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THE COSINE HYPERBOLIC-AND SINE HYPERBOLIC RULES FOR DUAL HYPERBOLIC SPHERICAL TRIGONOMETRY

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Abstract-The dual hyperbolic unit sphere H_0^2 is the set of all dual time-like units vectors in the dual Lorentzian space D_1^3 with signature (+,+,-). In this paper, we give the cosine hyperbolic and sine hyperbolic-rules for a dual dual hyperbolic spherical triangle $\widetilde{A}\widetilde{B}\widetilde{C}$ which its sides are great-circle-arcs.

INTRODUCTION

Dual numbers had been introduced by W. K. Clifford (1849-1879) as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on line geometry and kinematics [1]. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in a Euclidean three-dimension space R^3 is one-to-one correspondence with the points of a dual unit sphere in the dual space D^3 of triples of dual numbers.

If we consider the Minkowski 3-space R_1^3 instead of R^3 the correspondence of E. Study mapping can be given as follows: "Oriented time-like and space-like lines in R_1^3 may be represented by time-like and space-like unit vectors with three-components in the dual Lorentzian space D_1^3, respectively [3]". A differentiable curve on the dual hyperbolic unit sphere H_0^2 corresponds to a time-like ruled surface while a differentiable curve on the dual Lorentzian unit sphere S_1^2 corresponds to any ruled surface. This correspondence is one-to-one and allows the geometry of Lorentzian ruled surface to be represented by the geometry of dual hyperbolic and Lorentzian spherical curves on H_0^2 and S_1^2, respectively. Dual hyperbolic spherical geometry, expressed with the help of dual time-like unit vectors is analogous to real hyperbolic spherical geometry, expressed with the help of real time-like unit vectors.

This paper gives some formulae and facts about the geometry of dual hyperbolic spherical curves.

1.Dual Numbers.

A dual number has the form $a+\varepsilon a^*$, where a and a^* are reel numbers and ε stands for the dual unit which is subjected to the rules:

$\varepsilon \neq 0 \quad 0\varepsilon = \varepsilon 0 = 0\ , 1\varepsilon = \varepsilon 1 = \varepsilon\ , \varepsilon^2 = 0$

we denote the dual numbers by notations $\widetilde{a} = a+\varepsilon a^*$, $\widetilde{x} = x+\varepsilon x^*$ and $\widetilde{\alpha} = \alpha+\varepsilon\alpha^*$. The composition rules for dual numbers result from the definitions:

i) Equality: $\widetilde{x} = \widetilde{y}$ *iff* $x = y$ *and* $x^* = y^*$

ii) Addition: $(x+\varepsilon x^*)+(y+\varepsilon y^*) = (x+y)+\varepsilon(x^*+y^*)$

ii) Multiplication: $(x+\varepsilon x^*).(y+\varepsilon y^*) = xy+\varepsilon(x^*y+xy^*)$

The set of all dual numbers is a commutative map having the numbers εc^* as divisions of zero. The division $\frac{\widetilde{a}}{\widetilde{b}}$ is a possible and unambiguous if $b \neq 0$ and it is easily seen that

$$\frac{\widetilde{a}}{\widetilde{b}} = \frac{a+\varepsilon a^*}{b+\varepsilon b^*} = \frac{a}{b} + \varepsilon(\frac{a^*}{b} - \frac{ab^*}{b^2}) \tag{1}$$

In all other cases division is either impossible or ambiguous. We define for a differentiable function f:

$$f(\widetilde{x}) = f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x) \tag{2}$$

where f' is the derivative of f. Thus:

$$\sinh(x+\varepsilon x^*) = \sinh x + \varepsilon x^* \cosh x \tag{3}$$

$$\cosh(x+\varepsilon x^*) = \cosh x + \varepsilon x^* \sinh x \tag{4}$$

and

$$\sqrt{\widetilde{x}} = \sqrt{x+\varepsilon x^*} = \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}}, \quad (x>0) \tag{5}$$

Now we define:

$$|\widetilde{x}| = \sqrt{<\widetilde{x},\widetilde{x}>} = \sqrt{<x,x> + 2\varepsilon <x,x^*>} \tag{6}$$

we have therefore in view at (6):

$$|\widetilde{x}| \doteq |x| + \varepsilon x^* \frac{x}{|x|} \quad (x \neq 0) \tag{7}$$

ànd consequently:

$$|\widetilde{x}| = \widetilde{x} \quad (x>0) \quad ; \quad |\widetilde{x}| = -\widetilde{x} \quad (x<0). \quad \textit{and} \quad |\widetilde{x}| = 0 \quad (\widetilde{x}=0) \tag{8}$$

2. Dual Vectors

Let $\{o; x_1, x_2, x_3 (time-like)\}$ be a right-handed orthonormal frame of reference in a three-dimensional Minkowski space R_1^3. The unit vector indicating the positive sense on the x_k axis will be denoted by $i_k (k=1,2,3)$. An ordered triple of dual numbers $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$. The numbers $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ are called the coordinates of $\widetilde{x}$; If these numbers are real the vector is said to be real. We define:

i) Lorentzian inner product:

$$<\widetilde{x},\widetilde{y}> = \widetilde{x}_1\widetilde{y}_1 + \widetilde{x}_2\widetilde{y}_2 - \widetilde{x}_3\widetilde{y}_3$$

ii) Lorentzian cross-product:

$$\widetilde{x} \wedge \widetilde{y} = (\widetilde{x}_2\widetilde{y}_3 - \widetilde{x}_3\widetilde{y}_2, \widetilde{x}_3\widetilde{y}_1 - \widetilde{x}_1\widetilde{y}_3, \widetilde{x}_1\widetilde{y}_2 - \widetilde{x}_2\widetilde{y}_1).$$

An a consequence of these definitions the well-known rules of vector algebra apply to dual vectors. That, we-have

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = \widetilde{x}_1 i_1 + \widetilde{x}_2 i_2 + \widetilde{x}_3 i_3 \tag{9}$$

$$\widetilde{x} = (x_1 + \varepsilon x_1^*, x_1 + \varepsilon x_2^*, x_2 + \varepsilon x_3^* x_3) = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*)$$

Introducing the reel vectors

$$x = (x_1, x_2, x_3) \; ve \; x^* = (x_1^*, x_2^*, x_3^*)$$

we can write

$$\widetilde{x} = x + \varepsilon x^*. \tag{10}$$

Let $\widetilde{x}$ be dual vector. $\widetilde{x}$ is said to be space-like if the vector x is space-like, time-like if vector x is time-like, and light-like(or null) if the vector x is light-like.

If $x \neq 0$ the norm $|\widetilde{x}|$ of $\widetilde{x}$ is defined by $(|<\widetilde{x},\widetilde{x}>|)^{1/2}$. From (6) we obtain

$$|\widetilde{x}| = |x| + \varepsilon \frac{<x,x^*>}{|x|} \quad (x \neq 0). \tag{11}$$

A dual vector $\widetilde{x}$ with norm 1 is called a dual unit vector. It follows from that $\widetilde{x}$ is a dual time-like unit vector (resp., dual space-like unit vector) if the relations

$$<x,x> = -1 \quad (resp., <x,x> = 1) \quad and \quad <x,x^*> = 0 \tag{12}$$

hold. Furthermore it is seen that for any vector

$$\widetilde{x} = x + \varepsilon x^*$$

with $x \neq 0$ the vector.

$$\widetilde{x}_0 = \frac{\widetilde{x}}{|x|} + \varepsilon\left(\frac{x^*}{|x|} - \frac{<x,x^*>}{|x|^2}.x\right)$$

or

$$\widetilde{x}_0 = \frac{\widetilde{x}}{|x|} + \varepsilon\left(\frac{x^*}{|x|} - \frac{<x,x^*>}{|x|^2}.x\right) = \frac{x}{|x|} + \varepsilon\,\frac{(x \wedge x^*) \wedge x}{|x|^{\frac{3}{2}}} \tag{13}$$

is a dual unit vector.

3. Dual Time-like Unit Vectors and Oriented Time-like Lines

Let x_0 and x_0^* be two free vectors satisfying. It is well-known that x_0 and x_0^* may be interpreted as the Plücker vectors of on unambiguously determined line l having x_0 as its time-like direction vector and passing through the point $p = x \wedge x_0^*$. This line becomes a directed time-like line. We may call l the carrier of directed time-like line and x the directed vector. The vector x_0^* is usually called the moment of the directed time-like with respect to o. We denote the directed time-like line by $[x_0, x_0^*]$. This notation implies

$$<x_0,x_0> = -1 \text{ and } <x_0,x_0^*> = 0.$$

It follows from the above that there exists a one-to-one correspondence between the set of all oriented time-like lines in three dimensional Minkowski space R_1^3 and the set of all dual time like unit vectors:

$$[x_0, x_0^*] \leftrightarrow \widetilde{x} = x + \varepsilon x^*.$$

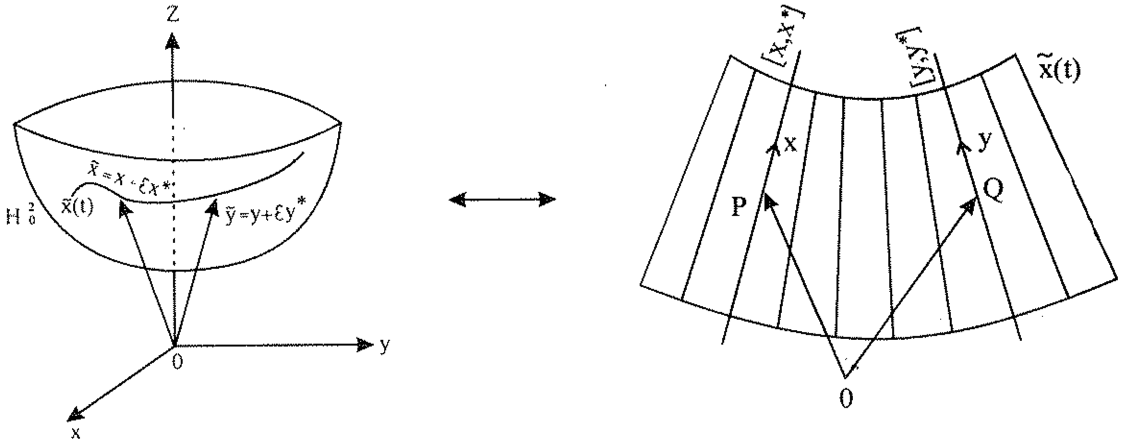

Figure 3.1. E. Study mapping for oriented time-like lines in R_1^3

4. Dual Inner and Cross-Products of Dual Time-like Unit Vectors.

Let $\widetilde{a} = a + \varepsilon\, a^*$ and $\widetilde{b} = b + \varepsilon\, b^*$ be two dual time-like unit vectors. Under the mapping (14) these vectors are the images of the oriented time-like lines respectively, having the lines, a and b as their carries. These lines are supposed to be different. Let n be the line intersecting a and b at right angles at the points P and Q, respectively. If $a \wedge b = 0$ the vector n stands for the unit vector having the sense $P\widetilde{Q}$. We have therefore $P\widetilde{Q} = q - p$ where p and q are the position vectors of P and Q respectively. The scalar will be called the oriented distance between a and b . Then we have: $a^* = p \wedge a$, $b^* = q \wedge b$. Let φ be the hyperbolic angle between a and b $(\varphi \in R)$. The image under the mapping (14) oriented space-like line S_n having n as its carrier and n^* is the d.s.1.u.v with the same as $\widetilde{a} \wedge \widetilde{b}$ now we obtain:

$$< \widetilde{a}, \vec{b} > = < a,b > + \varepsilon \{ < a, b^* > + < a^*, b > \} = < a,b > + \varepsilon \{ \det(a,q,b) + \det(p,a,b) \}$$

or

$$< \widetilde{a}, \vec{b} > = -\cosh\varphi - \varepsilon \det(q-p,a,b) = -\cosh\varphi - \varepsilon\varphi^* \sinh\varphi$$

putting $\varphi + \varepsilon\varphi^* = \widetilde{\varphi}$ we find:

$$< \widetilde{a}, \vec{b} > = -\cosh(\varphi + \varepsilon\,\varphi^*) = -\cosh\widetilde{\varphi}\,. \qquad (15)$$

The dual number $\widetilde{\varphi}$ is called dual hyperbolic angle $\widetilde{a}$ and $\widetilde{b}$. Obviously, $< \widetilde{a}, \widetilde{b} > = 0$ is a necessary and sufficient condition for a and b to intersect at right angles. We find furthermore:

$$\widetilde{a} \wedge \widetilde{b} = a \wedge b + \varepsilon(a \wedge b^* + a^* \wedge b) = n \sin\varphi + \varepsilon\{\ a \wedge (q \wedge b) + (p \wedge a) \wedge b\ \}.$$

Observing that $q = p + \varphi^* n$ we get

$$\widetilde{a} \wedge \widetilde{b} = n \sin\varphi + \varepsilon\{\ a \wedge (p \wedge b) + (p \wedge a) \wedge b + \varphi^* a \wedge (n \wedge b\ \}.$$

Since $a \wedge (p \wedge b) + p \wedge (b \wedge a) + b \wedge (a \wedge p) = 0$ we find:

$$\widetilde{a} \wedge \widetilde{b} = n \sinh\varphi + \varepsilon\{\ p \wedge (a \wedge b) + \varphi^* < a,b > n\ \} = n \sinh\varphi + \varepsilon(n^* \sinh\varphi + \varphi^* n \cosh\varphi)$$

or

$$\widetilde{a} \wedge \widetilde{b} = (n + \varepsilon\, n^*).(\sinh\varphi + \varepsilon\,\varphi^* \cosh\varphi)\,.$$

Therefore:

$$\widetilde{a} \wedge \widetilde{b} = \widetilde{n} \sinh\widetilde{\varphi}\,. \qquad (16)$$

For coincident a and b we find $\widetilde{a} \wedge \widetilde{b} = 0$. We shall suppose that $\widetilde{a} \wedge \widetilde{b} \neq 0$.

5. Dual Hyperbolic Spherical Triangle

Let x be the position vector with respect to Lorentzian coordinate system $(0; x_1, x_2, x_3)$ of a real point with coordinates (x_1, x_2, x_3).The set of all points x with $< x,x > = -1$ is the hyperbolic unit sphere. It is called as hyperbolic space in [5]. If the dual time-like vector $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ is not real we call $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ the coordinates of a dual hyperbolic point. The set of all dual time-like unit vectors is called hyperbolic unit sphere (d.h.u.s for short) with O as its centre. The real hyperbolic unit sphere is subset of the d.h.u.s. The mapping (14) induces a one to-one correspondence between the points of the d.h.u.s and the oriented time-like lines of three dimensional Minkowski space R_1^3. The real points on the d.h.u.s correspond to the carries of which pass through o. Then it is easy to develop dual hyperbolic spherical trigonometry. To prepare such a development we consider two points $\widetilde{A}$ and $\widetilde{B}$ on the d.h.u.s given by the dual time-like vectors $\widetilde{a} = a + \varepsilon\, a^*$ and $\widetilde{b} = b + \varepsilon\, b^*$, respectively. We introduce the set of all dual time-like vectors given by

$\widetilde{c}_\lambda = c_\lambda + \varepsilon c_\lambda^* = (1-\widetilde{\lambda})\widetilde{a} + \widetilde{\lambda}\widetilde{b}$. Where $\widetilde{\lambda} = \lambda + \varepsilon\lambda^*$ and $0 \le \lambda \le 1$. We put $\widetilde{c}_\lambda = |\widetilde{c}_\lambda|.\widetilde{e}_\lambda$; then $\widetilde{e}_\lambda$ is a point $\widetilde{c}_\lambda$ on the d.h.u.s. The set of all points $\widetilde{c}_\lambda$ with $0 \le \lambda \le 1$ is called the dual hyperbolic great circle arch $\widetilde{A}\widetilde{B}$. We shall say that $\widetilde{c}_\lambda$ runs along from $\widetilde{A}$ to $\widetilde{B}$ if λ increases from 0 to 1. With the arc $\widetilde{A}\widetilde{B}$ we shall always mean this are in the sense from to $\widetilde{B}$. The point P on the d.h.u.s. indicated by the d.u.v. with the same sense as $\widetilde{a} \wedge \widetilde{b}$ will be called the pole of $\widetilde{A}\widetilde{B}$. Let $,\widetilde{B},\widetilde{C}$ be three points on the d.h.u.s. given by the linearly independent dual time-like unit vectors $\widetilde{a} = a + \varepsilon a^*, \widetilde{b} = b + \varepsilon b^*$ and $\widetilde{c} = c + \varepsilon c^*$, respectively . These points together with the great circle $A\widetilde{B}, B\widetilde{C}, C\widetilde{A}$ form a dual hyperbolic spherical triangle $\widetilde{A}\widetilde{B}\widetilde{C}$ (Figure 5.1).

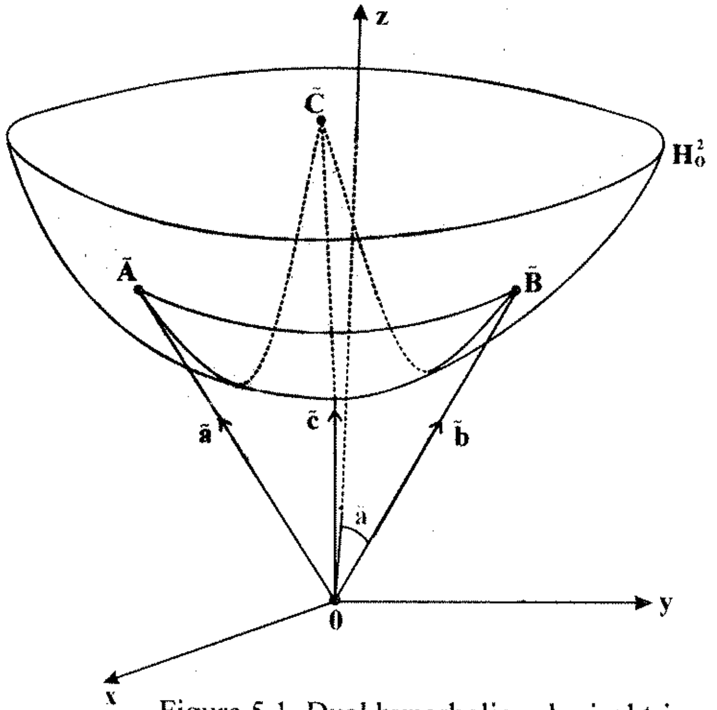

Figure 5.1. Dual hyperbolic spherical triangle $\widetilde{A}\widetilde{B}\widetilde{C}$

We shall always suppose that the notation is such that $\det(a,b,c) > 0$. The dual space-like unit vectors with the same sense as $\widetilde{b} \wedge \widetilde{c}, \widetilde{c} \wedge \widetilde{a}$ and $\widetilde{a} \wedge \widetilde{b}$ will be denoted by $\widetilde{n}_a, \widetilde{n}_b$ and $\widetilde{n}_c$, respectively. We define the side $\widetilde{a}$ of $\widetilde{A}\widetilde{B}\widetilde{C}$ as the *dual hyperbolic angle* for which

$$< \widetilde{b}, \widetilde{c} > = -\cosh\widetilde{a} \quad , \quad \widetilde{b} \wedge \widetilde{c} = \widetilde{n}_a \sinh\widetilde{a} .$$

Similar definitions are given for the sides $\widetilde{b}$ and $\widetilde{c}$. Putting $\widetilde{n}_a = n_a + \varepsilon n_a^*$ we observe that n_a is the space-like unit vector having the same sense as $b \wedge c$. If $\widetilde{a} = a + \varepsilon a^*$ we have consequently $\sinh a > 0$.

This implies $|\sinh\widetilde{a}| = \sin\widetilde{a}$; similarly $|\sinh\widetilde{b}| = \sinh\widetilde{b}, |\sinh\widetilde{c}| = \sinh\widetilde{c}$. It is easily seen that $\widetilde{a},\widetilde{b}$ and $\widetilde{c}$ are the dual unit vectors having the same sense as $\widetilde{n}_b \wedge \widetilde{n}_c$, $\widetilde{n}_c \wedge \widetilde{n}_a$ and $\widetilde{n}_a \wedge \widetilde{n}_b$, respectively. The angle $\widetilde{a}$ of $\widetilde{A}\widetilde{B}\widetilde{C}$ is defined as the *dual central angle* given by

$$< \widetilde{n}_b, \widetilde{n}_c > = \cosh\widetilde{\alpha}, \widetilde{n}_b \wedge \widetilde{n}_c = \widetilde{a}\sinh\widetilde{\alpha} \qquad (18)$$

with similar definitions for the angles $\widetilde{\beta}$ and $\widetilde{\gamma}$

By means of these definitions we find

$$<\tilde{a}\wedge\tilde{c},\tilde{a}\wedge\tilde{b}>=-\sin\tilde{b}.\sinh\tilde{c}<\tilde{n}_b,\tilde{n}_c>=-\sinh\tilde{b}.\sinh\tilde{c}.\cosh\tilde{\alpha}. \tag{19}$$

We have on the other hand:

$$<\tilde{a}\wedge\tilde{c},\tilde{a}\wedge\tilde{b}>=<\tilde{a},\tilde{c}\wedge\left(\tilde{a}\wedge\tilde{b}\right)>=<\tilde{a},<\tilde{c},\tilde{b}>\tilde{a}-<\tilde{c},\tilde{a}>\tilde{b}>$$

$$=-\cosh\tilde{a}+\cosh\tilde{b}\cosh\tilde{c}. \tag{20}$$

It follows in view of (18) that

$$\cosh\tilde{a}=\cosh\tilde{b}.\cosh\tilde{c}-\sinh\tilde{b}.\sinh\tilde{c}.\cosh\tilde{\alpha}. \tag{21}$$

We obtain moreover:

$$a\sinh\tilde{\alpha}.\sinh\tilde{b}.\sinh\tilde{c}=\tilde{n}_b\wedge\tilde{n}_c\sinh\tilde{b}\sinh\tilde{c}$$

$$=(\tilde{c}\wedge\tilde{a})\wedge(\tilde{a}\wedge\tilde{b})=-<\tilde{c}\wedge\tilde{a},\tilde{b}>\tilde{a}$$

$$=-\det(\tilde{a},\tilde{b},\tilde{c})\tilde{a}.$$

Therefore

$$\frac{\sinh\tilde{\alpha}}{\sinh\tilde{a}}=\frac{-\det(\tilde{a},\tilde{b},\tilde{c})}{\sinh\tilde{a}\sinh\tilde{b}\sinh\tilde{c}} \tag{22}$$

from which we infer

$$\frac{\sinh\tilde{\alpha}}{\sinh\tilde{a}}=\frac{\sinh\tilde{\beta}}{\sinh\tilde{b}}=\frac{\sinh\tilde{\gamma}}{\sinh\tilde{c}}$$

The formulae (21) and (22) are the cosine and sine-rules for dual hyperbolic spherical trigonometry.

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