Mathematical & Computational Applications, Vol. 4, No. 3, pp. 283-287, 1999.

ON STATISTICALLY CONVERGENT SEQUENCES OF FUZZY NUMBERS

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Abstract-In this paper, some equivalent alternative conditions are obtained for a sequence of fuzzy numbers to be statistically Cauchy.

1. PRELIMINARIES

Let D denote the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line R where $\underline{A}$ and $\overline{A}$ denote the end points of A. For $A, B \in D$ define

$$A \le B \text{ iff } \underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B},$$
$$d(A, B) = \max\left(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\right).$$

It is not hard to see that d defines a metric on D and (D, d) is complete metric space and $d(A, B)$ is called the distance between the intervals A and B. Also it is easy to see that $\le$ defined above is a partial order relation in D.

A fuzzy number is a fuzzy subset of a real line R which is bounded, convex and normal. Let $L(R)$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support. In other words, if $X \in L(R)$ then for any $\alpha \in [0, 1]$, X^α is compact where

$$X^\alpha = \begin{cases} t : X(t) \ge \alpha & \text{if } \alpha \in (0, 1], \\ t : X(t) > \alpha & \text{if } \alpha = 0. \end{cases}$$

Define a map $\overline{d} : L(R) \times L(R) \to R$ by

$$\overline{d}(X, Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

It is straightforward to see that d is a metric in $L(R)$.

For $X, Y \in L(R)$ define

$$X \le Y \text{ iff } X^\alpha \le Y^\alpha \text{ for any } \alpha \in [0, 1].$$

We now quote the following definitions which will be needed in the sequel (see, [3]).

Definition 1. A sequence $X = (X_k)$ of fuzzy numbers is function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term of the sequence.

Definition 2. A sequence $X = (X_k)$ of fuzzy numbers is said to be converge to a fuzzy number X_0 if for every $\varepsilon \rangle 0$ there is a positive integer N_0 such that $d(X_k, X_0) < \varepsilon$ for $k > N_0$. And $X = (X_k)$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there is a positive integer N_0 such that $d(X_k, X_l) < \varepsilon$ for $k, l > N_0$.

2. STATISTICAL CONVERGENCE

The notation of statistical convergence was introduced by Fast [1] and also independently Schoenberg [8] for real and complex sequences. Fridy [2] obtained an equivalent criterion for statistically convergent real sequences similar to the Cauchy criterion of convergence. The concept of statistical convergence for sequences of fuzzy numbers was introduced and studied by Nuray and Savaş [5]. In this paper, some equivalent alternative conditions are obtained for a sequence of fuzzy numbers to be statistically Cauchy.

The natural density of a set K of positive integers (See, e.g., [4, p. 290]) is defined by $\delta(K) = \lim_n \frac{1}{n}|\{k \le n : k \in K\}|$, where $|\{k \le n : k \in K\}|$ denotes the number of elements of K not exceeding n. We shall be concerned with integer sets having natural density zero.

Definition 3. If $X = (X_k)$ is a sequence that satisfies a property P for all n except a set of natural density zero, then we say that X_k satisfies P for almost all k and we write by "a.a.k".

Definition 4. A sequence $X = (X_k)$ is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0 \lim_n \frac{1}{n}|\{k \le n : d(X_k, X_0) \ge \varepsilon\}| = 0$. i.e., $d(X_k, X_0) < \varepsilon$ a.a.k. In this case we write st- $\lim X_k = X_0$.

Note that $\lim X_k = X_0$ implies st- $\lim X_k = X_0$ but the converse does not. For example, define $X_k = A$ a fixed fuzzy number, if k is a square of a natural number and $X_k = 0$ otherwise. Then st- $\lim X_k = A$ and $\lim X_k \neq A$.

Definition 5. A sequence $X = (X_k)$ is said to be statistically Cauchy, if for every $\varepsilon > 0$ there is a positive integer N_0 such that $\lim_n \frac{1}{n}|\{k \le n : d(X_k, X_{N_0}) \ge \varepsilon\}| = 0$. i.e., $d(X_k, X_{N_0})$ a.a.k.

Every statistically convergent sequences of fuzzy numbers is statistically Cauchy sequence of fuzzy numbers. For any sequence $X=(X_k)$ of fuzzy numbers $\text{supp } X=\{k\in N: X_k\neq 0$.

In the next theorem, some equivalent alternative conditions are obtained for a sequence of fuzzy numbers to be statistically Cauchy.

Theorem 1. For any sequence $X=(X_k)$of fuzzy numbers, the following are equivalent.

(a) X is a statistically Cauchy sequence of fuzzy numbers.

(b) There exists a subsequence $K=(k_n)$ of N with $\delta(K)=1$ such that $\bar{d}(X_{k_m}, X_{k_n})\to 0$ as $m, n\to\infty$.

(c) There exists the sequences $Y=(Y_k)$ and $Z=(Z_k)$ of fuzzy numbers such that $X=Y+Z$, $\bar{d}(Y_m, Y_n)\to 0$ as $m, n\to\infty$ and $\delta(\text{supp } Z)=0$.

Proof. To prove that (a) implies (b), let $X=(X_k)$ be a statistically Cauchy sequence of fuzzy numbers. By definition, we can find $m_1\in N$ such that if K_1 is the set $\{k\in N: d(X_{m_1}, X_k)<1/2$, then $\delta(K_1)=1$ Similarly, we can find $m_2\in N$ such that $\delta(B_1)=1$, where $B_1=\{k\in N: \bar{d}(X_k, X_m)<1/4$. If we set $K_2=K_1\cap B_1$, then $\delta(K_2)=1$, $K_2\subset K_1$ and $\bar{d}(X_{k_1}, X_{k_2})<1/2$ for all $k_1, k_2\in K_2$. Proceeding inductively, we can construct a decreasing sequence (K_j) of subsets of N such that

$$\delta(K_j)=1 \text{ for each } j\in N \qquad \text{.......... (1)}$$

and

$$\bar{d}(X_{k_1}, X_{k_2})<1/j \text{ for all } k_1, k_2\in K_j,\ j\in N \qquad \text{.......... (2)}$$

Next, we construct a subsequence (v_j) of N as follows. Let $v_1\in K_1$. By (2), we can find $v_2\in K_2$ with $v_2>v_1$ such that for each $n\geq v_2$, $\frac{|K_2(n)|}{n}>1/2$ The subsequence (v_j) can be defined inductively such that $v_j\in K_j$ for each $j\in N$ and

$$\frac{|K_j(n)|}{n}>\frac{j-1}{j} \text{ for each } n\geq v_j \qquad \text{.......... (3)}$$

If we set

$$K = \{k \in N : 1 \le k < v_1\} \cup \left[\bigcup_{j \in N} \{k : v_j \le k < v_{j+1}\} \cap K_j\right]$$

then for any $j \in N$ and $v_j \le n \le v_{j+1}$,

$$K_j(n) = \{k \le n : k \in K_j\} \subset \{k \le n : k \in K\} = K(n)$$

which implies by (3) that for all $j \in N$,

$$\frac{|K(n)|}{n} \ge \frac{|K_j(n)|}{n} > \frac{j-1}{j}$$

and hence $\delta(K) = 1$.

Now we show that $\bar{d}(X_m, X_n) \to 0$ as $m, n \to 0$ and $m, n \in K$. For this, let $\varepsilon > 0$ and let $j \in N$ be such that $j > \frac{1}{\varepsilon}$. If $m, n \in K$ and $m, n > v_j$, we can find $r, s \ge j$ such that $v_r \le m \le v_{r+1}$, $v_s \le n \le v_{s+1}$, so that $m \in K_r$, $n \in K_s$. Suppose that $r \le s$. Then $K_s \subset K_r$, so that $m, n \in K_r$. By (2)

$$\bar{d}(X_m, X_n) \le \frac{1}{r} \le \frac{1}{j} < \varepsilon$$

which proves our assertion. Thus (a) implies (b).

Next, let (b) hold. Then there is $K = (k_n)$ such that $\delta(K) = 1$ and $\bar{d}(X_{k_n}, X_{k_m}) \to 0$ as $m, n \to \infty$. Define,

$Y_k = X_k$, $Z_k = \theta$, if $1 \le k \le k_1$ and

$Y_k = X_{k_n}$, $Z_k = X_k - X_{k_n}$ if $k_n \le k \le k_{n+1}$, $n \in N$.

Then $X = Y + Z$, $\delta(\text{supp } Z) \le \delta(K') = 0$ and construction shows that $\bar{d}(Y_{k_n}, Y_{k_m}) \to 0$ as $m, n \to \infty$. This proves (c).

Lastly, if (c) is assumed to hold and Y and Z are as determined there, then $X_k = Y_k$ for all $k \in N = N - \text{supp } Z = (k_n)$ with $\delta(K) = 1$.

For a given $\varepsilon > 0$, let $m \in N$ be such that

$\bar{d}(X_{k_m}, X_{k_n}) < \varepsilon$ for all $n \ge m$.

Then

$$\delta\left(\left\{k \in N : \bar{d}\left(X_k, X_{k_m}\right) \geq \varepsilon\right\}\right) \leq \delta\left(N - \{k_{m+1}, k_{m+2}, \ldots\}\right) = 0,$$

so that (a) holds. This completes the proof of the theorem.

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