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SPACES OF DOUBLE SEQUENCES OF FUZZY NUMBERS

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Abstract : *In this paper, we introduce some new double sequence spaces of fuzzy numbers and show that they are complete metric spaces.*

1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [6] and subsequently several authors including Zadeh have discussed various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [1] where it was shown that every convergent sequence is bounded. In [2] Nanda have studied the space of bounded and convergent sequences of fuzzy numbers and shown that they are complete metric spaces. Recently, Savas and Nuray [5] have introduced statistical convergence of sequence of fuzzy numbers. More recently, Savas has discussed the double convergent sequences of fuzzy numbers.

2. PRELIMINARIES

Let D denote the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line R where denote the end points of A. For $A, B \in D$ define,

$$A \leq B \text{ iff } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B},$$
$$d(A,B) = \max\left(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\right).$$

It is not hard to see that d defines a metric on D and d (A, B) is called the distance between the intervals A and B. Also it is easy to see that $\leq$ defined above is a partial order relation in D.

A fuzzy number is a subset of the real line R which is bounded, convex and normal. Let L(R) denote the set of all fuzzy numbers which are upper semi continuous and have compact support. In other words, *if* $X \in L(R)$ then for any $\alpha \in [0,1]$, X^α is compact set in R where,

$$X^\alpha = \begin{cases} t: X(t) \geq \alpha, & \text{if } \alpha \in (0,1] \\ t: X(t) > 0 & \text{if } \alpha = 0 \end{cases}$$

Define a map $\bar{d}: L(R) \, x \, L(R) \to R$ by the rule

$$\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

It is straightforward to see that $\bar{d}$ is a metric in L (R).

For $X, Y \in L(R)$ define

$$X \le Y \text{ iff } X^\alpha \le Y^\alpha \text{ for any } \alpha \in [0,1].$$

A subset E of L(R) is said to be bounded above if there exists a fuzzy number C, called an upper bounded of E, such that $X \le C$ for every $X \in E$. C is called the least upper bounded (l. u. b. sup) of E, if C is an upper bounded and is the smallest of all upper bounds. A lower bound and the greatest lower bound (g. l. b. or inf.) are defined similarly. E is said to be bounded if it is both bounded above and bounded below.

It is known (see, [1], [3]) that L(R) is a complete metric space with the metric $\bar{d}$.

We now quote the following definitions, which will be needed in the sequel $(see, [4])$.

Definition 1 : A double sequence $X = (X_{nm})$ of fuzzy numbers is function X from NxN (N is the set of all positive integers) into L(R). The fuzzy number X_{nm} denotes the value of the function at a point $(n,m) \in NxN$ and is called the (n,m)- term of the double sequence.

Definition 2 : A double sequence of $X = (X_{nm})$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_{n,m} X_{nm} = X_0$, if for every $\varepsilon \rangle 0$ there exists a positive integer n_0 such that $\bar{d}(X_{nm}, X_0) \langle \varepsilon$ for $n, m \ge n_0$

We define $(see, [1])$

c = The set of all convergent sequences of fuzzy numbers.

m = The set of all bounded sequences of fuzzy numbers

We now have the following sets :

$(\Gamma : m) =$ The set of all double sequences of fuzzy numbers such that

$\left[\bar{d}\left(X_{nm,}\, 0\right)\right]^{1/m} \leq M$, independent of n, m

$(\Gamma\colon c) =$ The set of all double sequences of fuzzy numbers such that

$\left[\bar{d}\left(X_{nm,}\, 0\right)\right]^{1/m} \leq M$, independent of n, m; and the columns of $X = \left(X_{nm}\right)$ converge.

$(c\colon \Lambda) =$ The set of all double sequences of fuzzy numbers such that $\left(\sigma_n\right)$ is bounded, where

$$\sigma_n = \sum_{m=1}^{\infty}\left[\bar{d}\left(X_{nm},\, 0\right)\right]^{1/m} \qquad (n = 1, 2 \ldots)$$

$(c\colon \Gamma) =$ The set of all double sequences of fuzzy numbers such that $\left(\sigma_n\right)$ is a null sequence.

3. MAIN RESULTS

We have the following results :

Theorem 1 : The space $(\Gamma\colon m)$ is a complete metric space with the metric ρ defined by

$$\rho(X, Y) = \sup\left\{\left[\bar{d}\left(X_{nm,}\, Y_{nm}\right)\right]^{1/m}, n,\, m = 1, 2 \ldots\right\}$$

where $X = \left(X_{nm}\right)$ and $Y = \left(Y_{nm}\right)$ are convergent sequences of fuzzy numbers.

Proof . Let $\left(X^i : i = 1, 2 \ldots\right)$ with $X^{(i)} = \left(X^{(i)}{}_{nm}\right)$ be a Cauchy sequence in $(\Gamma\colon m)$ Then given any $\varepsilon > 0$ there is positive integer i_0 such that

$$\rho\left(X^{(i)}, Y^{(j)}\right) < \varepsilon, \left(i \geq i_0,\ j \geq i_0\right)$$

so that

$$\bar{d}\left(X^i{}_{nm},\, Y^j{}_{nm}\right) < \varepsilon^m \qquad \left(i \geq i_0,\ j \geq i_0\right) \qquad \ldots (1)$$

Hence, for each fixed n and fixed m we have

$$X^{(i)}{}_{nm} \to X_{nm} \qquad (i \to \infty).$$

Let $X = (X_{nm})$ $\quad (n, m = 1, 2 \ldots)$ But then, by (1) we infer that

$$\rho\left(X^{(i)}, X\right) \to 0 \qquad \text{as } i \to \infty$$

We show that double sequence $X = (X_{nm})$ belongs to $(\Gamma : m)$. Letting $j \to \infty$ in (1), we get,

$$\bar{d}\left(X_{nm}, X^{(i)}{}_{nm}\right) \le \varepsilon^{m}, \qquad (i \ge i_0)$$

and so,

$$\left[\bar{d}\left(X_{nm}, 0\right)\right]^{1/m} \le \left[\bar{d}\left(X^{i_0}{}_{nm}, 0\right)\right]^{1/m} + \varepsilon.$$

But the double sequence $X^{(i_0)}$ is in $(\Gamma : m)$, so that $\left[\bar{d}\left(X^{i_0}_{nm}, 0\right)\right]^{1/m} \le M$ independently of n, m. Hence we have

$$\left[\bar{d}\left(X_{nm}, 0\right)\right]^{1/m} \le M + \varepsilon = H,$$

which is independent of n, m.

This completes the proof.

Theorem 2 : The space $(\Gamma : c)$ is a closed subset of the complete the fuzzy metric space $(\Gamma : m)$; Consequently, $(\Gamma : c)$ is also a complete fuzzy metric space.

Proof. It is known (see [1], Theorem 3 and 4) that c $\subset$ m. Therefore $(\Gamma : c)$ is a subset of $(\Gamma : m)$. Left Cl$(\Gamma : c)$ denote the closure of $(\Gamma : c)$ in fuzzy metric topology given by the metric ρ for $(\Gamma : m)$. If X is in Cl$(\Gamma : c)$ then there exists a sequence $(X^{(i)})$ in $(\Gamma : c)$ such that

$$\rho(X^{(i)}, X) \to 0 \text{ as } \quad i \to \infty$$

By theorem 1, it follows that

$$\left[\bar{d}\left(X_{nm}, 0\right)\right]^{1/m} \le H$$

which is independent of n, m.

Also, the column limits of $X^{(i_0)}$ exist. Hence, given any $\varepsilon > 0$ there is a positive integer n_0 such that

$$\bar{d}(X_{nm}^{i_0}, X_{km}^{i_0}) \leq \varepsilon^m \qquad (k \geq n_0,\ n \geq n_0) \qquad \ldots (2)$$

for each fixed m. Since $\rho(X^{(i)}, X) \to 0$ as $i \to \infty$, there is a positive integer i_0 such that

$$\bar{d}(X_{nm}, X_{nm}^{(i_0)}) \leq \varepsilon^m \qquad \ldots (3)$$

for each fixed n and m. We now invoke the inequality

$$\bar{d}\left(X_{nm}, X_{km}\right) \leq \bar{d}\left(X_{nm}, X_{nm}^{(i_0)}\right) + \bar{d}\left(X_{nm}^{(i_0)}, X_{km}^{(i_0)}\right) + \bar{d}\left(X_{km}^{(i_0)}, X_{km}\right)$$

and use (2) and (3) to conclude that the column limits of the double sequence $X = \left(X_{nm}\right)$ exist. Thus, the double sequence $X = \left(X_{nm}\right)$ belongs to $(\Gamma{:}c)$. Since X is arbitrary in Cl $(\Gamma{:}c)$, it follows that Cl $(\Gamma{:}c)$ is contained in $(\Gamma{:}c)$. Therefore, $(\Gamma{:}c)$ is closed in the complete metric fuzzy space $(\Gamma{:}m)$

Finally we conclude this paper by stating Theorems 3 and 4. We omit their proofs since they are analogous to theorems 1 and 2 respectively.

Theorem 3 : The space $(c{:}\Lambda)$ is a complete metric space with the metric D for $(c{:}\Lambda)$ given by,

$$D(X,Y) = \sup\left(\sum_{m=1}^{\infty}\left[\bar{d}\left(X_{nm}, Y_{nm}\right)\right]^{1/m}\right), \qquad n = 1, 2, \ldots$$

where $X = \left(X_{nm}\right)$ and $Y = \left(Y_{nm}\right)$ are convergent sequences of fuzzy numbers which are in $(c{:}\Lambda)$.

Theorem 4 : The space $(c{:}\Gamma)$ is a closed subset of $(c{:}\Lambda)$. Consequently, $(c{:}\Gamma)$ is a complete fuzzy metric space.

REFERENCES

1. M. Matloka, Sequences of Fuzzy Numbers, BUSEFAL 28, 28-37, 1986

2. S. Nanda, On Sequences of Fuzzy Numbers, Fuzzy Sets and systems. 33, 123-126, 1989

3. M. L. Puri and D. A. Ralescu, Differentials for fuzzy functions, J. Math. Anal. Appl. 91 1983.

4. E. Savas , A note on double sequences of fuzzy Numbers, T. Journal of Maths, 20. 1996.

5. E. Savas and F. Nuray, Statistical convergence of sequence of fuzzy numbers, Math. Slovaca, 44, 1994.

6. L. Zadeh, Fuzzy Sets, Inform and Control 8, 338- 353, 1965.