Mathematical & Computational Applications, Vol. 5, No. 2, pp. 113-117, 2000

NON – PERTURBATIVE SOLUTION OF THE GINZBURG – LANDAU EQUATION

M. İNÇ
Firat University, Faculty of Arts and Sciences, Department of Mathematics,
23119 Elazığ / TURKEY
N.BİLDİK
Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics,
45140 Manisa / TURKEY

Abstract - The Ginzburg – Landau equation is solved by decomposition method.

INTRODUCTION AND DISCUSSION

The decomposition method can be an effective method for solving a wide class of problems providing generally a rapidly convergent series solution[3 – 7].It has some distinct advantages over usually approximation methods in that it is computationally convenient, The method makes unnecessary the massive computation of discretized methods for solution of partial differential equations. No linearization or perturbation is required. It provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems (as opposed to pathological mathematical systems).There are some quite significant advantages over methods which must assume linearity "smallness'', deterministic behaviour, stationary, restricted kinds of stochastic behaviour, uncoupled boundary conditions etc. The method has features in common with many other methods but it is distinctly different on close examination and one should not be misled by apparent simplicity into superficial conclusions [8 – 16].

In this paper, consider the Ginzburg – Landau equation, it is well known [1,2]:

$$\frac{\partial u}{\partial t}=u-(1+iR)|u|^2 u+(1+ib)\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial condition

$$u(x,0)=g(x)\ ,0\le x\le 1\ ,t\rangle 0 \tag{2}$$

where R,b are real parameters. To apply the decomposition method , we write equation (1) in an operator form

$$L_t u=u-(1+iR)|u|^2 u+(1+ib)L_x u \tag{3}$$

where L_t and L_x are the differential operators defined by

$$L_t=\frac{\partial}{\partial t}\ \text{ and }\ L_x=\frac{\partial^2}{\partial x^2}\ . \tag{4}$$

It is clear that L_t is inevitable and L_t^{-1} is the one-fold integration from 0 to t, i.e., $L_t^{-1}()=\int_0^t()\,dt$.

Operating with L_t^{-1} on both sides of (3), and using the condition (2) yield an equivalent expression for (3) given by

$$L_t^{-1} L_t\, u = L_t^{-1}\, u - (1+iR)\, L_t^{-1}\, |u|^2 u + (1+ib)\, L_t^{-1} L_x u$$
$$u(x,t) - u(x,0) = L_t^{-1}\, u - (1+iR)\, L_t^{-1}\, |u|^2 u + (1+ib)\, L_t^{-1} L_x u$$
$$u\,(x,t) = g(x) + L_t^{-1}\, u - (1+iR)\, L_t^{-1}\, |u|^2 u + (1+ib)\, L_t^{-1} L_x u\,. \qquad (5)$$

Following , the analysis of Adomian $[2-5]$, we write u (x , t) in the decomposition form :

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad . \qquad (6)$$

Substituting (6) into (5) gives

$$\sum_{n=0}^{\infty} u\,(x,t) = g(x) + L_t^{-1}\left(\sum_{n=0}^{\infty} u_n\right) - (1+iR)\, L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\, \{|u|^2 u\}\right) + (1+ib) L_t^{-1} L_x\left(\sum_{n=0}^{\infty} u_n\right) \quad (7)$$

where A_n are the Adomian polynomials $[2-5]$, the $A\{|u|^2\} = A\{u.u\}$ are given by

$$A_0 = u_0^2$$
$$A_1 = 2u_0 u_1$$
$$A_2 = u_1^2 + 2u_0u_2$$
$$A_3 = 2u_1u_2 + 2u_0u_3$$
$$A_4 = u_2^2 + 2u_1u_3 + 2u_0u_4$$
$$A_5 = 2u_2u_3 + 2u_1u_4 + 2u_0u_5$$
$$A_6 = u_3^2 + 2u_0u_6 + 2u_1u_5 + 2u_2u_4$$

The $A\{|u|^2 u\} = A\{(u.u).u\}$ are given by

$$A_0 = (u_0.u_0).u_0$$
$$A_1 = (2u_0.u_1).u_0 + (u_0.u_0).u_1$$
$$A_2 = (2u_0.u_2 + u_1.u_1).u_0 + (2u_0.u_1).u_1 + (u_0.u_0).u_2$$
$$A_3 = (2u_0.u_3 + 2u_1.u_2).u_0 + (2u_0.u_2 + u_1.u_1).u_1 + (2u_0.u_1).u_2 + (u_0.u_0).u_3$$
$$A_4 = (2u_0.u_4 + u_2u_2 + 2u_1.u_3).u_0 + (2u_1.u_2 + 2u_0.u_3).u_1 + (u_1.u_1 + 2u_0.u_2).u_2 + (2u_0.u_1).u_3 + (u_0.u_0).u_4$$
$$A_5 = (2u_0.u_5 + 2u_1u_4 + 2u_2.u_3).u_0 + (2u_1.u_3 + u_2u_2 + 2u_0.u_4).u_1 + (2u_1.u_2 + 2u_0.u_3).u_2 + (u_1.u_1 + 2u_0.u_2).u_3 + (2u_0.u_1).u_4 + (u_0.u_0).u_5$$
$$A_6 = (2u_0.u_6 + 2u_1u_5 + 2u_2.u_4 + u_3.u_3).u_0 + (2u_0.u_5 + 2u_1u_4 + 2u_2.u_3).u_1 + (2u_0.u_4 + 2u_1u_3 + u_2.u_2).u_2 + (2u_0.u_3 + 2u_1u_2).u_3 + (2u_0.u_2 + u_1.u_1).u_4 + (2u_0.u_1).u_5 + (u_0.u_0).u_6 \qquad (8)$$

Substituting (6) and (8) into (5) leads to the determination of all components of u by

$$u_0 = g(x) = u(x,0) \quad , \qquad (9)$$
$$u_1 = L_t^{-1}\, u_0 - (1+iR)\, L_t^{-1}\, (A_0) + (1+ib)\, L_t^{-1} L_x u_0 \quad , \qquad (10)$$
$$u_2 = L_t^{-1}\, u_1 - (1+iR)\, L_t^{-1}\, (A_1) + (1+ib)\, L_t^{-1} L_x u_1 \quad , \qquad (11)$$

...

......

$$u_{n+1} = L_t^{-1}\, u_n - (1+iR)\, L_t^{-1}\, (A_n) + (1+ib)\, L_t^{-1} L_x u_n \quad , \qquad (12)$$

Where A_n are Adomian polynomials defined above by (8). In conjunction of (9) and (12), all components of u (x , t) in (6) can be easily determined.

As a result, the terms of the series $u_0, u_1, u_2, \dots$ are identified and the exact solution may be entirely determined by using the approximation

$$u(x,t) = \lim_{n \to \infty} \phi_n \qquad , \tag{13}$$

where

$$\phi_n = \sum_{k=0}^{n-1} u_k(x,t) \qquad , \tag{14}$$

or

$$\begin{aligned}
&\phi_1 = u_0 \\
&\phi_2 = u_0 + u_1 \\
&\phi_3 = u_0 + u_1 + u_2 \\
&\dots \\
&\dots\dots\dots \\
&\phi_n = u_0 + u_1 + u_2 + \dots + u_{n-1} \quad , \quad n \geq 0.
\end{aligned}$$

We applied this method to many numerical examples, and the results we obtained have shown a high degree of accuracy. In many problems, the practical solution ϕ_n,the n-term approximation, is converging and accurate for low values of n. Convergence is established in $[17-20]$.

Adomian and Rach $[3]$and Wazwaz $[15]$ have investigate the phenomena of the self-cancelling "noise" terms whose sum vanishes in the limit. An important observation was made that "noise" terms appear for homogeneous cases only. Further, it was formally justified that if terms in u_0 are cancelled by terms in u_1, even though $u_1$includes further terms, then the remaining non-noise terms in constitute the exact solution of the equation.

It is worthwhile to mention that the non-homogeneous equations is quickly solved by observing the self-cancelling ''noise'' terms whose sum vanishes in the limit. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques and does not require unjustified assumptions, linearization or perturbation.

To give a clear overview of the methodology ,as an illustrative example, we take the Ginzburg-Landau equation to present one non-homogeneous example.

Example:

Let's assume R and b are constant and equal to zero ,consider the following the Ginzburg-Landau equation described by

$$\frac{\partial u}{\partial t} - u - |u|^2 u - \frac{\partial^2 u}{\partial x^2} = f(x,t). \tag{15}$$

We consider the problem

$$u_t - u + |u|^2 u - u_{xx} = x\left(1 - t + x^2 t^3\right) \tag{16}$$

with initial condition

$$u(x,0) = 0. \tag{17}$$

Following the outline scheme , (16) is rewritten in a standard form

$$L_t u - u + |u|^2 u - L_x u = f(x,t) \tag{18}$$

Thus,

$$u\ (x,t) = u(x,0) + L_t^{-1}\ u_n - L_t^{-1}\ A_n + L_t^{-1}\ L_x u_n + L_t^{-1}(f(x,t)). \qquad (19)$$

The decomposition series solution $u(x,t)$ into $\sum_{n=0}^{\infty} u_n(x,t)$ yield the term-by-term

$$u_0 = u(x,0) + L_t^{-1}(f(x,t)) = xt - \frac{1}{2}xt^2 + \frac{1}{4}x^3t^4 \quad , \qquad (20)$$

$$u_1 = L_t^{-1}u_0 - L_t^{-1}(A_0) + L_t^{-1}L_x u_0 = \frac{1}{2}xt^2 - \frac{1}{6}xt^3 - \frac{1}{4}x^3t^4 - \frac{1}{10}x^3t^5 + \frac{1}{5}x^4t^5 + \cdots . \qquad (21)$$

and so on we can calculate to the other components. It is obvious that the last two term in u_0 and also the first and the third term in u_1 are self-cancelling noise terms. Keeping the remaining non-noise term in u_0 leads to the exact solution of (16) given by

$$u(x,t) = xt. \qquad (22)$$

The solution that satisfies the given problem (16) and the condition (17).

REFERENCES

[1] P.G.Drazin, *Non-Linear Systems*, Cambridge University Press, New York NY 1992.
[2] Y.Kuramoto and S.Koga, Anomalous Period-doubling Bifurcation's Leading to Chemical Turbulence, Phys.Lett.A 92, 1-4, 1982.
[3] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluer Academic Publishers , Boston ,1994.
[4] G. Adomian and R.Rach, Noise Terms in Decomposition Solution, Series, Computers Math.Appl., **24** (11), 61-64, 1992.
[5] G. Adomian, A Review of the Decomposition Method in Applied Mathematics, J.Math. Anal.Appl, **135**, 501-544, 1988.
[6] G. Adomian, *Nonlineer Stochastic Operator Equations*, Academic Press, San Diego, CA.,1986.
[7] G. Adomian, Optical Propagation in Random Media, Applied Math.and Computation, **88**, 127-129, 1997.
[8] G. Adomian,R.C.Rach and R.E.Meyers, Numerical Integration Analytic Continuation and Decomposition, Applied Math.and Computation, **88**, 95-116, 1997.
[9] G. Adomian and R.Rach, Transformation of Series, App. Math.Lett., **4**, 69-71, 1991.
[10] G. Adomian, On KdV Type Equations, Applied Math. and Computation, **88**, 131-135, 1997
[11] G. Adomian, The Kadomtsev – Petviashvili Equation, Applied Math. and Computation , **76**, 76-95, 1996.
[12] G. Adomian, Explicit Solutions of Nonlinear Partial Differential Equations, Applied Math.and Computation, **88**, 117-126, 1997.
[13] A.M.Wazwaz, Equality of Partial Solution in the Decomposition Method for Partial Differential Equations, Intern.J. Computer Math., **65**, 293-308, 1997.
[14] A.M.Wazwaz, On the Solution of the Fourth Order Parabolic Equation by the Decomposition Method, Intern. J.Computer Math., **57**, 213-217, 1995.
[15] A.M.Wazwaz, A New Approach to the Nonlinear Advection Problem:An Application of the Decomposition Method, Applied Math. and Computation, **72**, 175-181, 1995.

[16] A.M.Wazwaz, Necessary Conditions for the Appearance of Noise Terms in Decomposition Solution Series, Applied Math. and Computation, **5**, 265-274, 1997.
[17] Y.Cherruault,G.Saccomandi and B.Some, New Results for Convergence of Adomian's Method Applied to Integral Equations, Math. Comput. Modelling, **16** (2), 85-93, 1992.
[18] Y.Cherruault and G.Adomian, Decomposition method: A New Proof of Convergence, Math .Comput.Modelling, **18** (12), 103-106, 1993.
[19] K.Abbaoui and Y.Cherruault, New Ideas for Proving Convergence of Decomposition Methods, Comput.Math . Applic., **29** (7), 103-108, 1995.
[20] K.Abbaoui and Y.Cherruault, Convergence of Adomian's Method Applied to Differential Equations, Comput. Math.Applic., **28** (5), 103-109, 1994.