Mathematical & Computational Applications, Vol. 5, No. 2, pp. 67-75, 2000

CALCULATION OF THE NATURAL FREQUENCIES OF A BEAM-MASS SYSTEM USING FINITE ELEMENT METHOD

H. R. Öz

Department of Mechanical Engineering, Celal Bayar University, 45140 Manisa, Turkey

Abstract - In this study, the natural frequencies of an Euler-Bernoulli type beam with a mass are calculated. The beam is supported with different end conditions. The mass is located on different locations. The linear natural frequencies are calculated by using finite element method for the first five modes. Results are compared with those of exact and other approximate methods.

1. INTRODUCTION

Beam-mass systems have been investigated by many researchers. The natural frequencies of beam-mass systems, plates carrying concentrated masses under different boundary conditions were calculated by using approximate and exact analysis [1-4] and two fundamental theories of beam vibrations were compared [5]. Chai and Low [6] investigated the natural frequencies of a beam with a mass near the beam's ends. Low *et al.* [7] found that the results of experiments and the theory did not match well for beams of large slenderness ratio for centre loaded beams. Chai *et al.* [8] and Low *et al.* [9] presented both experimental and theoretical results using Rayleigh-Ritz procedure and showed that the correlation between theory and experiments was much improved when stretching effects were included [9]. Özkaya *et al.* [10] analyzed non-linear free and forced vibrations of a beam-mass system by considering five different sets of boundary conditions. The authors investigated the effects of the location and the magnitude of the mass on the natural frequencies. They used a perturbation technique in the analysis. Low [11] used different assumed shape functions to obtain the kinetic and potential energies of the three classical beams carrying a concentrated mass. Low and Dubey [12] presented shape functions for calculating the frequencies. Low [13] compared different models for simply supported beam. Özkaya and Pakdemirli [14] obtained the frequencies for the clamped-clamped beam with mass and searched approximate solutions of free and forced non-linear vibrations using a perturbation method. They compared the results with the solutions of artificial neural network method. Turhan [15] considered the longitudinal and transverse vibrations of bars and beams with a mass and torsional vibration of shafts with a disk. The natural frequencies were obtained using Rayleigh method and were compared with the exact solutions.

In this study, an Euler-Bernoulli type beam carrying a mass on different locations is considered. Transverse vibrations of the beam is investigated. For support conditions, six different cases are discussed. These cases are simple-simple, simple-sliding, clamped-simple, clamped-clamped, clamped-sliding, sliding-sliding supports. In the analysis, finite element method is used to calculate the first five natural frequencies.

Cubic interpolation function is assumed for the vertical displacement of the beam. Also linear bending is assumed. Results are compared with exact, approximate closed-form and Rayleigh's method solutions. It is found that the accuracy of finite element results are very high and closer to the exact frequency values than closed-form and Rayleigh solutions.

2. FINITE ELEMENT FORMULATION

In this part, the kinetic and elastic potential energies of the beam in transverse vibration will be given. The equations for natural frequencies will be obtained by using finite element method. Some assumptions are made in deriving the equations and in search of the solutions. The beam is assumed to have only vertical displacement, no elongation of neutral axis or movement in longitudinal direction is allowed. Cubic interpolation function for vertical displacement is assumed. For the beam shown in Figure 1, the kinetic and elastic potential energies without a concentrated mass are

$$T = \frac{1}{2}\rho A \int_0^{L_T} \dot{v}^2 dx \tag{1}$$

$$U = \frac{1}{2} EI \int_0^{L_T} v''^2 dx \tag{2}$$

where ρ is density, A is cross-sectional area, L_T is total length, EI is flexural rigidity. x and z denote longitudinal and vertical directions respectively and v denotes the displacement in z direction. $(\dot{\ })$ and $(\ )'$ denote differentiations with respect to time t and x respectively. Since the longitudinal velocity component is very small, it is neglected in the kinetic energy equation. x_m is the location of the concentrated mass M_c.

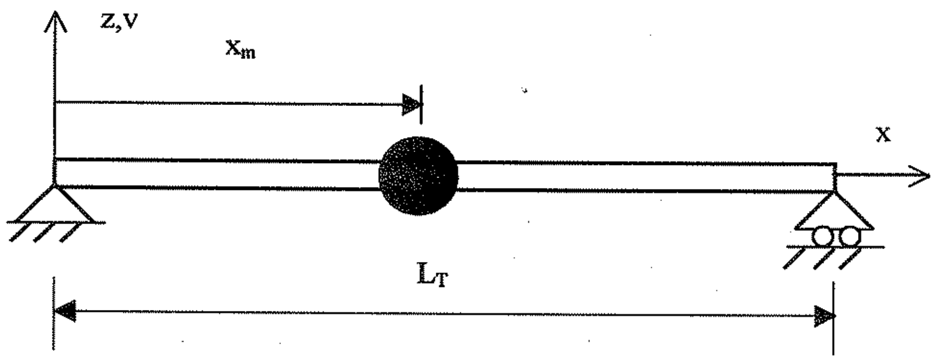

Figure 1. A beam with a concentrated mass on simple supports

For the finite element formulation, as a first step, one defines the degrees of freedom for a bending element in Figure 2. v_1, v_2 are the vertical displacements at the nodes 1 and 2 of the beam element respectively. θ_1 and θ_2 represent rotations of the nodes. L is the length of the beam element.

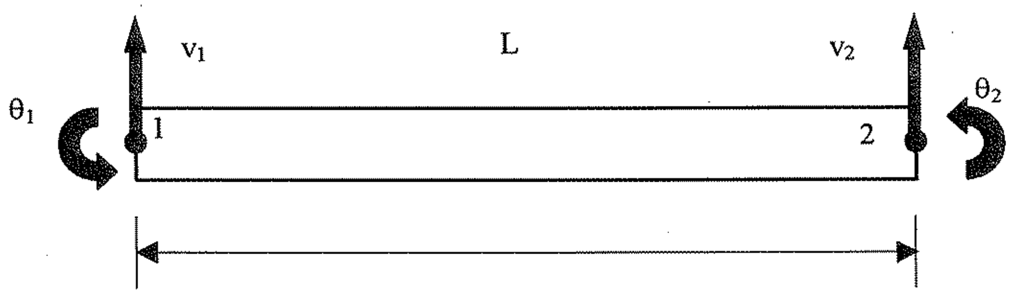

Figure 2. Beam bending element

Cubic interpolation function for the vertical displacement is

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{3}$$

Equation (3) can be written in matrix form as follows

$$\{v\} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{4}$$

The nodal displacement vector for a bending element consists of vertical displacements and rotations at nodes 1 and 2

$$\{v\}_e = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{5}$$

where rotation or slope of the beam is

$$\theta_r = \frac{\partial v_r}{\partial x} \qquad r = 1,2 \tag{6}$$

After applying finite element formulation, the kinetic and elastic potential energies for one element can be written as follows

$$T = \frac{1}{2}\{\dot{v}\}_e^T [M] \{\dot{v}\}_e \tag{7}$$

$$U = \frac{1}{2}\{v\}_e^T [K]\{v\}_e \tag{8}$$

where $\{\ \}^T$ denotes transpose of the matrix and

$$[K]_e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{9}$$

is the element stiffness matrix between nodes 1 and 2, and

$$[M]_e = \frac{mL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{10}$$

is the element inertia matrix where m is mass per element length.

If the total length of the beam is considered, the element inertia and stiffness matrices must be combined and only one inertia and one stiffness matrix must be obtained. For the concentrated mass, a matrix with only one element is written for the location and this matrix is combined with system inertia matrix. The equation of motion can be obtained for free vibrations by applying Lagrange equation

$$[M]\{\ddot{v}\} + [K]\{v\} = \{0\} \tag{11}$$

where [M] and [K] are the system inertia and stiffness matrices respectively, $\{v\}$ is system displacement vector. Assume a solution in the form of

$$\{v\} = \{V\}e^{j\omega_n t} \tag{12}$$

where j and ω_n denote $\sqrt{-1}$ and natural frequencies and $\{V\}$ is displacement amplitude vector. Substituting equation (12) into equation (11), one writes

$$\left[[K] - \omega_n^2 [M]\right]\{V\} = \{0\} \tag{13}$$

For non-trivial solution, the determinant of coefficient matrix must be zero. This gives

$$\left|[K] - \omega_n^2 [M]\right| = 0 \tag{14}$$

Equation (14) is an eigenvalue problem and is to be solved for different end conditions and mass locations in the next section. The boundary conditions used in finite element analysis are as follows

Case 1. Simple-Simple Supports
$$v_i=0,\ v_f=0 \tag{15}$$

Case 2. Simple-Sliding Supports
$$v_i=0,\ \theta_f=0 \tag{16}$$

Case 3. Clamped-Simple Supports
$$v_i=0,\ \theta_i=0,\ v_f=0 \tag{17}$$

Case 4. Clamped-Clamped Supports
$$v_i=0,\ \theta_i=0,\ v_f=0,\ \theta_f=0 \tag{18}$$

Case 5. Clamped-Sliding Supports
$$v_i=0,\ \theta_i=0,\ \theta_f=0 \tag{19}$$

Case 6. Sliding-Sliding Supports
$$\theta_i=0,\ \theta_f=0 \tag{20}$$

where i and f denote initial and final nodes of the beam.

3. NUMERICAL SOLUTIONS

Numerical values for the natural frequencies for the first five modes will be given in this section. Solutions of the eigenvalue problem (equation (14)) for different end conditions (equations (15-20)) and mass locations are presented in Tables 1-6. The dimensionless concentrated mass and its location can be written as follows

$$\alpha = \frac{M_c}{\rho A L_T} \tag{21}$$

$$\eta = \frac{x_m}{L_T} \tag{22}$$

In this study, the mass on the beam is assumed to have the same weight with the beam, namely $\alpha=1$. The analytical values (exact) were given by Özkaya *et al.* [10] and Özkaya and Pakdemirli [14]. The approximate closed form solutions were given by Low [13]. Rayleigh's quotient solutions were given by Turhan [15]. 40 elements are used in the finite element analysis. As can be seen from the tables, the finite element solutions are very accurate. If the number of elements are increased then the analysis will give better results. In ref. [11,12,13] the natural frequencies for simply supported beams were given but their accuracy are lower than the finite element analysis. Also, frequency values obtained by Rayleigh's method in ref. [15] have less accuracy.

Table 1. The natural frequencies of a beam carrying a mass on simple-simple supports

α	η	ω_1			ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [10]	FEM	Ref. [13]	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM
1	0.0	9.8695	9.8696	9.8767	39.4784	39.4784	88.8264	88.8267	157.9144	157.9147	246.7413	246.7442
	0.1	8.9962	8.9962	9.0432	29.8891	29.8892	66.0691	66.0691	127.2135	127.2149	213.3439	213.3457
	0.2	7.4541	7.4541	7.4575	26.9462	26.9463	73.5140	73.5141	149.3992	149.3993	246.7413	246.7442
	0.3	6.3946	6.3947	6.3958	29.7503	29.7504	86.7293	86.7296	143.2258	143.2267	209.3172	209.3763
	0.4	5.8468	5.8468	5.8482	35.2374	35.2375	79.9788	79.9790	132.6574	132.6588	246.7413	246.7442
	0.5	5.6795	5.6796	5.6809	39.4784	39.4784	67.8883	67.8885	157.9144	157.9147	206.7901	206.7913

Table 2. The natural frequencies of a beam carrying a mass on simple-sliding supports

α	η	ω_1		ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM
1	0.0	2.4674	2.4674	22.2066	22.2066	61.6850	61.6851	120.9032	120.9031	199.8604	199.8617
	0.1	2.4087	2.4087	18.3454	18.3454	45.1359	45.1360	93.4431	93.4434	167.2211	167.2214
	0.2	2.2578	2.2578	14.8086	14.8085	46.3928	46.3929	108.0103	108.0104	196.6417	196.6442
	0.3	2.0706	2.0706	14.2145	14.2145	54.4452	54.4453	120.0471	120.0479	166.6578	166.5895
	0.4	1.8920	1.8920	15.3836	15.3836	61.6850	61.6851	96.2916	96.2918	188.1808	188.1824
	0.5	1.7415	1.7415	17.9539	17.9539	53.0106	53.0106	107.5473	107.5469	181.7185	181.7188
	0.6	1.6226	1.6226	21.2279	21.2280	45.5640	45.5640	118.1939	118.1947	173.8152	173.8172
	0.7	1.5332	1.5332	21.9816	21.9816	50.9158	50.9159	98.6861	98.6863	193.0932	193.0939
	0.8	1.4706	1.4706	19.8790	19.8791	61.6850	61.6851	106.6180	106.6175	165.0326	165.0348
	0.9	1.4328	1.4329	17.8328	17.8329	55.9844	55.9845	116.7804	116.7806	198.9933	198.9955
	1.0	---------	1.4199	----------	16.9721	----------	51.6973	-----------	106.0585	-----------	180.1248

Table 3. The natural frequencies of a beam carrying a mass on clamped-simple supports

α	η	ω_1			ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [10]	FEM	Ref [15]	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM
1	0.1	15.2752	15.2752	15.6374	45.5767	45.5766	79.3377	79.3377	133.4672	133.4684	217.8576	217.8558
	0.2	13.8203	13.8203	13.9618	33.2808	33.2808	77.0176	77.0178	153.7460	153.9472	259.8318	259.8411
	0.3	11.3683	11.3683	11.3769	33.0378	33.0377	92.2403	92.2404	178.0890	178.0914	234.5798	234.5850
	0.4	9.0600	9.6093	9.6125	38.6505	38.6505	103.6283	103.6293	145.8877	145.8875	263.2084	263.2148
	0.5	8.6977	8.6977	8.7015	47.2840	47.2841	84.6891	84.6893	172.7437	172.7447	236.1355	236.1381
	0.6	8.4780	8.4749	8.4797	48.5385	48.5386	87.0356	87.0357	158.8255	158.8255	266.7995	266.8061
	0.7	8.9482	8.9482	8.9543	41.1580	41.1580	104.0726	104.0729	153.4427	153.4442	237.9214	237.9235
	0.8	10.3964	10.3964	10.4022	35.8279	35.8279	89.8535	89.8537	172.3995	172.4018	270.5268	270.5310
	0.9	13.2773	13.2815	13.3335	36.9648	36.9648	78.9377	78.9378	146.4463	146.4479	238.6747	238.6769
	1.0	15.4182	15.4182		49.9648	49.9649	104.2482	104.2480	178.2706	178.2713	272.0322	272.0364

Table 4. The natural frequencies of a beam carrying a mass on clamped-clamped supports

α	η	ω_1			ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [14]	FEM	Ref. [15]	Exact Ref. [14]	FEM	Exact Ref. [14]	FEM	Exact Ref. [14]	FEM	Exact Ref. [14]	FEM
1	0.0	22.3733	22.3733		61.6728	61.6729	120.9032	120.9039	199.8604	199.8616	298.5569	298.5627
	0.1	21.9474	21.9474	22.6154	53.8427	53.8428	89.8598	89.8599	151.9623	151.9631	243.0824	243.0849
	0.2	18.3360	18.3360	18.4060	40.9434	40.9434	93.3305	93.3307	177.8542	177.8546	290.1980	290.2042
	0.3	14.4030	14.4030	14.4138	44.2995	44.2996	112.5615	112.5613	195.4739	195.4752	254.3674	254.3715
	0.4	12.4047	12.4047	12.4150	53.5218	53.5218	114.5992	114.6006	167.6507	167.6513	297.2762	297.2847
	0.5	11.8182	11.8182	11.8273	61.6727	61.6729	95.7568	95.7570	199.8604	199.8616	253.7298	253.7347

Table 5. The natural frequencies of a beam carrying a mass on clamped-sliding supports

α	η	ω_1		ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM
1	0.1	5.5858	5.5858	59.2073	29.2074	61.9712	61.9712	102.3233	102.3244	172.2446	172.2447
	0.2	5.4925	5.4925	22.7446	22.7447	50.6636	50.6637	111.6425	111.6432	203.5245	203.5262
	0.3	5.1950	5.1950	18.7343	18.7344	57.8091	57.8091	134.4440	134.4454	208.7476	208.7507
	0.4	4.7130	4.7130	18.4833	18.4833	70.3074	70.3075	122.8417	122.8417	195.9552	195.9572
	0.5	4.1970	4.1971	20.9093	20.9093	71.6210	71.6211	115.0728	115.0743	216.3253	216.3279
	0.6	3.7522	3.7522	25.7223	25.7223	59.1356	59.1358	138.7920	138.7920	196.5491	186.5506
	0.7	3.4075	3.4075	30.1129	30.1129	58.3002	58.3002	118.7904	118.7908	220.4928	220.4959
	0.8	3.1614	3.1614	28.4668	28.4668	74.0070	74.0071	117.3647	117.3655	187.0877	187.0884
	0.9	3.0090	3.0091	25.3661	25.3662	69.0642	69.0643	135.4291	135.4300	223.2036	222.4318
	1.0	2.9545	2.9546	23.9392	23.9392	63.4326	63.4327	122.7165	122.7272	201.7195	201.7231

Table 6. The natural frequencies of a beam carrying a mass on sliding-sliding supports

α	η	ω_1		ω_2		ω_3		ω_4		ω_5	
		Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM	Exact Ref. [10]	FEM
1	0.0	6.9049	6.9049	31.8900	31.8899	76.4168	76.4170	140.6264	140.6268	224.5502	224.5545
	0.1	7.1089	7.1090	34.0789	34.0789	83.5771	83.5772	155.3912	155.3921	246.7413	246.7442
	0.2	7.6615	7.6616	38.3503	38.3503	85.7404	85.7405	130.1379	130.1373	211.7200	211.7228
	0.3	8.4912	8.4912	37.8853	37.8852	68.0983	68.0984	141.5267	141.5280	246.7413	246.7442
	0.4	9.4084	9.4084	30.6404	30.6404	76.1398	76.1399	154.1645	154.1643	207.5299	207.5327
	0.5	9.8696	9.8696	27.6195	27.6196	88.8265	88.8266	127.5589	127.5603	246.7413	246.7442

4. CONCLUDING REMARKS

The transverse vibrations of an Euler-Bernoulli type beam carrying a concentrated mass is considered. Six different boundary conditions and eleven different mass locations are assumed for the calculation of natural frequencies. The finite element method is used in the analysis. Cubic interpolation function for the vertical displacement is taken. The approximate results are compared with exact and other approximate solutions. The finite element method results are very close to the exact frequency values. Also increasing the number of elements in the analysis increases the accuracy of the natural frequencies.

5. REFERENCES

1. L. S. Srinath and Y. C. Das, Vibrations of beams carrying mass, *Transaction of the American Society of Mechanical Engineers, Journal of Applied Mechanics, Series* E 784-785, 1967.
2. J. H. Lau, Fundamental frequency of a constrained beam, *Journal of Sound and Vibration* **78**, 154-157, 1981.
3. P. A. A. Laura, C. Filipich and V. H. Cortinez, Vibrations of beams and plates carrying concentrated masses, *Journal of Sound and Vibration* **117**, 459-465, 1987.
4. W. H. Liu and F. H. Yeh, Free vibration of a restrained-uniform beam with intermediate masses, *Journal of Sound and Vibration* **117**, 555-570, 1987.
5. M. J. Maurizi and P. M. Belles, Natural frequencies of the beam-mass system: comparison of the two fundamental theories of beam vibrations, *Journal of Sound and Vibration* **150**, 330-334, 1991.
6. G. B. Chai and K. H. Low, On the natural frequencies of beams carrying a concentrated mass, *Journal of Sound and Vibration* **160**, 161-166, 1993.
7. K. H. Low, T. M. Lim and G. B. Chai, Experimental and analytical investigations of vibration frequencies for center-loaded beams, *Computers and Structures* **48**, 1157-1162, 1993.
8. G. B. Chai, K. H. Low and T. M. Lim, Tension effects on the natural frequencies of centre-loaded clamped beams, *Journal of Sound and Vibration* **181**, 727-736, 1995.
9. K. H. Low, R. N. Dubey and G. B. Chai, Experimental investigation and shape functions for lateral vibration of axially constrained beams with a concentrated mass at the centre, *Journal of Sound and Vibration* **202**, 446-451, 1997.
10. E. Özkaya, M. Pakdemirli and H. R. Öz, Non-linear vibrations of a beam-mass system under different boundary conditions, *Journal of Sound and Vibration* **199**, 679-696, 1997.
11. K. H. Low, Closed-form formulas for fundamental vibration frequency of beams under off-centre load, *Journal of Sound and Vibration* **201**, 528-533, 1997.
12. K. H. Low and R. N. Dubey, A note on the fundamental shape function and frequency of beams under off-center load, *Journal of Sound and Vibration* **202**, 134-138, 1997.
13. K. H. Low, Comments on "Non-linear vibrations of a beam-mass system under different boundary conditions", *Journal of Sound and Vibration* **207**, 284-286, 1997.
14. E. Özkaya and M. Pakdemirli, Non-linear vibrations of a beam-mass system with both ends clamped, *Journal of Sound and Vibration* **221**, 491-503, 1999.
15. Ö. Turhan, E. Altınsoy and S. İnceoğlu, Üzerinde tekil kütle taşıyan kiriş, çubuk ve millerin temel frekansları için Rayleigh yaklaşıklıkları ve egzakt çözümler, *9. Ulusal Makina Teorisi Sempozyumu* 206-220, 1999.