Mathematical & Computational Applications, Vol. 5, No. 2, pp. 129-138, 2000

AN ELASTIC-PLASTIC STRESS ANALYSIS IN AN ORTHOTROPIC COMPOSITE CANTILEVER BEAM LOADED UNIFORMLY

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Abstract-In this study an elastic-plastic stress analysis is carried out in an orthotropic composite cantilever beam loaded uniformly at the upper surface. An analytical solution is found satisfying both the governing differential equation in two dimensional case and boundary conditions. In this solution, the intensity of the uniform force is chosen small, therefore σ_y is neglected in comparison with the other stress components. The angles between the first principal axis and x axis are chosen as 0°, 30°, 45°, 60° and 90°. The plastic region begins first at the upper surface of the beam for 30° and 45° orientation angles. However, it starts earlier at the lower surface for 60° orientation angle. The intensity of the residual stress component of σ_x is maximum at the upper and/or lower surfaces in the beam. The intensity of the residual stress component of τ_{xy} is maximum on or around the longitudinal axis of the beam.

1. INTRODUCTION

Residual stresses in the composites are particularly important because they may lead to premature failure. Prediction and measurement of residual stresses are therefore important in relation to production, design and performance of composite components. Akay *et al.* [1] measured the thermal residual stresses in injection moulded thermoplastics by removing thin layers from specimens and measuring the resultant curvature or the bending moment in the remainder of the specimens. Jeronimidis and Parkyn [2] investigated residual stresses in carbon fibre-thermoplastic matrix laminates. The finite element technique gives an excellent elasto-plastic stress analysis in composites structures [3,4,5,6].

Karakuzu *et al.* [7] carried out an elasto-plastic stress analysis in an aluminum matrix composite cantilever beam loaded by a single force or uniformly distributed forces by using an exact analytical solution. They determined the expansion of the plastic zone and residual stresses in that beam.

Sayman *et al.* [8] carried out an analytical solution for a composite cantilever beam loaded by a single force at its free end.

In the present study, an elastic-plastic stress analysis is carried out in an orthotropic composite beam loaded by a uniformly distributed force on the upper surface. A closed form solution is performed by satisfying both governing differential equation for an anisotropic system and boundary conditions.

2.ELASTIC SOLUTION

The elastic solution of an orthotropic cantilever beam loaded uniformly, as shown in Figure 1, is given by Lekhnitskii [9]. The governing differential equation of a plane stress case is written as,

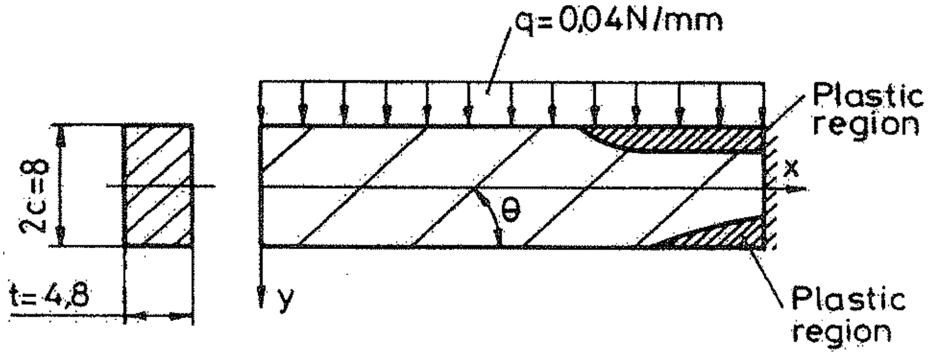

Figure 1. Composite beam.

$$a_{22}\frac{\partial^4 F}{\partial x^4}-2a_{26}\frac{\partial^4 F}{\partial x^3\partial y}+(2a_{12}+a_{66})\frac{\partial^4 F}{\partial x^2\partial y^2}-2a_{16}\frac{\partial^4 F}{\partial x\,\partial y^3}+a_{11}\frac{\partial^4 F}{\partial y^4}=0 \tag{1}$$

where F is a stress function and a_{ij} is the component of the compliance matrix [10],

$$\begin{aligned}
a_{11} &= S_{11}m^4+(2S_{12}+S_{66})m^2n^2+S_{22}n^4\\
a_{12} &= S_{12}(m^4+n^4)+(S_{11}+S_{22}-S_{66})m^2n^2\\
a_{22} &= S_{11}n^4+(2S_{12}+S_{66})m^2n^2+S_{22}m^4\\
a_{16} &= (2S_{11}-2S_{12}-S_{66})nm^3-(2S_{22}-2S_{12}-S_{66})n^3m\\
a_{26} &= (2S_{11}-2S_{12}-S_{66})n^3m-(2S_{22}-2S_{12}-S_{66})nm^3\\
a_{66} &= 2(2S_{11}+2S_{22}-4S_{12}-S_{66})m^2n^2+S_{66}(m^4+n^4)
\end{aligned} \tag{2}$$

where $m=\cos\theta$, $n=\sin\theta$, $S_{11}=1/E_1$, $S_{22}=1/E_2$, $S_{12}=-\nu_{12}/E_1$, $S_{66}=1/G_{12}$.

The polynomial for the elastic solution of the problem is chosen as,

$$F=\frac{d}{6}x^2y^3+\frac{e}{12}xy^4+\frac{f}{20}y^5+\frac{k}{2}xy^2+\frac{g}{6}y^3+\frac{b}{2}x^2y+\frac{a}{2}x^2 \tag{3}$$

Substituting in Equation (1), gives,

$x(-4a_{16}d+2ea_{11})+y(4a_{12}d+2a_{66}d-4a_{16}e+6fa_{11})=0$

For satisfying the equation, every term that is a function of x and y must be equal to zero as,

$-4a_{16}d+2ea_{11}=0, \qquad e=sd$ (4)

where $s=2a_{16}/a_{11}$

$2a_{12}d+a_{66}d-2a_{16}e+3fa_{11}=0, \qquad f=rd$

where

$$r=\frac{2a_{16}s-2a_{12}-a_{66}}{3a_{11}} \tag{5}$$

Now, the stress components become

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = dx^2 y + sdy^2 + rdy^3 + kx + gy$$
$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = \frac{d}{3} y^3 + by + a \qquad (6)$$
$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -dxy^2 - \frac{sd}{3} y^3 - ky - bx$$

The boundary conditions are:

$$\sigma_y = -\frac{q}{t} \quad at \quad y = -c$$
$$\sigma_y = 0 \quad at \quad y = +c$$
$$\tau_{xy} = 0 \quad at \quad y = \pm c \qquad (7)$$
$$\int_{-c}^{c} \sigma_x t dy = 0 \quad \text{at the free end}$$
$$\int_{-c}^{c} \sigma_x t y dy = 0 \quad \text{at the free end}$$

where t is the thickness of the beam and q is the intensity of the uniform load given as Newton/mm.

By solving the above equations, the unknown parameters are determined, and the stress components become,

$$\sigma_x = -\frac{q}{2I}\left(x^2 y + ry^3 + sxy^2 - \frac{1}{3}c^2 sx - \frac{3}{5}rc^2 y\right)$$
$$\sigma_y = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2 y\right) - \frac{q}{2t} \qquad (8)$$
$$\tau_{xy} = -\frac{q}{2I}\left(-xy^2 - \frac{1}{3}sy^3 + c^2 x + \frac{1}{3}sc^2 y\right)$$

where I is the inertia moment of the cross section of the beam.

3. ELASTIC-PLASTIC SOLUTION

In this study, it is assumed that the material in the plastic region is perfectly plastic for the simplicity of the investigation. In the plastic region stress components have to satisfy the equations of equilibrium for a plane-stress case as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad (9)$$

The uniform force q is chosen as a small value, therefore σ_y can be neglected in comparison with the other stress components. Putting $\sigma_y=0$ in the second equation gives the shear stress component τ_{xy} as a function of y or a constant.

The Tsai-Hill Theory is used as a yield criterion in this solution. X and Y are the yield points in the 1st and 2nd principal axes, respectively. S is the yield point in the 1-2 plane for the simple pure shear. For this condition it can be written as,

$$\left[\frac{\sigma_1^{\,2}}{X^2}-\frac{\sigma_1\sigma_2}{X^2}+\left(\frac{\sigma_2}{Y}\right)^2+\left(\frac{\tau_{12}}{S}\right)^2\right]^{\frac{1}{2}}=1$$
$$\bar{\sigma}_e=\left[\sigma_1^{\,2}-\sigma_1\sigma_2+\left(\frac{\sigma_2 X}{Y}\right)^2+\left(\frac{\tau_{12}X}{S}\right)^2\right]^{\frac{1}{2}}=X \qquad (10)$$

where σ_e is the equivalent stress, and σ_1, σ_2 and τ_{12} are the stress components in principal material directions as,

$$\sigma_1=\sigma_x\cos^2\theta+2\tau_{xy}\sin\theta\ \cos\theta$$
$$\sigma_2=\sigma_x\sin^2\theta-2\tau_{xy}\sin\theta\ \cos\theta \qquad (11)$$
$$\tau_{12}=-\sigma_x\sin\theta\ \cos\theta+\tau_{xy}\left(\cos^2\theta-\sin^2\theta\right)$$

Substituting the stress components in equation (10) and deriving it with respect to x, gives $\frac{\partial\sigma_x}{\partial x}=0$. Putting in the first equation of equilibrium, it is obtained τ_{xy} as a constant. Yielding begins at the upper or lower surfaces of the beam, $\tau_{xy}=0$ on these surfaces. From this condition it is found that τ_{xy} is equal to zero in the plastic region. From the yield criterion, the stress component σ_x which causes the material to yield is determined as,

$$X_1=\frac{X}{\sqrt{\cos^4\theta-\sin^2\theta\cos^2\theta+\frac{X^2\sin^4\theta}{Y^2}+\frac{X^2\sin^2\theta\cos^2\theta}{S^2}}} \qquad (12)$$

where θ is the angle between the first principal material direction and the x axis. As a result of this, in the plastic region the stress components σ_x and τ_{xy} are equal to X_1 and zero, respectively.

3.1.Elastic Part

If the plastic region expands at the upper and lower surfaces of the beam, consider a section from the free end x, as shown in Figure 2. The boundary conditions for this section are written as,

$$\tau_{xy}=0 \qquad at \quad y=-h_1$$
$$\tau_{xy}=0 \qquad at \quad y=h_2$$
$$\int_{-h_1}^{h_2}\tau_{xy}t\,dy=-qx$$

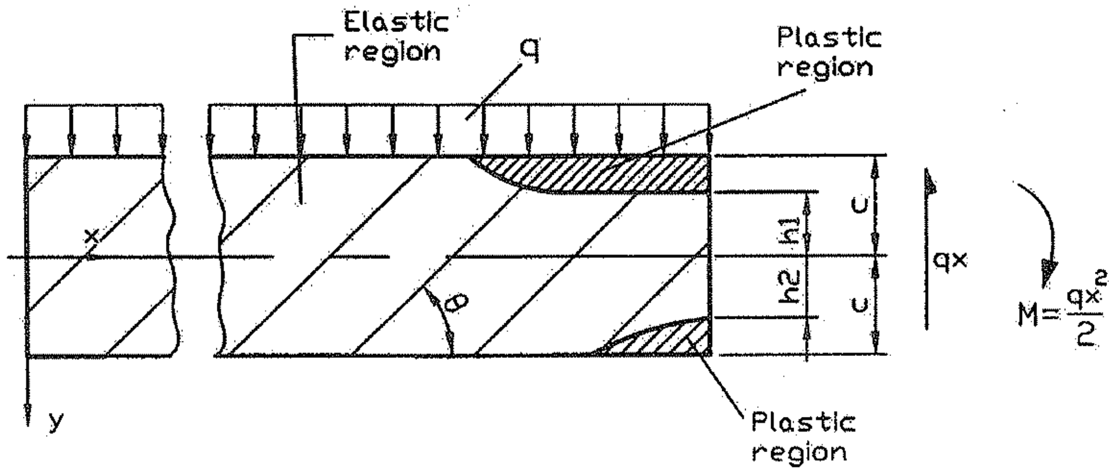

Figure 2. Expansion of the plastic region.

where the positive τ_{xy} is in the opposite direction of qx, therefore qx takes a negative sign.

$$\begin{aligned} \sigma_x &= X_1 \quad at \quad y = -h_1 \\ \sigma_x &= -X_1 \quad at \quad y = h_2 \end{aligned} \tag{13}$$

The resultant of σ_x at this section is equal to zero:

$$X_1(c-h_1)t - X_1(c-h_2)t + \int_{-h_1}^{h_2} \sigma_x t dy = 0$$

The moment in this section is equal to,

$$X_1(c-h_1)\frac{c+h_1}{2}t + X_1(c-h_2)\frac{c+h_2}{2}t - \int_{-h_1}^{h_2} \sigma_x t y dy = \frac{qx^2}{2}$$

where positive σ_x produces an opposite moment of $qx^2/2$.

For satisfying both the differential equation (1) and the boundary conditions (13) in the elastic part of the elastic-plastic zone, the stress function is chosen as:

$$F = \frac{d}{6}x^2y^3 + \frac{e}{12}xy^4 + \frac{f}{20}y^5 + \frac{p}{2}xy^2 + \frac{g}{6}y^3 + \frac{b}{2}x^2y + \frac{k}{2}y^2 \tag{14}$$

The stress components from this polynomial are found as,

$$\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} = dx^2y + sdy^2x + rdy^3 + px + gy + k \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} = -dxy^2 - \frac{sd}{3}y^3 - py - bx \end{aligned} \tag{15}$$

q is chosen as a small value, therefore σ_y can be neglected in comparison with the other stress components.

The stress function satisfies both the governing differential equation and the boundary conditions. From the boundary conditions the seven unknown parameters are determined as,

$$
\begin{aligned}
&p=-dx(h_2-h_1)-\frac{sd}{3}\left(h_1^{2}-h_1h_2+h_2^{2}\right)\\
&b=-dh_1h_2+\frac{sdh_1h_2}{3x}(h_1-h_2)\\
&g=-\frac{2X_1}{h_1+h_2}-dx^2-rd\left(h_1^{2}-h_1h_2+h_2^{2}\right)+sdx(h_1-h_2)\\
&k=rdh_1h_2(h_1-h_2)-dx^2(h_1-h_2)+\frac{sdx}{3}\left(h_1^{2}-4h_1h_2+h_2^{2}\right)-\frac{X_1(h_1-h_2)}{h_1+h_2}\\
&d=\frac{-\frac{qx}{t}}{\frac{x}{6}(h_1+h_2)^3+\frac{s}{12}(h_1+h_2)^3(h_2-h_1)}\\
&u=\frac{3qxr+2X_1xt-\sqrt{(3qxr+2X_1xt)^2-8qX_1ts^2x^2}}{2X_1st}
\end{aligned}
\tag{16}
$$

$$h_1=h_2+u$$

$$
\frac{d}{60}\left\{r(2h_2+u)\left[8(h_2+u)^4+8h_2^{4}+2h_2(h_2+u)^3+2h_2^{3}(h_2+u)-12h_2^{2}(h_2+u)^2\right]-5sxu(2h_2+u)^3\right.
$$
$$
\left.-\frac{X_1}{3}\left[(h_2+u)^2+h_2^{2}-h_2(h_2+u)\right]-\frac{qx^2}{2t}+X_1c^2=0\right.
$$

u is determined by using the given constants. Subsequently, solving the last equation by using the Newton-Raphson method gives h_2 and subsequently the other parameters are determined. h_1 is found to be equal to h_2 for the 0° and 90° orientation angles.

4. A SAMPLE

The mechanical properties of an orthotropic composite beam are given in Table 1. The thickness and the height of the beam are chosen as 4,8 mm and 8 mm, respectively.

Table 1. Mechanical properties and yield strengths of the composite beam.

E_1 [MPa]	E_2 [MPa]	G_{12} [MPa]	ν_{12}	Axial strength (X) [MPa]	Transverse strength (Y) [MPa]	Shear strength (S) [MPa]
4300	960	240	0.4	21.0	5.2	5.8

5. RESULTS AND DISCUSSION

The composite cantilever beam is loaded uniformly. The intensity of the uniform load is chosen as q=0.04 N/mm. The yield points at the upper and lower surfaces of the beam from the analytical solution are given at Table 2. As seen from this Table, yielding occurs first at upper surface for 30° and 45° orientation angles. However, it starts earlier at the lower surface for the 60° orientation angle. The yield point has the same distances from the free end for 0° and 90° orientation angles because of the symmetry of the material properties.

Table 2. The distance between the free end and yield points.

Orientation angles	0°	30°	45°	60°	90°
At the upper surface (mm)	232.00	162.90	140.00	126.40	115.50
At the lower surface (mm)	232.00	168.20	141.70	125.50	115.50

The expansion of the plastic region and the stress component of σ_x are given in Table 3, for 0°, 30°, 45°, 60° and 90° orientation angles. As seen from this Table, the intensity of the residual stress of σ_x at the upper surface is greater than that at the lower surface for 30°, 45° orientation angles. It is the same for 0° and 90° orientation angles. However the intensity of the residual stress on the lower surface is greater than that on the upper surface for 60° orientation angle. When the orientation angle is increased, the plastic region spreads rapidly as seen for the beam of 90° orientation angle. Also the residual stress component of τ_{xy} is given on the x axis, because it is maximum on or around this axis.

Table 3. Expansion of the plastic region and the residual stress component of σ_x at the upper and lower surfaces and the residual stress components of τ_{xy} on the x axis.

				At the upper surface (MPa)			At the lower surface (MPa)			
θ	X (mm)	h_1 (mm)	h_2 (mm)	$(\sigma_x)_p$	$(\sigma_x)_e$	$(\sigma_x)_r$	$(\sigma_x)_p$	$(\sigma_x)_e$	$(\sigma_x)_r$	$(\tau_{xy})_r$
	240	3.71	3.71	21.00	22.49	-1.49	-21.00	-22.49	1.49	-0.03
0°	250	3.29	3.29	21.00	24.40	-3.40	-21.00	-24.40	3.40	-0.09
	260	2.79	2.79	21.00	26.39	-5.39	-21.00	-26.39	5.39	-0.17
	270	2.15	2.15	21.00	28.46	-7.46	-21.00	-28.46	7.46	-0.27
	170	3.65	3.91	10.70	11.64	-0.94	-10.70	-10.94	0.24	-0.01
	180	3.11	3.39	10.70	13.03	-2.33	-10.70	-12.29	1.59	-0.07
30°	190	1.54	2.84	10.70	14.50	-3.80	-10.70	-13.71	3.01	-0.14
	200	1.94	2.25	10.70	16.04	-5.34	-10.70	-15.21	4.51	-0.29
	150	3.40	3.50	7.74	8.88	-1.14	-7.74	-8.69	0.95	-0.04
	160	2.77	3.87	7.74	10.10	-2.36	-7.74	-9.89	2.15	-0.10
45°	170	2.09	2.20	7.74	11.39	-3.65	-7.74	-11.18	3.44	-0.23
	180	1.37	1.49	7.74	12.77	-5.03	-7.74	-12.54	4.80	-0.51
	130	3.77	3.71	6.19	6.55	-0.36	-6.19	-6.65	0.46	-0.02
	140	3.09	3.03	6.19	7.60	-1.41	-6.19	-7.71	1.52	-0.06
60°	150	2.37	2.30	6.19	8.73	-2.54	-6.19	-8.84	2.65	-0.15
	160	1.60	1.52	6.19	9.94	-3.75	-6.19	10.06	3.87	-0.39
	120	3.66	3.66	5.20	5.62	-0.42	-5.20	-5.62	0.42	-0.01
90°	125	3.23	3.23	5.20	6.10	-0.90	-5.20	-6.10	0.90	-0.04
	130	2.72	2.72	5.20	6.60	-1.40	-5.20	-6.60	1.40	-0.10
	135	2.05	2.05	5.20	7.12	-1.92	-5.20	-7.12	1.92	-0.20

The expansion of the plastic region and the residual stress distribution of σ_x along the sections of the beam are given in Figure 3, for 0° orientation angle. As seen from this Figure, the plastic zone expands slowly along the beam and the intensity of the residual stress of σ_x is maximum at the upper and lower surfaces.

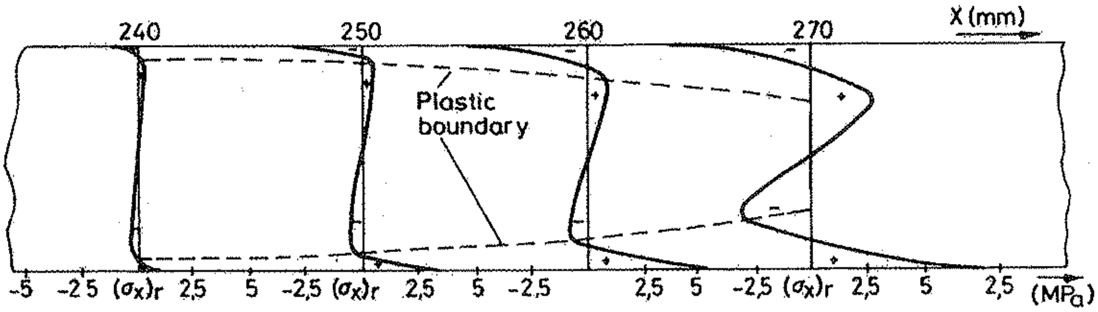

Figure 3. Distribution of the residual stress component of σ_x and expansion of the plastic region for 0° orientation angle.

The distribution of the residual stress component of σ_x and the expansion of the plastic zone are given in Figure 4, for 30° oriented beam. As seen from this Figure, the intensity of σ_x at the upper surface is maximum along the sections of the beam.

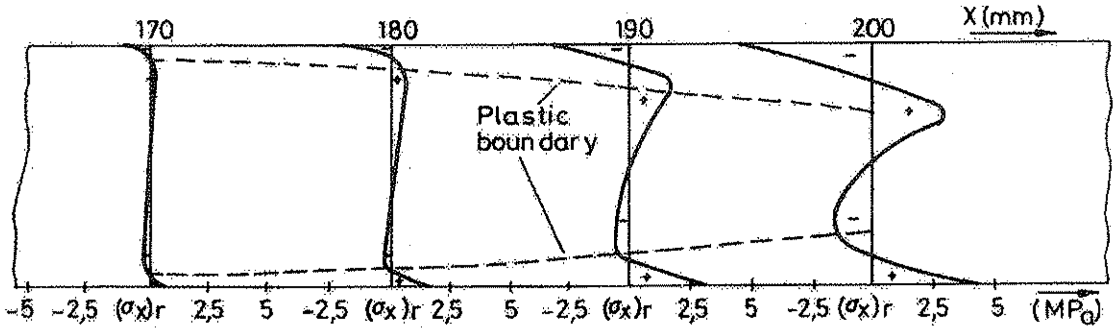

Figure 4. Distribution of the residual stress component of σ_x and expansion of the plastic region for 30° orientation angle.

The expansion of the plastic zone and the distribution of σ_x are given in Figures 5, 6 and 7, for 45°, 60° and 90° orientation angles, respectively. As seen from these Figures, the intensity of σ_x is maximum at the upper surface for 45° orientation angle and it is maximum at the lower surface of the beam for 60° orientation angle at the sections of the beam. The distribution of the residual stress component of σ_x is symmetric for 90° orientation angle.

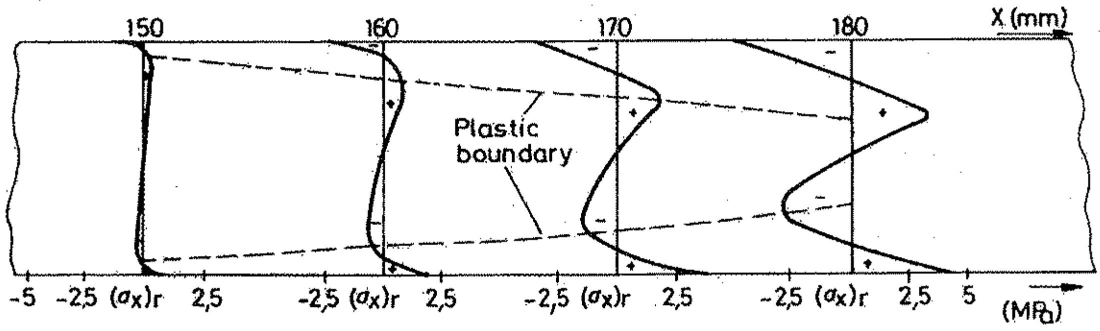

Figure 5. Distribution of the residual stress component of σ_x and expansion of the plastic region for 45° orientation angle.

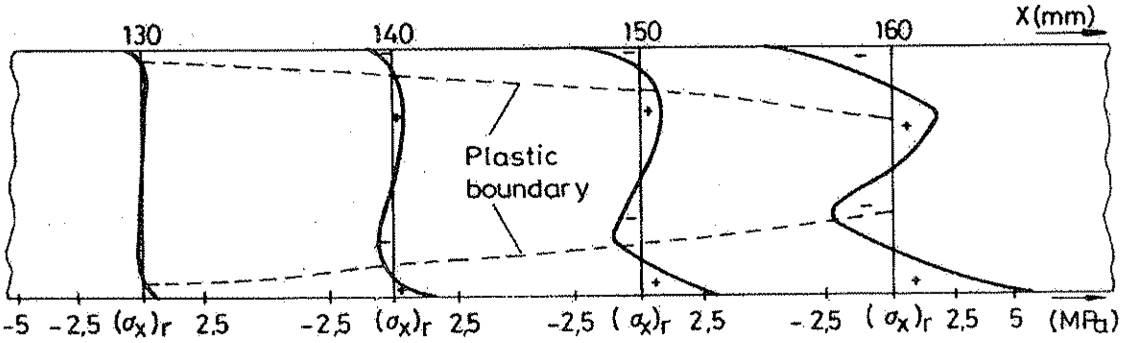

Figure 6. Distribution of the residual stress component of σ_x and expansion of the plastic region for 60° orientation angle.

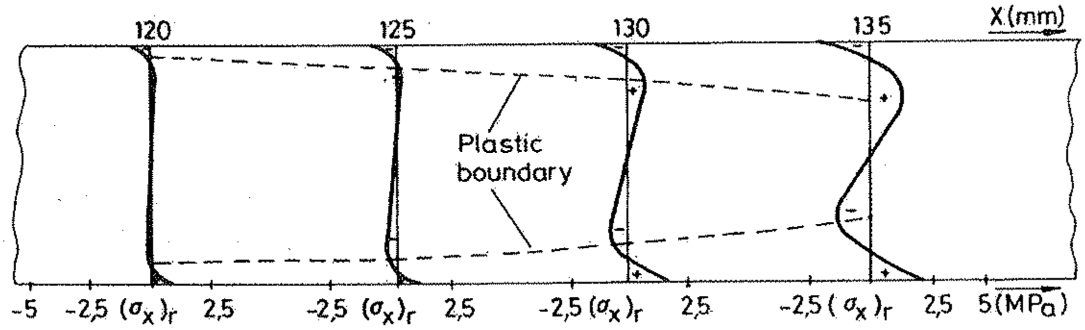

Figure 7. Distribution of the residual stress component of σ_x and expansion of the plastic region for 90° orientation angle.

6.CONCLUSIONS

In the present study, an elastic-plastic stress analysis is given for the orthotropic composite cantilever beam loaded uniformly on its upper surface.

1. The plastic region begins earlier at the upper surface of the beam for 30° and 45° orientation angles. However, it starts first at the lower surface for 60° orientation angle.
2. The plastic region begins at the upper and lower surfaces at the same distances from the free end for 0° and 90° orientation angles.
3. The plastic region expands rapidly on the upper side of the beam for 30° and 45° orientation angles. However, it is contrary for 60° oriented beam.
4. The intensity of the residual stress component of σ_x is maximum at the upper or lower surfaces in the beam. It is greatest at the upper surface for 30° and 45° orientation angles.
5. The intensity of the residual stress of τ_{xy} is maximum on or around the x axis. It is maximum on the x axis for 0° and 90° orientation angles due to symmetry of the material properties of the beam, with respect to x axis.
6. The intensity of the residual stress component of σ_x is much greater than that of τ_{xy}.

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