ON THE MOTION OF THE FRENET VECTORS AND SPACELIKE RULED SURFACES IN THE MINKOWSKI 3-SPACE

Yusuf YAYLI

Ankara University, Faculty of Sciences

Department of Mathematics, Ankara, Turkey

Abstract- In this paper, we obtained the distribution parameter of a spacelike ruled surface generated by a spacelike straight line in Frenet trihedron moving along a spacelike curve. We show that the spacelike ruled surface is developable if and only if the base curve is a helix (inclined curve). Furthermore, some theorems are given for the special cases which the line is being tangent, the principal normal and binormal of the base curve.

1. INTRODUCTION

A surface in the 3-dimensional Minkowski space $\mathbb{R}_1^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is called a spacelike surface if the induced Lorentzian metric on the surface is a positive [1]. A ruled surface is a surface swept out by a straight line X moving along a curve α . The various positions of the generating line X are called the rulings of the surface. Such a surface, thus, has a parametrization in ruled form as follows,

$$\varphi(t, v) = \alpha(t) + vX(t).$$

We call α to be the base curve, and X to be the director curve. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces [2]. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction

The spacelike ruled surface M is given by the parametrization

$$\begin{aligned} \varphi: I \times \mathbb{R} &\longrightarrow \mathbb{R}_1^3 \\ (t, v) &\longrightarrow \varphi(t, v) = \alpha(t) + vX(t) \end{aligned}$$

In $\mathbb{R}_1^3$ where $\alpha: I \longrightarrow \mathbb{R}_1^3$ is a differentiable spacelike curve parameterized by its arc-length in $\mathbb{R}_1^3$ that is, $\langle \alpha'(t), \alpha'(t) \rangle = 1$ and $X(t)$ is the director vector of the director curve such that X is orthogonal to the tangent vector field T of the base curve α . $\{T, X, \bar{N}\}$ is an orthonormal frame field along α in $\mathbb{R}_1^3$ where $\bar{N}$ is time like.

$$\langle T, T \rangle = \langle X, X \rangle = 1, \langle \bar{N}, \bar{N} \rangle = -1.$$

The curve of striction of a skew spacelike surface is given by

$$\bar{\alpha}(t) = \alpha(t) - \frac{\langle T, D_T X \rangle}{\langle D_T X, D_T X \rangle} X(t) \quad (1)$$

$\bar{\alpha}$ is a spacelike curve [2]. Let P_x be distribution parameter of spacelike ruled surface, then

$$P_x = \frac{\det(T, X, D_T X)}{\langle D_T X, D_T X \rangle} \quad (2)$$

where D is Levi-Civita connection on $\mathbb{R}_1^3$.

Theorem 1.1. A spacelike ruled surface is a developable surface if and only if the distribution parameter of the spacelike ruled surface is zero.

Proof. $D_T X \in S_p \{T, X, \bar{N}\}$ then $D_T X = -aT - c\bar{N}$. Where $a = \langle D_T T, X \rangle, c = \langle D_T X, \bar{N} \rangle$.

Furthermore, if we fix the parameter v , then the curve

$$\varphi_v : I \times \{v\} \rightarrow M$$

the tangent vector field of which is given by $A = (1-av)T - cv\bar{N}$.

The spacelike ruled surface is developable if and only if $c=0$.

$$\det(T, X, D_T X) = \langle T \wedge X, D_T X \rangle = \langle \bar{N}, D_T X \rangle = c$$

thus; $P_x = 0$ if and only if $c=0$.

2. THE FRENET VECTORS FOR SPACELIKE CURVES

If the principal vector field N of a spacelike curve $\alpha(t)$ is timelike and the binormal vector field B is spacelike, then we have the following Frenet formula along $\alpha(t)$ [4]

$$\begin{aligned} \alpha'(t) &= T \\ D_T T &= \frac{dT}{dt} = k_1(t)N \\ D_T N &= \frac{dN}{dt} = k_1(t)T + k_2(t)B \\ D_T B &= \frac{dB}{dt} = k_2(t)N \end{aligned} \quad (3)$$

If the principal vector field N of a spacelike curve $\alpha(t)$ is spacelike and the binormal vector field B is timelike, then we have the following Frenet formula along $\alpha(t)$ [6]

$$D_T T = k_1 N$$

$$D_T N = -k_1 T + k_2 B \quad (4)$$

$$D_T B = k_2 N$$

3. ONE-PARAMETER SPATIAL MOTION IN $\mathbb{R}_1^3$

Let $\alpha: I \longrightarrow \mathbb{R}_1^3$ be a spacelike curve and $\{T, N, B\}$ be Frenet vector. Where T , N and B are the tangent, principal normal and binormal vectors of the curve, respectively. T is spacelike and N or B is timelike vectors.

The two coordinate systems $\{0; T, N, B\}$ and $\{0'; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ are orthogonal coordinate systems in $\mathbb{R}_1^3$ which represent the moving space H and the fixed space H' , respectively. Let us express the displacements (H/H') of H with respect to H' . During the one parameter spatial motion H/H' , each fixed line X of the moving space H , generates, in generally, a spacelike ruled surface in the fixed space H' .

Let X be a unit spacelike vector and X is fixed

$$X \in \text{Sp}\{T, N, B\} \text{ and } X = x_1 T + x_2 N + x_3 B \quad (5)$$

we have

$$\langle X, X \rangle = +1$$

We can obtain the distribution parameter of the timelike ruled surface generated by line X of the moving space H . Let N be a timelike vector. T and B are spacelike. From (5)

$$D_T X = x_1 T' + x_2 N' + x_3 B' \quad (6)$$

substituting (3) into (6)

$$D_T X = x_1 k_2 T + (x_1 k_1 + x_3 k_2) N + x_2 k_2 B$$

From (2) we obtain

$$\begin{aligned} P_x &= + \frac{\det(T, X, D_T X)}{\langle D_T X, D_T X \rangle} \\ &= \frac{k_2 x_2^2 - k_2 x_3^2 + x_1 k_1 x_3}{k_1^2 x_2^2 - (k_1 x_1 + k_2 x_3)^2 + k_1^2 x_2^2}, x_1^2 - x_2^2 + x_3^2 = +1 \\ P_x &= \frac{x_1 k_1 x_3 - k_2 (1 - x_1^2)}{x_2^2 (k_1^2 + k_2^2) - (x_1 k_1 + x_3 k_2)^2} \end{aligned} \quad (7)$$

(6) Let B be timelike vector then T and N are spacelike vectors. Substituting (4) into

$$D_T X = -k_1 x_2 T + (x_1 k_1 + x_3 k_2) N + x_2 k_2 B$$

from (2) we obtain

$$P_x = \frac{x_1 k_1 x_3 + k_2 (-x_3^2 + x_2^2)}{x_2^2 (k_1^2 - k_2^2) + (x_1 k_1 + x_3 k_2)^2}, \quad x_1^2 + x_2^2 - x_3^2 = +1 \quad (8)$$

The ruled surface developable if and only if P_x is zero. From (7) (or (8))

$$P_x = 0 \text{ if and only if } \frac{k_1}{k_2} = \frac{x_2^2 - x_3^2}{x_1 x_3}$$

Hence we state the following theorem

Theorem 3.1: During the one-parameter spatial motion H/H' the spacelike ruled surface in the fixed space H' generated by a fixed line X of the moving space H is developable if and only if $\alpha(t)$ is a helix such that the harmonic curvature h of the base curve $\alpha(t)$ satisfies the equality

$$h = \frac{k_1}{k_2} = \frac{x_2^2 - x_3^2}{x_1 x_3}$$

4. SPECIAL CASES

4.1 The Case $X=T$ (Spacelike)

In this case, $x_1 = 1, x_2 = x_3 = 0$ thus from (7) (or (8))

$$P_x = 0$$

We can give following theorem

Theorem 4.1. During the one-parameter spatial motion H/H' , the spacelike ruled surface in the fixed space H' generated by the tangent line T of the curve $\alpha(t)$ in the moving space H is developable.

4.2. The Case $X=N$ (Spacelike):

In this case, $x_1 = x_3 = 0$ and $x_2 = 1$

Thus, from (8)

$$P_N = \frac{+k_2}{k_1^2 - k_2^2} \quad (9)$$

4.3 The Case $X=B$ (Spacelike)

In this case, $x_1 = x_2 = 0$ and $x_3 = 1$. Thus from (7)

$$P_B = \frac{k_2}{k_2^2} = \frac{1}{k_2} \quad (10)$$

By (9) and (10) we can give the relation between P_B and P_N

$$\left(\frac{k_1}{k_2}\right)^2 = \left(1 + \frac{P_B}{P_N}\right) \quad (11)$$

Hence the following theorem is hold.

Theorem 4.2. During the one-parameter spatial motion H/H' the curve of base $\alpha: I \rightarrow E^3$ is helix (inclined curve) if and only if P_N/P_B is constant, where P_N and P_B are the distribution parameters of the surfaces generated by the principal normal and binormal.

4.4. The Case, X is in the normal plane.

In this case x_1 is zero. The timelike surface is developable surface since from (7) (or (8))

$$k_2(-x_3^2 + x_2^2) = 0 \quad (12)$$

$X = x_2 N + x_3 B$ and if N is timelike then $-x_2^2 + x_3^2 = +1$ if B is timelike then $x_2^2 - x_3^2 = +1$. Hence, from (12) $k_2 = 0$. Thus $\alpha(t)$ is a planar curve. Hence the following theorem is hold.

Theorem 4.3. During the one-parameter spatial motion H/H' , the spacelike ruled surface in the fixed space H' generated by a fixed line X in the normal plane of H is developable if and only if $\alpha(t)$ is a planar curve in osculating plane.

4.5. The Case, X is in the Osculating plane.

In this case x_3 is zero. Thus the spacelike ruled surface is developable since from (7) (or (8))

$$k_2 x_2^2 = 0 \quad (13)$$

if $x_2 = 0$ then $X = T$. This is the case (4.1). If $k_2 = 0$ then $\alpha(t)$ is a planar curve. Therefore Theorem 4.3 can be restated.

4.6. The Case X is in the rectifying plane.

In this case x_2 is zero. From (7) (or (8)) for distribution parameter we can write that

$$P_X = \mp \frac{x_3}{k_1 x_1 + k_2 x_3} \quad (14)$$

$P_X = 0$ if and only if x_3 is zero. Then $X = T$. This is the case (4.1)

Furthermore, the following theorems can be stated.

Theorem 4.4 During the one-parameter spatial motion H/H' the distribution parameters the spacelike ruled surfaces in the fixed space H' generated by a fixed line X in the rectifying plane of H are the same if and only if the base curves are Bertrand curves.

Proof. If $\alpha(s)$ has a Bertrand curve then

$$x_1 k_1 + x_3 k_2 = \text{constant}$$

thus from (14) P_X is constant. If P_X is constant then from (14), $x_1 k_1 + x_3 k_2$ is constant. Hence $\alpha(s)$ is Bertrand curve.

4.7. The Case, the curve $\alpha(t)$ of base the same the curve $\bar{\alpha}(t)$ of striction.

In this case, from (1), $\langle T, D_T X \rangle = 0$

$$D_T X = \mp k_1 x_2 T + (x_1 k_1 + x_3 k_2) N + x_2 k_2 B$$

we have $k_1 x_2 = 0$, $k_1 \neq 0$ x_2 is zero.

Hence the following theorem is hold.

Theorem 4.5. If the curve $\alpha(t)$ of base is the same as the curve $\bar{\alpha}(t)$ of striction, then the director curves of the spacelike ruled surfaces are in the rectifying plane of $\bar{\alpha}(t)$.

If $\alpha(t) = \bar{\alpha}(t)$ then $X(t)$ is rectifying plane. Thus theorem (4.4) can be repeated for the striction curve $\bar{\alpha}(t)$.

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