

MONODROMY PRINCIPLE AND UNIVERSAL COVERINGS

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Abstract - Let G be a star topological groupoid [Definition 2.2]. In this paper using the structure of monodromy groupoid we obtain a result which is more general than a result given in [1] and known as monodromy principle. Further if G is a topological group and W is a suitable open neighbourhood of the identity element then we obtain a universal covering of G .

1 INTRODUCTION

A groupoid can be thought of as a group with many objects, or with many identities. A precise definition is given in Section 2. A groupoid with one object is essentially just a group. So the notion of groupoid is an extension of that of group. This gives a range of applications. Monodromy Principle is an important way of extending local morphisms on simply connected topological groups.

In this paper using the construction of the monodromy groupoid given in Section 2 we have a result for star topological groupoids.

Let G be a Lie group and W an open neighbourhood of the identity in G . Douady and Lazard in [2] obtained a group $\tilde{G}$ which has a topology such that left (hence right) translations are homeomorphisms. In general $\tilde{G}$ is not a topological group. However they proved that the projection map $p: \tilde{G} \rightarrow G$ is a covering map. In this paper we also prove that if G is a topological group and W is a path connected neighbourhood of the identity then $\tilde{G}$ becomes a topological group and find the conditions on W and G for the projection map $p: \tilde{G} \rightarrow G$ to be a universal covering.

All spaces are assumed to be locally path connected.

2 MONODROMY PRINCIPLE

A groupoid G on O_G is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, which we call just elements of G , a set O_G of objects together with functions $\alpha, \beta: G \rightarrow O_G$, $\varepsilon: O_G \rightarrow G$ such that $\alpha\varepsilon = \beta\varepsilon = 1_{O_G}$, the identity map. The functions α, β are called initial and final point maps respectively. If $a, b \in G$ and $\beta(a) = \alpha(b)$, then the product or composite ba exists such that $\alpha(ba) = \alpha(a)$ and $\beta(ba) = \beta(b)$. Further, this composite is associative, for $x \in O_G$ the element $\varepsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$,

$a^{-1}a = (\varepsilon\alpha)(a)$, $aa^{-1} = (\varepsilon\beta)(a)$. The map $G \rightarrow G$, $a \mapsto a^{-1}$ is called the inversion

In a groupoid G for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y . For $x \in O_G$ we denote the *star* $\{a \in G : \alpha(a) = x\}$ of x by G_x and the *costar* $\{a \in G : \beta(a) = x\}$ of x by G^x . In G the set O_G is mapped bijectively to the set of identities by $\varepsilon : O_G \rightarrow G$. So we sometimes write O_G for the set of identities. Let G be a groupoid and W a subset of G . We say G is *generated* by W if each element of G is written as a union of the elements in W .

Let G be a groupoid. A *subgroupoid* of G is a pair of subsets $H \subseteq G$ and $O_H \subseteq O_G$ such that $\alpha(H) \subseteq O_H$, $\beta(H) \subseteq O_H$, $1_x \in H$ for each $x \in O_H$ and H is closed under the partial multiplication and inversion in G .

Let G and H be groupoids. A *local morphism* of groupoid is a map $f : W \rightarrow H$ from a subset of G containing all the identities in G such that for $u \in W$ $\alpha_H(fu) = f(\alpha_G u)$, $\beta_H(fu) = f(\beta_G u)$ and $f(vu) = f(v)f(u)$ whenever $u, v \in W$, vu is defined and belongs to W .

A *morphism* from G to H is a pair of maps $f : G \rightarrow H$ and $O_f : O_G \rightarrow O_H$ such that $\alpha_H \circ f = O_f \circ \alpha_G$, $\beta_H \circ f = O_f \circ \beta_G$ and $f(vu) = f(v)f(u)$ for all $v, u \in G$ with $\alpha v = \beta u$.

Since the following definition is related to the star topological groupoids, we give here for completeness.

Definition 2.1 ([3]) A *topological groupoid* is a groupoid G on O_G together with topologies on G and O_G such that the maps which define the groupoid structure are continuous, namely the projections $\alpha, \beta : G \rightarrow O_G$, the object inclusion map $\varepsilon : O_G \rightarrow G$, $x \mapsto \varepsilon(x)$, the inversion $G \rightarrow G$, $a \mapsto a^{-1}$ and the partial multiplication $G_\alpha \times_\beta G \rightarrow G$, where the pullback

$$G_\alpha \times_\beta G = \{(b, a) \in G \times G : \alpha(b) = \beta(a)\}$$

has the subspace topology from $G \times G$.

Note that in a topological groupoid G if one of the initial and final point maps α, β and the inversion are continuous, then the other map is continuous. Also in a topological groupoid G the inversion is clearly a homeomorphism and α, β are identification maps, since they have object inclusion map ε as a right inverse.

A *morphism* of topological groupoids $f : H \rightarrow G$ is a morphism of groupoids in which both maps $f : H \rightarrow G$ and $O_f : O_H \rightarrow O_G$ are continuous.

Note that in a topological groupoid G for each $a \in G(x, y)$ the right translation $R_a : G_y \rightarrow G_x$, $b \mapsto ba$ and left translation $L_a : G^x \rightarrow G^y$, $b \mapsto ab$ are homeomorphisms.

The following definition is due to Pradines [4].

Definition 2.2 A *locally star topological groupoid* is a pair (G, W) of a groupoid G and a topological space W such that:

- (i) $O_G \subseteq W$, that is, W contains all identities;
- (ii) W is the topological sum of the subspaces $W_x = W \cap G_x$, $x \in O_G$;
- (iii) if $a \in G(x, y)$, then for the right translation $R_a : W_y \rightarrow W_y a$, $b \mapsto ba$ the sets $R_a^{-1}(W)$ and $W \cap W_y a$ are open in W_y and the map

$$R_a : R_a^{-1}(W) \rightarrow W \cap W_y a, b \mapsto ba,$$

the restriction of the right translation, is a homeomorphism.

Such a locally star topological groupoid (G, W) is said to be a *star topological groupoid* if G and W have the same underlying set. So a star topological groupoid is a groupoid in which each star G_x has a topology such that right translations (and hence left translations) are homeomorphisms.

Remark that a topological groupoid has an underlying star topological groupoid structure, but a star topological groupoid is not necessarily a topological groupoid.

Let G be a star topological groupoid and W a subset of G . We say W is *star open* (resp. *star connected*, *star simply connected*) if $W_x = W \cap G_x$ is open (resp. connected and simply connected) in G_x for each $x \in O_G$. So a star topological groupoid G is said to be *star connected* (resp. *simply connected*) if G_x is star connected (resp. simply connected) for each $x \in O_G$.

The following proposition shows that a locally star topological groupoid is extended to a star topological groupoid. This result is due to Pradines [4] (see also [5] for the poof in details).

Proposition 2.3 Let (G, W) be a locally star topological groupoid. Then (G, G) may be given the structure of a star topological groupoid such that W is star open in G .

For the construction of the monodromy groupoid we need the notions of normal subgroupoid and quotient groupoid ([6] and [7]).

A normal subgroupoid of G is a subgroupoid N of G such that $O_N = O_H$ and for each $x, y \in O_G$ and $a \in G(x, y)$ we have $aN(x) = N(y)a$. Let G be a groupoid and N a normal subgroupoid of G such that $N(x, y) = \emptyset$ if $x \neq y$. Define a groupoid G/N on O_G by $G/N(x, y) = \{aN(x) : a \in G(x, y)\}$ for any $x, y \in O_G$ with the multiplication that if $a \in G(x, y)$ and $b \in G(y, z)$ then

$$bN(y)aN(x) = baN(x)N(y) = baN(x).$$

This groupoid is called quotient groupoid of G by N .

The construction of the free groupoid is as follows [7]: Let W be a directed graph. Let $p = (a_n, \dots, a_1)$ be a sequence of the edges such that the target of a_i is equal to the source of a_{i+1} . Such a p is called directed path. Write $()_x$ for the empty

path associated to x . The composition of two paths and $q = (b_m, \dots, b_1)$ is defined by $qp = (b_m, \dots, b_1, a_n, \dots, a_1)$ if the target of a_n is the source of b_1 . Then we have a category $P(W)$. Let $\hat{a}$ denote the converse path of a in W . Define an equivalence relation on $P(W)$ as follows: Two directed paths p, q are equivalent if we can obtain one from the other by adding or deleting a number of $a\hat{a}$ or $\hat{a}a$. This is an equivalence relation. The set of equivalence classes $[p]$ is denoted by $F(W)$. A groupoid multiplication on $F(W)$ is defined by $[q][p] = [qp]$. So $F(W)$ becomes a groupoid which is called *free groupoid* on the graph W .

Let G be a groupoid and R a subgroupoid of G . By the normal subgroupoid $N(R)$ generated by R we mean the smallest normal groupoid of G including R . A direct construction of $N(R)$ is given in [6] as follows: Let G be a groupoid and R a subset of G such that $O_R = O_G$ and $R(x, y) = \emptyset$ if $x \neq y$. Let $N(x)$ be the set of all elements $r = a_n^{-1} r_n a_n \dots a_1^{-1} r_1 a_1$ for $a_i \in G(x, x_i)$ and r_i or r_i^{-1} an element of $R(x_i)$. Let $N(R)$ be the family of $N(x)$ for all $x \in O_G$. Then $N(R)$ is a normal subgroupoid of G such that $O_{N(R)} = O_G$ and $NR(x, y) = \emptyset$ if $x \neq y$. $N(R)$ is called the normal subgroupoid generated by R .

Now let G be a groupoid and W a subset of G such that $O_G \subseteq W$, i.e. W contains all the identities. Let $F(W)$ be the free groupoid on W , N the normal subgroupoid of $F(W)$ generated by the elements in the form $[vu]^{-1}[v][u]$ for $u, v \in W$ such that vu is defined and $v, u \in W$ and let $\tilde{G}$ be the quotient groupoid $F(W)/N$. The quotient groupoid $\tilde{G}$ is called *monodromy groupoid*. By the construction of $\tilde{G}$ we have an inclusion map $\tilde{\tau} : W \rightarrow \tilde{G}$ and a projection morphism $p : \tilde{G} \rightarrow G$ induced by the inclusion map $i : W \rightarrow G$. Let $\tilde{W} = \tilde{\tau}(W)$.

Following the method of Douady and Lazard in [2], it was proved in [8] that if W has a topological structure then the monodromy groupoid $\tilde{G}$ has a star topological groupoid structure.

We now prove the following proposition for star topological groupoids, which is given in [9] for topological groupoids and restated for Lie groupoids in [10] (see also [11]).

Proposition 2.4 Let G be a star topological groupoid and W a star open subset of G containing all the identities in G . Then $\tilde{G}$ becomes a star topological groupoid such that $\tilde{W}$ is star open in $\tilde{G}$. Further the restriction $p_x : \tilde{G}_x \rightarrow G_x$ of the projection map $p : \tilde{G} \rightarrow G$ is a covering map.

Proof: Let $\tilde{W}$ be the image of W under the inclusion map $\tilde{\tau} : W \rightarrow \tilde{G}$. First of all we prove that the pair $(\tilde{G}, \tilde{W})$ has a locally star topological groupoid and then by Proposition 2.3 this locally star topological groupoid extends to a star topological groupoid.

- (i) Since by the construction of $\tilde{G}$ obviously $O_{\tilde{G}} = O_G$, and by the assumption $O_G \subseteq W$, we have $O_{\tilde{G}} \subseteq \tilde{W}$;
- (ii) $\tilde{W}$ generates $\tilde{G}$ as a groupoid;
- (iii) retopologise $\tilde{W}$ as the topological sum of $\tilde{W}_x$'s for $x \in O_{\tilde{G}}$;
- (iv) since G is a star topological groupoid for $a \in G(x, y)$ the right translation

$$R_a : W_y \rightarrow W_y a, b \mapsto ba$$

is a homeomorphism. So for $a \in G(x, y)$, the set $(R_a)^{-1}(\tilde{W} \cap \tilde{W}_y a)$ is open in $\tilde{W}_y$ and the restriction of R_a

$$R_a : (R_a)^{-1}(\tilde{W} \cap \tilde{W}_y a) \rightarrow \tilde{W} \cap \tilde{W}_y a$$

is a homeomorphism.

Hence the pair $(\tilde{G}, \tilde{W})$ becomes a locally star topological groupoid, and by Proposition 2.3 this extends to a star topological groupoid. To prove that the restriction map $p_x : \tilde{G}_x \rightarrow G_x$ is a covering map, let $a \in G(x, y)$. Then $W_y a$ is an open neighbourhood of a in G_x , and

$$p^{-1}(W_y a) = \bigcup \{ \tilde{W}_y b : b \in p^{-1}(a) \}.$$

But for each $b \in p^{-1}(a)$, the restriction $p_x : \tilde{W}_y b \rightarrow W_y a$ is a homeomorphism and the sets $\tilde{W}_y b$ for all b such that $pb=a$ are disjoint. Since the sets $\tilde{W}_y b$ are open, the restriction $p_x : \tilde{G}_x \rightarrow G_x$ is a covering map. $\square$

The following theorem generalises a result called Monodromy Principle in [1]
Theorem 2.5 (Monodromy Principle) Let G be a star topological groupoid which is star connected and star simply connected and W a star open and star connected subset of G . Then any local morphism of groupoids $f : W \rightarrow H$ extends to a morphism of groupoids $\phi : G \rightarrow H$, i.e, there exists a morphism $\phi : G \rightarrow H$ of groupoids such that $\phi \tilde{\iota} = f$.

Proof: Note that by the universal property of $\tilde{G}$ there is a morphism $\phi : \tilde{G} \rightarrow H$ such that $\phi \tilde{\iota} = f$. On the other hand the projection morphism $p : \tilde{G} \rightarrow G$, induced from the inclusion map $i : W \rightarrow G$, is reduced to a covering on each fibre $p_x : \tilde{G}_x \rightarrow G_x$. By Corollary 3.4.8 in [9] when W is star connected, $\tilde{G}$ is also star connected. So here both spaces G_x and $\tilde{G}_x$ are connected and G_x is simply connected. Hence $p_x : \tilde{G}_x \rightarrow G_x$ becomes a homeomorphism. Further since $O_p : O_{\tilde{G}} \rightarrow O_G$ is the identity, the morphism $p : \tilde{G} \rightarrow G$ is an isomorphism of groupoids. So we have a morphism $\phi : G \rightarrow H$ as required. $\square$

3 UNIVERSAL COVERING

A covering map $p: \tilde{X} \rightarrow X$ of topological spaces is called *universal* if it covers every covering of X in the sense that if $q: \tilde{Y} \rightarrow X$ is another covering of X then there exists a map $r: \tilde{X} \rightarrow \tilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p: \tilde{X} \rightarrow X$ is called *simply connected* if $\tilde{X}$ is simply connected. Note that a simply connected covering is a universal covering.

It is a standard result in the theory of covering spaces that if $p: \tilde{X} \rightarrow X$ is a simply connected covering, X is a topological group with identity e and $\tilde{e} \in \tilde{X}$ such that $p(\tilde{e}) = e$, then there exists a group structure on $\tilde{X}$ with identity $\tilde{e}$ such that $\tilde{X}$ is a topological group with this group and $p: \tilde{X} \rightarrow X$ is a morphism of topological groups (see for example [1]). This is still true when $\tilde{X}$ is path connected but not necessarily simply connected.

In non connected case the situation is quite different and was first studied in [12]. He proved that the topological group X determines an obstruction class k_X and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a simply connected covering. In [9] an analogous algebraic result is given in terms of crossed modules and group objects in the category of groupoids (see also [13] for a revised version).

Let G be a topological group and W an open neighbourhood of the identity. Let $F(W)$ be the free group on W and N the normal subgroup of $F(W)$ generated by the elements $[vu]^{-1}[v][u]$ for $u, v \in W$ such that $vu \in W$ and let $\tilde{G}$ be the quotient group $F(W)/N$. The results given in this section form some parts of [9].

Proposition 3.1 Let G be a topological group and W an open, path connected neighbourhood of the identity e such that W generates G . Then the group $\tilde{G}$ becomes a topological group.

Proof: Note that since W generates G , if W is path connected then G becomes path connected. Similarly if W is path connected then $\tilde{G}$ becomes path connected. So $\tilde{G}$ becomes a topological group by the result stated above. $\square$

Let X be a topological space which has a simply connected covering. We call a subset W of X *liftable* if it is open, path connected and W lifts to each covering of X , that is, if $p: \tilde{X} \rightarrow X$ is a covering map, and $\tilde{x} \in \tilde{X}$ satisfies $p(\tilde{x}) = x \in W$, then there exists a unique map $\hat{i}: W \rightarrow \tilde{X}$ such that $p\hat{i} = i$ where $i: W \rightarrow X$ is the inclusion map.

It is easy to see that W is liftable if and only if it is open, path connected and for all $x \in W$, $i_*\pi_1(W, x) = \{1\}$ where $\pi_1(W, x)$ is the fundamental group of W at the base point x and i_* is the map induced by the inclusion map $i: W \rightarrow X$. Remark that

if a topological space X has a simply connected covering then each point $x \in X$ has a liftable neighbourhood.

Proposition 3.2 If G is a topological group and V is an open neighbourhood of e in G , then there is an open neighbourhood W of e such that $W = W^{-1}$ and $W^2 \subseteq V$.

Proof: In a topological group G , the group difference map

$$G \times G \rightarrow G, (a, b) \mapsto ab^{-1}$$

is continuous and hence there is an open neighbourhood U of e in G such that $U \times U \subseteq \delta^{-1}(V)$. Therefore it is enough to take $W = U \cap U^{-1}$. If V is liftable then W can be chosen as liftable. For if V is liftable then for each $x \in W$, the fundamental group $\pi_1(W, x)$ is mapped to the singleton by the morphism induced by the inclusion map $i: W \rightarrow G$. But W is not necessarily path connected and hence not liftable. But since the component $C_e(W)$ of e in W is liftable and satisfies these conditions we can replace W by $C_e(W)$ and suppose that W is liftable. $\square$

The following definition is very standard in the theory of topological groups.

Definition 3.3 Let G and H be topological groups and W an open neighbourhood of the identity e of G . A continuous map $\phi: W \rightarrow H$ is called *local morphism* if $\phi(uv) = \phi(u)\phi(v)$ for $u, v \in W$ such that $uv \in W$.

Proposition 3.4 Let G be a topological group and $q: H \rightarrow G$ a covering morphism of topological groups. Let W be an open, path connected neighbourhood of the identity e of G such that W^2 is contained in a liftable neighbourhood V of e . Then the inclusion map $i: W \rightarrow G$ lifts to a local morphism $\hat{i}: W \rightarrow H$.

Proof: Let e' be the identity of H . Since V lifts to H then W lifts to H by We now prove that $\hat{i}$ is a local morphism. We know by the lifting theorem that $\hat{i}$ is continuous. Let $u, v \in W$ such that $uv \in W$. Let a and b be the paths from u and v to e in W respectively and let c be the product path of a and b from uv to e . Suppose that $\tilde{a}, \tilde{b}$ and $\tilde{c}$ are the liftings of a, b and c in H respectively. Hence we have $\hat{i}(uv) = \hat{i}(u)\hat{i}(v)$. $\square$

By Proposition 3.2 and Proposition 3.4 we can obtain the following corollary.

Corollary 3.5 Let G be a topological group and $q: H \rightarrow G$ a covering morphism. Let V be a liftable neighbourhood of e in G . Then there exists a liftable neighbourhood W of e such that $W = W^{-1}$, $W^2 \subseteq V$ and hence the inclusion map $i: W \rightarrow G$ lifts to a local morphism $\hat{i}: W \rightarrow H$.

Theorem 3.6 Let G be a path connected topological group which admits a simply connected covering. Let W be an open, path connected neighbourhood of the identity e such that W^2 is contained in a liftable neighbourhood V of e . Then the covering

map $p: \tilde{G} \rightarrow G$ is a universal covering.

Proof: Note that by Proposition 3.2 there exists such a neighbourhood W . We know from [2] that the canonical projection map $p: \tilde{G} \rightarrow G$ is a covering map. Since W is path connected so also is $\tilde{G}$. So $\tilde{G}$ becomes a topological group. To prove that $p: \tilde{G} \rightarrow G$ is universal, we verify the universal property, that means $\tilde{G}$ covers every covering of G . For let $q: H \rightarrow G$ be a covering. Choose $e' \in H$ such that $q(e') = e$. Then e' defines a topological group structure on H such that e' is the identity element and q is a morphism of topological groups. By Proposition 3.4 the inclusion map $i: W \rightarrow G$ lifts to a local morphism $\hat{i}: W \rightarrow H$ and then $\hat{i}$ determines a morphism $\phi: \tilde{G} \rightarrow H$ which is continuous on $\tilde{W}$. Since $\tilde{W}$ are open in $\tilde{G}$ and generates $\tilde{G}$ the continuity of ϕ elsewhere follows. Hence $\phi: \tilde{G} \rightarrow H$ becomes a covering morphism. $\square$

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