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PRELIMINARY GROUP CLASSIFICATION OF $y'' = f(x)y^2$

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Abstract: Direct group classification of $y'' = f(x)y^2$ fails because the classifying condition is a third-order non-linear integro-differential equation for which a closed form solution is unknown. This fact prevents a complete group classification. In this paper we resort to the method of preliminary group classification in order to perform a partial group classification of the ordinary differential equation (ode) mentioned above and obtain new classes that admit symmetry.

1. INTRODUCTION

The ode

$$y'' = f(x)y^2 \tag{1}$$

arises in the study of non-static spherically symmetric perfect fluid solutions of Einstein's field equation [1, 2, 3]. The knowledge of the form of $f(x)$ for which Eq. (1) is solvable leads to a non-static spherically symmetric perfect fluid solutions. A unified way of looking for integrable cases of Eq. (1) is by Lie group classification which originated from the Norwegian mathematician Sophus Lie [4]. In their celebrated paper [1], Kustaanheimo and Qvist used the Lie method in order to find the form of $f(x)$ for which Eq. (1) can be integrated by quadratures (see also [3] for a recent work on Lie symmetry approach for this problem). But in this approach, the classifying condition, i.e. the condition under which Eq. (1) admits symmetry namely

$$\frac{d^3}{dx^3}\left(\alpha f^{-2/5} + \beta f^{-2/5}\int f^{2/5}dx\right) = (\gamma x + \delta)f, \tag{2}$$

where α, β, γ, δ are constants, is highly non-linear and does admit closed form solutions only in special cases. Kustaanheimo and Qvist considered the case when

$$f(x) = (ax^2 + bx + c)^{-5/2},$$

where a, b and c are constants. This does not provide a complete group classification of Eq. (1).

In this paper we aim at using the method of preliminary group classification [5] in order to find the forms of $f(x)$ for which Eq. (1) has symmetries. In doing so, we recover the Kustaanheimo and Qvist class and some new classes. It is worth mentioning that the method of preliminary group classification provides only partial classification. Nevertheless, it has the advantage of involving mainly algebraic manipulations, namely the construction of nonsimilar subalgebras of the principal Lie algebra. Thus the analysis of the classifying condition which is sometimes very non-linear (as in this case) and difficult to tackle is avoided. In the sequel we assume that the reader is familiar with Lie symmetry analysis of differential equations [6] and the method of preliminary group classification [5].

2. PRINCIPAL LIE ALGEBRA AND EQUIVALENCE TRANSFORMATIONS OF $y'' = f(x)y^2$

Straightfoward calculations show that for an arbitrary $f(x)$, Eq. (1) does not have symmetries. This is, the principal Lie algebra of Equation (1) is trivial.

By definition, an equivalence transformation of Eq. (1) is an invertible transformation that leaves its family form invariant. In general, finding all such transformations is very difficult. But those depending on a parameter and forming a group can be found algorithmically [6]. According to the Lie–Ovsiannikov algorithm, the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \omega(x,f)\frac{\partial}{\partial f} \tag{3}$$

defines an equivalence trasformation of Eq. (1) if and only if

$$X^{[2]}(y'' - f(x)y^2)|_{(1)} = 0, \tag{4}$$

where $X^{[2]}$ is the second prolongation of X obtained by the usual prolongation formulas. One has

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''}, \tag{5}$$

where

$$\begin{aligned} \zeta_1 &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 \\ \zeta_2 &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3y'\xi_y)y'' \end{aligned}$$

and where the subscripts stand for partial differentiations.

The expansion of Eq. (4) and the separation of the resulting equation with respect to powers of y' yield the following system:

$$\xi_{yy} = 0, \tag{6}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{7}$$

$$2\eta_{xy} - \xi_{xx} - 3fy^2\xi_y = 0, \tag{8}$$

$$\eta_{xx} - \omega y^2 - 2fy\eta + fy^2(\eta_y - 2\xi_x) = 0. \tag{9}$$

After some simple calculations, we find that the solution to Eq. (6)-(9) is given by:

$$\xi = \frac{\alpha}{2}x^2 + \beta x + \gamma, \tag{10}$$

$$\eta = \frac{\alpha}{2}xy + \left(\frac{\beta}{2} + \lambda\right)y, \tag{11}$$

$$\omega = -\left[\frac{5}{2}(\alpha x + \beta) + \lambda\right]f. \tag{12}$$

Therefore the Lie algebra $L_\mathcal{E}$ of equivalence transformations is generated by the following operators:

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - 5xf\frac{\partial}{\partial f}, \tag{13}$$

$$X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 5f\frac{\partial}{\partial f}, \tag{14}$$

$$X_3 = y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f}, \tag{15}$$

$$X_4 = \frac{\partial}{\partial x}. \tag{16}$$

3. PRELIMINARY GROUP CLASSIFICATION OF $y'' = f(x)y^2$

Before performing preliminary group classification of Eq. (1), note that this equation is linearizable via a point transformation if and only if $f \equiv 0$.

Now consider the projection L_4 of $L_\mathcal{E}$ on (x, f). It is the Lie algebra generated by the operators

$$Y_1 = x^2\frac{\partial}{\partial x} - 5xf\frac{\partial}{\partial f}, \tag{17}$$

$$Y_2 = 2x\frac{\partial}{\partial x} - 5f\frac{\partial}{\partial f}, \tag{18}$$

$$Y_3 = -f\frac{\partial}{\partial f}, \tag{19}$$

$$Y_4 = \frac{\partial}{\partial x}. \tag{20}$$

The essence of the method of preliminary group classification resides in the following statements [7].

Proposition 1 Let L be a subalgebra of the algebra L_4. Denote by Z_i, $i = 1, \ldots, k$ a basis of L and by W_i the elements of $L_\mathcal{E}$ such that $Z_i =$ projection of W_i on (x, f). If the equation

$$f = F(x) \tag{21}$$

is invariant with respect to L then the equation

$$y'' = F(x)y^2 \tag{22}$$

admits the operators

$$T_i = \text{ projection of } W_i \text{ on } (x, y)\,.$$

Proposition 2 Let Eq. (22) and the equation

$$y'' = \bar{F}(x)y^2 \tag{23}$$

be constructed according to Proposition 1 via subalgebras L and $\bar{L}$, respectively. If L and $\bar{L}$ are similar subalgebras of L_4 then Eqs. (22) and (23) are equivalent with respect to the equivalence group $G_\mathcal{E}$ generated by $L_\mathcal{E}$.

According to these propositions the problem of preliminary group classification of Eq. (1) is reduced to the algebraic problem of constructing nonsimilar subalgebras of L_4 or optimal systems of subalgebras [6]. Winternitz *et al.* [8] constructed all nonsimilar subalgebras for four-dimensional Lie algebras. We exploit their results. Before doing so we first change the basis of L_4. The new basis is defined by:

$$E_1 = Y_1, \;\; E_2 = -\frac{1}{2}Y_2, \;\; E_3 = -Y_4, \;\; E_4 = Y_3. \tag{24}$$

In the new basis, the nonzero Lie brackets read:

$$[E_3, E_1] = 2E_2, \;\; [E_1, E_2] = E_1, \;\; [E_2, E_3] = E_3. \tag{25}$$

This corresponds to $A_{3,8} \oplus A_1$ in [8]. If we denote by θ_i, $i = 1, 2, 3$; the set of nonsimilar subalgebras of $L_4 \equiv A_{3,8} \oplus A_1$ of dimension i, we have:

$$\begin{aligned}
\theta_1 &= \{< E_1 >, < E_4 >, < E_2 + \alpha E_4 >, < E_1 + E_2 + \beta E_4 >, < E_1 \pm E_4 >\}, \\
&\quad\; \alpha \geq 0, \; \beta \in \mathbb{R}, \\
\theta_2 &= \{< E_1, E_4 >, < E_2, E_4 >, < E_1 + E_3, E_4 >, < E_1, E_2 + \alpha E_4 >\}, \; \alpha \in \mathbb{R}, \\
\theta_3 &= \{A_2 \oplus A_1 \equiv < E_1, E_2, E_4 >, \; A_{3,8} \equiv < E_1, E_2, E_3 >\}.
\end{aligned}$$

Now we apply Proposition 1 to the optimal systems. Since this involves routine calculations (of invariants), we will treat only one case. The remaining cases are dealt with similarly. Consider the subalgebra $< E_1 >$, where

$$E_1 = x^2\frac{\partial}{\partial x} - 5xf\frac{\partial}{\partial f}. \tag{26}$$

A basis of invariants is found from the equation

$$\frac{dx}{x^2} = \frac{df}{-5xf}$$

and can be taken in the form

$$I_1 = fx^5. \tag{27}$$

From the invariant equation, taken in the form

$$I_1 = f_0 \equiv \text{ const.},$$

it follows that

$$f = f_0 x^{-5}. \tag{28}$$

By again applying Proposition 1, we find the symmetry

$$T_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

So the equation

$$y'' = f_0 x^{-5} y^2$$

admits T_1. Simple calculations show that this equation also admits

$$T_2 = x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}.$$

This can be explained by the fact that the optimal systems $< E_1 >$ and $< E_1, E_2 + \alpha E_4 >$ have the same invariant when $\alpha = -15/2$. The remaining cases are listed in Table 1. Note that we have excluded the case $f = 0$ which is trivial.

4. CONCLUSION

We have used the method of preliminary group classification to obtain a partial group classification of Eq. (1). New cases of Eq. (1) which have symmetries have been uncovered. But the problem of the complete group classification of Eq. (1) remains open. The direct classification being almost impossible for the reasons we mentioned earlier. A complete classification may require new techniques. As a matter of interest, in a recent paper [11] computer algebra was used in the analysis of solutions of (1). This was done by making an ansätz and certain cases that admit one symmetry. Among them one that gave rise to complete integrability of (1) was already considered in [1]. The cases of one symmetry that give rise to complete integrability are precisely when the symmetry is Noether and all of these are investigated in [12].

Table 1: The results of preliminary group classification of $y'' = f(x)y^2$

$f(x)$	Symmetries	Remarks
$f_0 x^{-5}$	$T_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$, $T_2 = x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$	The cases $f = f_0 x^m$ with $m \in \{0, -20/7, -15/7\}$ reduce to this case [9, 10].
$f_0 x^m$, $m \neq 0, -5, -20/7, -15/7$.	$T_1 = x\frac{\partial}{\partial x} - (m+2)y\frac{\partial}{\partial y}$	The equation reduces to Abel's equation of the second kind.
$f_0 x^{-5} e^{\epsilon/x}$, $\epsilon = \pm 1$.	$T_1 = x^2\frac{\partial}{\partial x} + (x+\epsilon)y\frac{\partial}{\partial y}$	The equation reduces to Abel's equation of the second kind.
$f_0(x-1)^{-m-5/2}x^{m-5/2}$	$T_1 = x(x-1)\frac{\partial}{\partial x} + (x-m-1/2)y\frac{\partial}{\partial y}$	The case $m = 0$ was considered by Kustaanheimo *et al.* [1]; also for $m \neq 0$, some cases are in [11].

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