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FREE 2-CROSSED COMPLEXES OF SIMPLICIAL ALGEBRAS

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Abstract–Using free simplicial algebras, it is shown that a free or totally free 2-crossed module can be constructed on suitable construction data in [2]. In this paper 2-crsossed complexes and freeness conditions for 2-crossed complexes are introduced and similar freeness results for these are discussed.

1. INTRODUCTION

Andre [19] uses simplicial methods to investigate homological properties of (commutative) algebras. Other techniques that can give linked results include those using the Koszul complex. Any simplicial algebra yields a crossed module derived from the Moore complex [6] and any finitely generated free crossed complex $C \to R$ of commutative algebras was shown in [14] to have $C \cong R^n/d(\wedge^2 R^n)$, i.e. the 2^{nd} Koszul complex term modulo the 2-boundaries.

Higher dimensional analogues of crossed modules of commutative algebras have been defined: 2-crossed modules by Grandjean and Vale [15] and crossed n-cubes of algebras by Ellis [13]. It would not be reasonable to expect a strong link between free 2-crossed modules or free crossed squares and Koszul-like constructions since the former record quadratic information, which is less evidently there in the Koszul complex. Nevertheless it seems to be useful to try to define what freeness of such "gadgets" should mean - for instance, to ask "free on what?". Arvasi and Porter [2] solution goes via free simplicial algebras as used by Andre.

It is a logical step to introduce an intermediate concept, namely 2-crossed complexes. These use a 2-crossed module plus a chain complex of modules and we will show how to derive such a thing from a simplicial algebras. We have therefore included a purely algebraic treatment of 2-crossed complexes giving explicit formulae for the structure involved in the passage from simplicial algebras to 2-crossed complexes and an explicit direct proof of a freeness result due to Baues in [12].

There is an alternative way of storing the information from a 2-crossed complex, namely as a "squared" complex as introduced by [8].

2. PRELIMINARIES

All algebras will be commutative and will be over the same fixed but unspecified ground ring.

2.1 Truncated simplicial algebras

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$ a subcategory $\Delta^{op}_{\leq k}$ determined by the object [j] of Δ with $j \leq k$. A simplicial algebra is a functor from the opposite category $\Delta^{op}_{\leq k}$ to Alg; a k-truncated simplicial algebra is a functor from $\Delta^{op}_{\leq k} \to$ Alg. We denote the category of k-truncated simplicial algebras by Tr_kSimpAlg. Recall

$$\text{cosk}_k : \text{Tr}_k\text{SimpAlg} \to \text{SimpAlg}$$

called the *k-coskeleton functor*, and a left adjoint

$$\text{sk}_k : \text{Tr}_k\text{SimpAlg} \to \text{SimpAlg},$$

called the *k-skeleton functor*.

We recall that ideal chain complex from [2]. By an *ideal chain complex* of algebras, (X, d) we mean one in which each Imd_{i+1} is an ideal of X_i. Given any ideal chain complex (X,d) of algebras and an integer n the truncation, $t_{n]}X$, of X at level n will be defined by

$$(t_{n]}X)_i = \begin{cases} X_i & \text{if } i < n, \\ X_i & \text{if } i = n, \\ 0 & \text{if } i > n. \end{cases}$$

The differential d of $t_{n]}X$ is that of X for i< n, d_n is induced from the n^{th} differential of X and all others are zero.

Recall that given a simplicial algebra A, the *Moore complex* (NA, ∂) of A is the chain complex defined by

$$(\text{NA})_n = \bigcap_{i=0}^{n-1} Kerd_i^n,$$

with $\partial_n : NA_n \to NA_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(A)$ of A is the n^{th} homology of the Moore complex of A, i.e.,

$$\pi_n(A) \cong H_n(NA, \partial) = \bigcap_{i=0}^{n} Kerd_i^n \,/\, d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} Kerd_i^n\right).$$

We say that the Moore complex NA of a simplicial algebra is of *length* k if NA_n=0 for all $n \geq k+1$ so that a Moore complex is of length k also of length r for $r \geq k$.

Lemma 2.1 *Let $tr_k(A)$ be a k-truncated simplicial algebra, and $cosk_k(tr_k(A))$, the algebra-theoretic k-coskeleton of $tr_k(A)$ (i.e. calculated within Alg). Then there is a natural epimorphism from $N(cosk_k(tr_k(A)))$ to $t_{k]}NA$ with acyclic kernel. Thus $cosk_k(tr_k(A))$ and $t_{k]}(NA)$ have the same weak homotopy type.*

Proof: Following Conduche [16], the Moore complex of $\text{cosk}_k(\text{tr}_k(\text{A}))$ is given by:

$$\text{N}(\text{cosk}_k(\text{tr}_k(\text{A})))_1 = 0 \quad \text{if} \quad \text{l>k+1},$$

$$\text{N}(\text{cosk}_k(\text{tr}_k(\text{A})))_{k+1} = \text{Ker}\,(\partial_k : NA_k \to NA_{k+1}),$$

$$\text{N}(\text{cosk}_k(\text{tr}_k(\text{A})))_\ell = \text{NA}_\ell \text{ if } \ell \leq \text{k}.$$

The natural epimorphism gives on Moore complexes

$$\begin{array}{ccccccccccc} N(\cos k_k(tr_k(A))) : & 0 & \longrightarrow & \partial NA_{k+1} & \longrightarrow & NA_k & \xrightarrow{\partial_k} & NA_{k-1} & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ t_{k]}(NA) & : 0 & \longrightarrow & 0 & \longrightarrow & NA_k / \partial NA_{k+1} & \longrightarrow & NA_{k-1} & \longrightarrow & \cdots \end{array}$$

and it is immediate that the kernel is acyclic as required. □

3 CROSSED MODULES OF ALGEBRAS

Crossed modules of groups were initially defined by Whitehead as models for (homotopy) 2-types in [20]. Conduche, [16], in 1984 described the notion of 2-crossed module as a model for 3-types. Both crossed modules and 2-crossed modules have been adapted for use in the context of commutative algebras (cf. [6], [14] and [15]). Throughout this paper we denote an action of $r \in R$ on $c \in C$ by $r \cdot c$.

A *crossed module* is an algebra morphism $\partial : C \to R$ with an action of R on C satisfying $\partial(r \cdot c) = r\partial c$ and $\partial(c) \cdot c' = cc'$ for all $c, c' \in C, r \in R$.

In this section, 2-crossed module is described in [15] and a free 2-crossed module of algebras will be introduced by using the second dimension Peiffer elements of [2].

We recall from Grandjean and Vale [15] the definition of 2-crossed module:

A 2-crossed module of k-algebras consists of a complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with ∂_2, ∂_1 morphisms of C_0-algebras, where the algebra C_0 acts on itself by multiplication, such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module in which C_1 acts on C_2 via C_0, (we require thus that for all $x \in C_2$, $y \in C_1$ and $z \in C_0$ that $(xy)z = x(yz)$), further, there is a C_0-bilinear function with

$$\{\otimes\} : C_1 \otimes C_1 \longrightarrow C_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{aligned}
&PL1: && \partial_2\{y_0 \otimes y_1\} = y_0 y_1 - y_0 \cdot \partial_1(y_1),\\
&PL2: && \{\partial_2(x_1) \otimes \partial_2(x_2)\} = x_1 x_2,\\
&PL3: && \{y_0 \otimes y_1 y_2\} = \{y_0 y_1 \otimes y_1\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\},\\
&PL4 && a)\ \{\partial_2(x_1) \otimes y\} = y \cdot x - \partial_1(y) \cdot x,\\
& && b)\ \{y \otimes \partial_2(x_1)\} = y \cdot x,\\
&PL5 && \{y_0 \otimes y_1\} \cdot z = \{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\},
\end{aligned}$$

for all $x, x_1, x_2 \in C_2$ $y, y_0, y_1, y_2 \in C_1$ *and* $z \in C_0$. One has

$$\begin{array}{ccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow & & \downarrow & & \downarrow \\
C_2' & \xrightarrow{\partial_2'} & C_1' & \xrightarrow{\partial_1'} & C_0',
\end{array}$$

such that $f_0\partial_1 = \partial_1' f_1$, $f_1\partial_2 = \partial_2' f_2$ and such that

$$f_1(c_0 \cdot c_1) = f_0(c_0) \cdot f(c_1), \quad f_2(c_0 \cdot c_2) = f_0(c_0) \cdot f_2(c_2),$$

and

$$\{\otimes\} f_1 \otimes f_1 = f_2 \{\otimes\},$$

for all $c_2 \in C_2$, $c_1 \in C_1$, $c_0 \in C_0$, where $f_2 : C_2 \to C_2'$, $f_1 : C_1 \to C_1'$, $f_0 : C_0 \to C_0'$,

We thus define the category of 2-crossed module denoting it by X_2Mod.

We denote the category of simplicial algebras with Moore complexes of length n by $\mathrm{SimpAlg}_{\leq n}$ in the following. First we recall from [2] the following results.

Proposition 3.1 *Let A be a simplicial commutative algebra with the Moore complex NA. Then the complex of algebras*

$$NA_2 / \partial_3(NA_3 \cap D_3) \xrightarrow{\overline{\partial_2}} NA_1 \xrightarrow{\partial_1} NA_0$$

is a 2-crossed module of algebras, where the Peiffer lifting map is defined as follows:

$$\{\otimes\}: NA_1 \otimes NA_1 \to NA_2 / \partial_3(NA_3 \cap D_3)$$

$$(y_0 \otimes y_1) \mapsto \overline{s_1 y_0 (s_1 y_1 - s_0 y_1)}.$$

Here the right hand side denotes a coset in $NA_2 / \partial_3(NA_3 \cap D_3)$ *represented by the corresponding element in* NA_2. □

3.1 Free 2-Crossed Modules

The definition of a free 2-crossed module is similar in some way to the corresponding definition of a free crossed module. However, the construction of a free 2-crossed module is naturally a little more complicated.

It will be helpful to have the notion of a *pre-crossed module*: this is just a homomorphism $\partial : C \to R$, with an action satisfying $\partial(r \cdot c) = r\partial c$ for c∈C, r∈R.

Let (C,R,∂) be a pre-crossed module, let Y be a set and let $\nu : Y \to C_2$ be a function, then (C,R,∂) is said to be a *free pre-crossed R-module with basis* ν or, alternatively, on the function $\partial\nu : Y \to R$ if for any pre-crossed R-module (C', R, ∂') and function $\nu' : Y \to C'$ with $\partial'\nu' = \partial\nu$ there is a unique morphism

$$\phi : (C, R, \partial) \to (C', R, \partial')$$

such that $\phi\nu = \nu'$

The pre-crossed module (C,R,∂) is *totally free*, if R is a free algebra.

We recall from [2] the following construction.

Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a 2-crossed module, let Y be a set and let $\nu : Y \to C_2$, then $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is said to be a *free 2-crossed module* with basis ν or, alternatively, on the function $\partial\nu : Y \to C_1$ if for any 2-crossed module $\{C_2', C_1, C_0, \partial_2', \partial_1\}$ and function $\nu' : Y \to C_2'$ with $\partial'\nu' = \partial\nu$ there is a unique morphism $\Phi : C_2 \to C_2'$ such that $\partial_2'\Phi = \partial_2$.

Remark:

"Freeness" in any setting corresponds to a left adjoint, so what are the categories involved here?

Let 2CM/PCM be the category whose object consists of a precrossed module (C, D,∂) and (Y,v) where v:Y→ C is simply a function to the underlying set of the algebra C such that ∂ v=0. Morphisms of such object consist of a pair ϕ, ϕ', where $\phi' : Y \to Y'$, is a function with $\nu'\phi' = \nu\phi$. Omittion of the algebra structure of the top algebra, C_2, of a 2-crossed module provides one with a functor from 2-crossed module to this category. The object of 2CM/PCM are thought of a s 2-(dimensional) construction data on given precrossed module.

Let (R, I) be ideal-pair case will be described only. Taking the kernel of this morphism $K \to R$ we pick a set of generators of K, $\nu_2 : Y \to K$ as a precrossed module and we have an object of 2CM/PCM. Thus to analyse an ideal homological pair, one natural method to use is to compare it via a free 2-crossed module, with a free precrossed module. This process is based some extent on the intuition of related CW-complex constructions in topology. Andre's use [19]

of simplicial resolutions provides the bridge between the two settings. The sort of construction data, one obtains from a simplicial resolution, corresponds to a special type of 2-crossed module:

A free 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is *totally free* if $\partial_1 : C_1 \to C_0$ is a totally free pre-crossed module.

The following notation and terminology is due to [2]. We give an explicit description of the construction of a totally free 2-crossed module. For this, we need to recall the 2-skeleton of the free simplicial algebra, which is

$$A^{(2)} : \ldots (R[s_0(X), s_1(X))[Y] \quad \underset{\leftarrow s_0}{\overset{d_0, d_1, d_2}{\rightarrow}} \quad R[X] \quad \underset{\leftarrow s_0}{\overset{d_0, d_1}{\rightarrow}} \quad R,$$

with the simplicial structure defined as in section 2.2 of [2]. Analysis of this 2-*dimensional construction data*, (cf. [2]), shows that it consists of some 1-dimensional data: namely the function $\varphi : X \to R$ that is used to induce $d_1 : R[X] \to R$, together with strictly 2-dimensional data consisting of the function $\psi : Y \to R^+[X]$ where $R^+[X]$ is the positively graded part of R[X] and which is used to induce d_2 from $R[s_0(X), s_1(X)][Y]$ to R[X]. We will denote this 2-dimensional construction data by $(Y, X; \psi, \varphi, , R)$.

Theorem 3.1 (See [2].) *A totally free 2-crossed module $\{L, A, R, \psi', \partial_1\}$ exists on the 2-dimensional construction data $(Y, X; \psi, \varphi, , R)$.*

Proof: See [2]. □

4. 2-CROSSED COMPLEXES AND SIMPLICIAL ALGEBRAS,

Any simplicial, $A_\bullet$, yields a normal chain complex of algebras, namely its Moore complex, (NA, ∂). Carrasco and Cegarra, [10], examined the extra structure inherent in a Moore complex that allows the reconstruction of $A_\bullet$ from NA. They gave the term hyper-crossed complex to the resulting structure. Crossed complexes themselves, (cf. Brown and Higgins, [17]) correspond to a class of hypercrossed complexes in which nearly all of the extra structure is trivial, so the only non-abelian algebras occur in dimensions 0 and 1 and are linked by a crossed module structure. The other terms are all modules over $NA_0/\partial NA_1$. Thus a crossed complex looks like a crossed module with a tail that is a chain complex of $\pi_0(A)$-modules. If the original simplicial algebra is the algebra of a reduced simplicial set, K, then it is well known that the corresponding complex has the "chains on the universal cover" in dimensions greater than 1 and a free crossed module in the bottom of dimension two. (This is implicit in much of the work of Baues on crossed (chain) complexes, [7, 11, 12], and was explicitly proved by Ehlers and Porter, [5].)

Crossed modules model algebraic 1-types (and hence topological 2-types) and we have recalled from Conduche's work, [16], that 2-crossed modules model algebraic 2-types (and hence topological 3-types). It is thus natural to give these latter models also a "tail" and to consider "2-crossed complexes". Such gadgets are related to the quadratic complexes of Baues, [11, 12], in an obvious way.

Definition:

A 2-crossed complex of algebras is a sequence of algebras

$$C: \ \ldots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

(i) C_0 acts on C_n, $n \geq 1$, the action of ∂C_1 being trivial on C_n for $n \geq 3$;

(ii) each ∂_n is a C_0-algebra homomorphism and $\partial_i \partial_{i+1} = 0$ for all $n \geq 1$;

and

(iii) $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a 2-crossed module.

Note that for any 2-crossed module,

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

the which $K = \mathrm{Ker}\,\partial_2$ is abelian, since $L \xrightarrow{\partial_2} M$ is a crossed module, but the much more is true. The action of M on L via N restricts to one on K, but by axiom PL4, the action is trivial. This implies that the action of N itself on K, factors through one of $N/\partial_1 M$. Thus in any 2-crossed complex,

$$\ldots \longrightarrow C_4 \longrightarrow C_3 \longrightarrow Ker\partial_2$$

is a chain complex of $C_0/\partial_1 C_1$-modules and a 2-crossed complex is just a 2-crossed module with a chain complex as a tail added on.

As usual there is an algebra version with C_0 an algebra and each C_n a family of algebras indexed by the objects of C_0.

Given a simplicial algebra or algebra, A, define

$$C_n = \begin{cases} NA_n & for \quad n = 0,1 \\ NA_2 / d_3(NA_3 \cap D_3) & for \quad n = 2 \\ NA_n /(NA_n \cap D_n) + d_{n+1}(NA_{n+1} \cap D_{n+1}) & for \quad n \geq 3 \end{cases}$$

with ∂_n induced by the differential of NA.

Proposition 4.1 *With the above structure, $(C_n\ \partial_n)$ is a 2-crossed complex.*

Proof: The only thing remaining is to check that $\partial_2 \partial_3$ is trivial. This composite

$$\frac{NA_3}{NA_3 \cap D_3 + d_4(NA_4 \cap D_4)} \to \frac{NA_2}{d_3(NA_3 \cap D_3)} \to NA_1$$

is induced from $\partial_2 \partial_3$ in NA_1, and so is trivial. Thus $\mathrm{Im}\,\partial_3 \subseteq Ker\partial_2$ as required. □

The notion of morphism for 2-crossed complexes should be clear. This will give us a category, X_2Comp of 2-crossed complexes and morphisms between them. We will similarly denote by X_1Comp the category of crossed complexes together with their morphisms. It is easily seen that the construction C is functorial from the category of simplicial algebras to that of 2-crossed complexes.

A 2-crossed complex C will be said to be *free* if for $n \geq 3$, the $C_0 / \partial C_1$-modules, C_n are free and the 2-crossed module at the base is a free 2-crossed module. It will be *totally free* if in addition the base 2-crossed module is totally free.

Before turning to a detailed examination of freeness in 2-crossed complexes, we will consider the relation between crossed complexes and 2-crossed complexes.

Let

$$C: C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a 2-truncated crossed complex, then (C_1, C_0, ∂_1) is a crossed module, C_2 is a module over C_0 on which $\partial_1 C_1$ acts trivially, and $\partial_1 \partial_2 = 0$.

Lemma 4.2 *The 2-truncated crossed complex yields a 2-crossed module by taking* $\{c \otimes c'\} = 0 \in C_2$ *for all* $c, c' \in C_1$ *and in which the actions of* C_1 *on* C_2 *are trivial.*

Proof: As (C_1, C_0, ∂_1) is a crossed module, if $c, c' \in C_1$, then

$$(\partial(c) \cdot c') = c \cdot c' = 0$$

and so 2CM1 is trivially satisfied.

If $c, c' \in C_2$, then as C_2 is, $\{c \otimes c'\} = 0$ and PL2 is also satisfied.

The vanishing of both sides in the various parts of PL3 makes this equally easy to check whilst triviality of the actions of C_1 on C_2 imply PL4. Finally PL5 is again trivially true. □

We will think of X_1Comp as a full subcategory of X_2Comp via this embedding.

Teorem 4.3 *The full subcategory of crossed complexes is a reflexive subcategory of* X_2Comp.

Proof: We have to show that the functor, E, thought of now as an inclusion, has a left adjoint, L. We first look at a slightly simpler situation.

Suppose that D is a 2-truncated crossed complex as above, and

$$C: L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

with morphisms, ∂_1, ∂_2 and Peiffer lifting, $\{\otimes\}$, is a 2-crossed module. If we are given a morphism, $\underline{f} = (f_2, f_1, f_0)$ of 2-crossed modules, $\underline{f}: C \to E(D)$, then if $m_1, m_2 \in M$, $f_2\{m_1 \otimes m_2\} = 0$ since within E(D), the Peiffer lifting is trivial. This in turn implies that $f_1 \langle m_1 \otimes m_2 \rangle = 0$, where $\langle m_1, m_2 \rangle = \partial(m_1)m_2 - m_1 m_2$ is the Peiffer commutator of m_1 and m_2. Thus any morphism $\underline{f}$ from C to E(D) has a kernel that contains the subgroup, $\{C_1 \otimes C_1\}$ generated by the Peiffer lifts in dimension 2, and the Peiffer subgroup, P_1 of the precrossed module, $\partial_1 : C_1 \to C_0$ in dimension 1.

We form L(C) as follows:

$$L(C)_0 = C_0 = C_1 / P_1 = C_2 / \{C_1 \otimes C_1\},$$

with the induced morphisms and actions. The previous discussion makes it clear that L(C) is a 2-truncated crossed complex, and L is clearly functorial. Of course $\underline{f}: C \to E(D)$ yields $L(\underline{f}) : L(C) \to LE(D) \cong D$, so L is the required reflection, at least on this subcategory of truncated objects.

Extending L to all crossed complexes is then simple as we take $L(C)_n = C_n$ if $n \geq 3$ $L(\partial)_n = \partial_n$ if n> 3 and

$$L(\partial)_3 : C_3 \to C_2 / \{C_1 \otimes C_1\} = L(C)_2$$

given by the composite of ∂_3 and the quotient from C_2 to $L(C)_2$. The details are easy so will be omitted. Thus the functor L preserves freeness. □

Proposition 4.4: *If C is a (totally) free 2-crossed complex, then L(C) is a (totally) free crossed complex.*

Proof: In the above the dimension is 2, L does nothing and as

$$C_0 / \partial C_1 \cong L(C)_0 / \partial L(C)_1$$

the freeness of the modules $L(C)_n$, $n \geq 3$ is not in doubt. In the base 2-crossed module, we have merely to check that $L(C)_2$ is a free $L(C)_0 / \partial L(C)_1$-module, as the behaviour of L on (C_1, C_0, ∂) is just that of the quotienting operation that turns a pre-crossed module into a crossed module and this preserves freeness.

Let therefore that $C: C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a free 2-crossed module with basis $\theta : Y_1 \to C_2$. Let also given a module M over $G = C_0/\partial C_1$ and a function $\phi : Y_1 \to M$. We need to show that ϕ extends over $L(C)_2$. To do this we construct a 2-crossed complex as follows: The base is the precrossed module (C_1, C_0, ∂), but this is completed by putting $\mathrm{Ker}\,\partial_1$ in dimension 2 with the inclusion as ∂_2. To this we add M with its G-action to form

$$D : Ker\partial_1 \times M \to C_1 \to C_0.$$

The boundary from level 2 to level 1 is trivial on M. The Peiffer lifting is just the Peiffer map from $C_1 \times C_1$ to $\mathrm{Ker}\,\partial_1$ and the axioms are easy to check.

Now define $\overline{\phi}$ from our given free 2-crossed module to this one, D, by defining $\overline{\phi}(y) = (\partial\theta y, \phi y)$ for $y \in Y_2$. Compose $\overline{\phi}$ with the obvious projection from D to the crossed complex

$$M \xrightarrow{\partial_2} 1 \xrightarrow{\partial_1} G$$

where as before, $G = C_0/\partial C_1$. The composed map factors through L(C) giving a morphism $L(C)_2 \to M$ extending $\overline{\phi}$. This is the unique extension of $\overline{\phi}$ since at each stage uniqueness was a consequence of the conditions. □

The functor $C^{(2)}$ has a right adjoint, just as $C^{(1)}$ does. Given a 2-crossed complex, C, one first construct the simplicial algebra corresponding to the 2-crossed module at the base, using Conduche's theorem. We also form the simplicial algebra from the chain complex given by all C_i, $i \geq 2$. The fact that C_2 may be non-abelian does not cause any problem, but does force semidirect products to be used rather than products. The two parts are then put together via a semidirect product as in Ehlers and Porter, [5], Proposition 2.4. An alternative but equivalent approach follows the route via hypercrossed complexes (cf. Carrasco and Cegarra, [10]), and the extension of the Dold-Kan theorem.

This variety is determined by a subset of the corresponding words for X_1Comp as given in [11].

Finally we return to the discussion of freeness for 2-crossed complexes. Given the linkages between the various categories above, one would expect the following:

Remarks

There are various things to note:

(i) The proof given in [6] that if F is a simplicial resolution of A then $C^{(1)}F$ is a free crossed resolution of G, can not be immediately extended to "2-crossed resolutions". Such a notion clearly would make sense and seems to be needed for handling certain problems in algebra extension theory, however we have not given a construction of a tensor product of a pair of 2-crossed complexes and the result for crossed resolutions used $\pi(1)\otimes\_$. This construction could be avoided by using $\Delta[1]\overline{\otimes}\_$, which should give the same result, but as we have not yet investigated colimits of 2-crossed complexes that construction must also be put off for a future date. It should be pointed out the Baues in [11] defined a tensor product totally free of quadratic complexes using a fairly obvious construction, so it seems unlikely that the conjectured constructions are technically difficult.

(ii) Although $C^{(2)}F$ is totally free for F free simplicial algebra, it seems almost certain that not all totally free 2-crossed complexes arise in this way. The problem is that in a CW-basis, the new generators are used to build $\pi_n F$ or $\pi_{n-1}F$ either as generators or relations. In a 2-crossed complex, the generators at each level influence the *relative* homotopy algebras, $\pi_n(F^{(n)}, F^{(n-1)})$. The differences here are subtle. This is of course more or less equivalent to the realisation problem of Whitehead discussed at length by Baues, [11], but occurring here in a purely algebraic context. Clearly this algebraic realisation problem is important for the analysis of the difference in the homotopical information that can be gleaned from crossed or 2-crossed as against simplicial methods.

REFERENCES

[1] A. MUTLU and T. PORTER, Freeness conditions for 2-crossed modules and complexes, *Theory and Applications of Categories*, **4**, 173-194, 1998.

[2] Z.ARVASI and T.PORTER, Freeness conditions for 2-Crossed modules of commutative algebras, *Applied Categorical Structure* **181,** 426-448, 1998.

[3] A. MUTLU and T. PORTER, Free crossed resolution from a simplicial group with given CW-basis, *University Wales Bangor, Bangor* Preprint 97.18, 1997.

[4] A. MUTLU, Peiffer Pairings in the Moore Complex of a Simplicial Group, *Ph.D. Thesis*, University of Wales Bangor, *Bangor* Preprint 97.11, 1997.

[5] P. J. EHLERS and T. PORTER, Varieties of simplicial groupoids, I: crossed complexes, *J. Pure Appl. Algebra*, **120,** 221-233, 1997; plus: Correction, same journal (to appear).

[6] Z. ARVASI and T. PORTER, Simplicial and crossed resolutions of commutative algebras, *J. Algebra*, **181**, 426-448, 1996.

[7] H. J. BAUES, Homotopy Types, *Handbook of Algebraic Topology*, Edited by I. M. James, Elsevier, 1-72, 1995.

[8] G.J. ELLIS, Crossed squares and combinatorial homotopy, *Math. Z.*, **214**, 93-110. 1993.

[10] P. CARRASCO and A.M. CEGARRA, Group-theoretic algebraic models for homotopy types, *J. Pure Applied Algebra*, **75**, 195-235, 1991.

[11] H. J. BAUES, Combinatorial homotopy and 4-dimensional complexes, *Walter de Gruyter*, 1991.

[12] H. J. BAUES, Algebraic Homotopy, *Cambridge Studies in Advanced Mathematics*, **15**, Cambridge Univ. Press. 1989.

[13] G.J. ELLIS, Higher dimensional crossed modules of algebras, *J. Pure Applied Algebra*, **52**, 277-282, 1988.

[14] T. PORTER, Homology of commutative algebras and an invariant of Simis and Vasconcelos, *J. Algebra*, **99**, 458-465, 1986.

[15] A.R. GRANDJEAN and M.J. VALE, 2-Modulos cruzados en la cohomologia de Andre-Quillen, *Memorias de la Real Academia de Ciencias*, **22**, 1-28, 1986.

[16] D. CONDUCHE, Modules croises generalises de longueur 2, *J. Pure Applied Algebra*, **34**, 155-178, 1984.

[17] R. BROWN and P. J. HIGGINS, Colimit-theorems for relative homotopy groups, *J. Pure Appl. Algebra*, **22**, 11-41, 1981.

[18] J. DUSKIN, Simplicial methods and the interpretation of triple cohomology, *Memoirs A.M.S.*, Vol. 3, **163**, 1975.

[19] M. ANDRE, Homologie des algebres commutatives, *Springer-Verlag*, Die Grundlehren Der Mathhematicschen Wisssenschaften in Einzeldarstellungen Band **206**, 1974.

[20] J.H.C. WHITEHEAD, Combinatorial homotopy I, II, *Bull. Amer. Math. Soc.*, **55**, 453-496, 1949.