.9) and (4.12), we get

$$\begin{aligned}||x_{n+1} - x|| &\le (1-\alpha_n)||x_n - x|| + \alpha_n\theta||x_n - x|| + \theta\alpha_n\beta_n(\varepsilon_n + ||r_n||) \\ &\quad + \alpha_n\varepsilon_n + ||e_n|| \\ &= (1 - \alpha_n(1-\theta))||x_n - x|| + \theta\alpha_n\beta_n(\varepsilon_n + ||r_n||) + \alpha_n\varepsilon_n + ||e_n|| \\ &\le \prod^n_{i=1}(1 - \alpha_i(1-\theta))||x_0 - x|| + \sum^n_{j=0}\alpha_j\prod^n_{i=j+1}(1 - \alpha_i(1-\theta))\varepsilon_j \\ &\quad + \theta\sum^n_{j=0}\alpha_j\beta_j\prod^n_{i=j+1}(1 - \alpha_j(1-\theta))(\varepsilon_j + ||r_j||) \\ &\quad + \sum^n_{j=0}\prod^n_{i=i+1}(1 - \alpha_j(1-\theta))||e_j||, \qquad (4.13)\end{aligned}$$

where $\prod^n_{i=j+1}(1 - \alpha_i(1-\theta)) = 1$, when $j = n$.

Now, Let B denote the lower triangular matrix with entries

$$b_{nj} = \alpha_j\prod^n_{i=j+1}(1 - \alpha_i(1-\theta)).$$

Then B is multiplicative, see Rhoades [6], so that

$$\lim_{n\to\infty} \sum^{n}_{j=0} \alpha_j \prod^{n}_{i=j+1} (1-\alpha_i(1-\theta))\, \varepsilon_j = 0$$

$$\lim_{n\to\infty} \theta \sum^{n}_{j=0} \alpha_j\, \beta_j \prod^{n}_{i=j+1} (1-\alpha_j(1-\theta))\, (\varepsilon_j + ||r_j||) = 0.$$

Since $\lim_{n\to\infty} \varepsilon_n = 0 = \lim_{n\to\infty} ||r_n||$.

Let D denote the lower triangular matrix with entries

$$d_{nj} = \prod^{n}_{i=j+1} (1-\alpha_i(1-\theta)).$$

Condition (b) implies that D is multiplicative, and hence

$$\lim_{n\to\infty} \sum^{n}_{j=0} \prod^{n}_{i=j+1} (1-\alpha_I(1-\theta))\, ||e_j|| = 0,$$

since $\lim_{n\to\infty} ||e_n|| = 0$.

Also, $\lim_{n\to\infty} \prod^{n}_{i=0} (1-\alpha_i(1-\theta)) = 0$,

Since $\sum^{n}_{i=0} \alpha_i = \infty$.

Hence, it follows from inequality (4.13) that $\lim_{n\to\infty} ||x_{n+1} - x|| = 0$, i.e., the sequence $\{x_n\}$ strongly converges to x in H. The inequality (4.12) implies that the sequences $\{y_n\}$ also converges to x. Since $w_n \in T(x_n)$, $w \in T(x)$ and T is Lipschitz continuous, we have

$$||w_n - w|| \leq m\, ||y_n - x||, \text{ where } m \geq 1,$$
$$\to 0, \text{ as } n \to \infty,$$

i.e., $\{w_n\}$ strongly converges to w.

We remark that if $\beta_n = 0$, for all $n \geq 0$, Theorem 4.1 gives the conditions under which the sequences $\{x_n\}$ and $\{w_n\}$ defined by MTPIA strongly converges to x and w, respectively.

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