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INVARIANT SOLUTIONS FOR SOIL WATER EQUATIONS

V.A. Baikov[1]
Institute for Symmetry Analysis and Mathematical Modelling, University of North-West, Private Bag X2046, Mmabatho 2735, South Africa

C.M. Khalique
Department of Mathematical Sciences,
University of North-West,
Private Bag X2046, Mmabatho 2735, South Africa

Abstract. We obtain exact solutions for a class of nonlinear partial differential equations which models soil water infiltration and redistribution in a bedded soil profile irrigated by a drip irrigation system. The solutions obtained are invariant under two-parameter symmetry groups.

1. INTRODUCTION

In [1] (see also [2]) a mathematical model was developed to simulate soil water infiltration and redistribution in a bedded soil profile irrigated by a drip irrigation system. This model is described by the class of equations

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi), \tag{1}$$

where ψ is soil moisture pressure head, $C(\psi)$ is specific water capacity, $K(\psi)$ is unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, t is time, x is the horizontal and z is the vertical axis which is considered positive downward. Because of the nonlinearity of equation (1), researchers have given analytical and numerical solutions for special cases when the functions $C(\psi)$ and $K(\psi)$ are constants and $S(\psi)$ are linear functions.

In this paper, using Lie group theory, we shall obtain exact/asymptotic invariant solutions of equation (1) for some special coefficients $C(\psi)$, $K(\psi)$ and $S(\psi)$ which are not constants nor linear.

In [4], all symmetries of equation (1) were found. The principal Lie algebra L_p (i.e., the Lie algebra of the Lie transformation group admitted by equation (1) for arbitrary functions $C(\psi)$, $K(\psi)$ and $S(\psi)$, see e.g. [3]) was found to be the three-dimensional Lie algebra spanned by the following three generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial z}. \tag{2}$$

For special cases of $C(\psi)$, $K(\psi)$ and $S(\psi)$), the algebra L_p is shown to extend by two or more operators. Also two examples of invariant solutions to equation (1) are given analytically and graphically in [4].

[1] On leave from Ufa State Aviation Technical University, 12 K. Marx st., Ufa 450025, Russia

2. INVARIANT SOLUTIONS

In this section we shall obtain exact/asymptotic (invariant) solutions of equation (1) for some special forms of the functions $C(\psi), K(\psi)$ and $S(\psi)$. We shall be considering those cases in which the principal Lie algebra L_p extends by one or more operators. For each case we shall look for solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra. Equation (1) is then reduced, in general, to second-order ordinary differential equations which are then solved to obtain solutions. We shall follow the general algorithm for constructing invariant solutions (see, e.g. [5] and [6]).

Here we consider examples of invariant solutions of equation (1) with $K(\psi) = 1$, $C(\psi) = \psi^{\sigma}$, where σ is an arbitrary constant and two forms of $S(\psi)$, viz. $S(\psi) = B\psi^{\gamma}$, and $S(\psi) = B\psi^{\sigma+1} + D\psi$, $B \neq 0$, $D \neq 0$ and $\gamma \neq \sigma + 1$.

We first consider the case when $S(\psi) = B\psi^{\gamma}$.

In this case equation (1) has the form

$$\psi_t = \psi^{-\sigma}\left\{\psi_{xx} + \psi_{zz}\right\} - B\psi^{\gamma-\sigma}. \tag{3}$$

According to the classification result, equation (3) admits a five-dimensional Lie algebra L_5 obtained by an extension of the principal Lie algebra L_p by the following two operators:

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$

and

$$X_5 = 2(1+\sigma-\gamma)t\frac{\partial}{\partial t} + (1-\gamma)x\frac{\partial}{\partial x} + (1-\gamma)z\frac{\partial}{\partial z} + 2\psi\frac{\partial}{\partial \psi}.$$

We now construct invariant solutions under these two operators. These operators span a two-dimensional subalgebra L_2 of the algebra L_5 and have two functionally independent invariants. We first calculate a basis of invariants $I(t, x, z, \psi)$ by solving the system of linear first-order partial differential equations:

$$X_4 I = 0, \quad X_5 I = 0.$$

Since we have $[X_4, X_5] = 0$, the subalgebra L_2 is Abelian. Therefore we can solve the equations $X_4 I = 0, X_5 I = 0$ successively in any order. The first equation provides three functionally independent solutions

$$J_1 = x^2 + z^2, \quad J_2 = t \quad \text{and} \quad J_3 = \psi.$$

Hence the common solution $I(t, x, z, \psi)$ of the system is defined as a function of J_1, J_2 and J_3 only. Writing the action of X_5 on the space of J_1, J_2 and J_3 we obtain

$$X_5 = 2(1-\gamma)J_1\frac{\partial}{\partial J_1} + 2(1+\sigma-\gamma)J_2\frac{\partial}{\partial J_2} + 2J_3\frac{\partial}{\partial J_3}.$$

Consequently, from the second equation $X_5 I = 0$ we obtain the following two functionally independent solutions (invariants):

$$I_1 = J_1 J_2^{\frac{\gamma-1}{1+\sigma-\gamma}} \equiv (x^2 + z^2)t^{\frac{\gamma-1}{1+\sigma-\gamma}},$$

and

$$I_2 = J_3 J_2^{\frac{-1}{1+\sigma-\gamma}} \equiv \psi t^{\frac{-1}{1+\sigma-\gamma}}.$$

The invariant solution is given by $I_2 = \Phi(I_1)$, that is

$$\psi t^{\frac{-1}{1+\sigma-\gamma}} = \Phi\left((x^2+z^2)t^{\frac{\gamma-1}{1+\sigma-\gamma}}\right)$$

or

$$\psi = t^{\frac{1}{1+\sigma-\gamma}}\Phi(\eta), \quad \eta = (x^2+z^2)t^{\frac{\gamma-1}{1+\sigma-\gamma}}. \tag{4}$$

Substituting this into equation (3) we obtain

$$\eta\Phi'' + \Phi' - \frac{\gamma-1}{4(1+\sigma-\gamma)}\eta\Phi^{\sigma}\Phi' + \frac{B}{4}\Phi^{\gamma} - \frac{\Phi^{1+\sigma}}{4(1+\sigma-\gamma)} = 0. \tag{5}$$

This is a second-order nonlinear differential equation and it can be shown that it has a special solution of the type

$$\Phi(\eta) = \left\{\frac{-4}{B(1-\gamma)^2}\right\}^{\frac{1}{\gamma-1}} \eta^{\frac{1}{1-\gamma}}$$

and consequently equation (4) yields

$$\psi = \left\{\frac{-4}{B(1-\gamma)^2}\right\}^{\frac{1}{\gamma-1}} (x^2+z^2)^{\frac{1}{1-\gamma}}$$

as an invariant solution of equation (3) which is a stationary (independent of time) solution.

Also, it can easily be seen that

$$\Phi_0 = \left(\frac{-1}{B(1+\sigma-\gamma)}\right)^{\frac{1}{\gamma-1-\sigma}}$$

is a constant solution of equation (5). We now obtain an approximate solution near Φ_0. By letting $\Phi = \Phi_0 + \Phi_1$ we linearize equation (5) near the constant solution Φ_0. We obtain

$$\Phi_1'' + \left(\frac{1}{\eta} - \frac{\gamma-1}{4(1+\sigma-\gamma)}\Phi_0^{\sigma}\right)\Phi_1' + \left(\frac{B\gamma}{4}\Phi_0^{\gamma-1} - \frac{\sigma+1}{4(1+\sigma-\gamma)}\Phi_0^{\sigma}\right)\frac{1}{\eta}\Phi_1 = 0. \tag{6}$$

If we let

$$\Phi_1 = y e^{-\frac{1}{2}\int\left(\frac{1}{\eta} - \frac{\gamma-1}{4(1+\sigma-\gamma)}\Phi_0^{\sigma}\right)d\eta}$$

and substitute in equation (6), it can be seen that y satisfies the second-order differential equation

$$y'' = P(\eta)y$$

where

$$P(\eta) = \frac{-1}{4\eta^2} + \left(\frac{\sigma+1}{4(1+\sigma-\gamma)}\Phi_0^{\sigma} - \frac{\gamma-1}{8(1+\sigma-\gamma)}\Phi_0^{2\sigma} - \frac{B\gamma}{4}\Phi_0^{\gamma-1}\right)\frac{1}{\eta} + \frac{(\gamma-1)^2}{64(1+\sigma-\gamma)}.$$

The Liouville-Green approximation for the general solution of $y'' = P(\eta)y$ is given by (see for example [7])

$$y = c_1 P^{\frac{-1}{4}}(\eta) e^{\int P^{\frac{1}{2}}(\eta)d\eta} + c_2 P^{\frac{-1}{4}}(\eta) e^{-\int P^{\frac{1}{2}}(\eta)d\eta},$$

where c_1 and c_2 are arbitrary constants. Hence an approximate invariant solution of equation (3) is

$$\psi = t^{\frac{1}{1+\sigma-\gamma}} \left\{ \left(\frac{-1}{B(1+\sigma-\gamma)}\right)^{\frac{1}{\gamma-1-\sigma}} + e^{\frac{-1}{2}\int\left(\frac{1}{\eta} - \frac{\gamma-1}{4(1+\sigma-\gamma)}\Phi_0^{\sigma}\right)d\eta}. \right.$$

$$\left. \left[c_1 P^{\frac{-1}{4}}(\eta) e^{\int P^{\frac{1}{2}}(\eta)d\eta} + c_2 P^{\frac{-1}{4}}(\eta) e^{-\int P^{\frac{1}{2}}(\eta)d\eta}\right] \right\}.$$

We now consider the second case when $S(\psi) = B\psi^{\sigma+1} + D\psi$, where $B \neq 0$ and $D \neq 0$ are arbitrary constants.
In this case equation (1) has the form

$$\psi_t = \psi^{-\sigma}(\psi_{xx} + \psi_{zz}) - B\psi - D\psi^{1-\sigma} \tag{7}$$

and the principal Lie algebra extends by two operators, namely

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

and

$$X_5 = e^{B\sigma t}\frac{\partial}{\partial t} + Be^{B\sigma t}\psi\frac{\partial}{\partial \psi}.$$

These operators span a two-dimensional subalgebra L_2 of the algebra L_5. We have $[X_4, X_5] = 0$. Hence the subalgebra L_2 is Abelian. We then solve the system $X_4 I = 0, X_5 I = 0$ for invariants, begining with the equation $X_4 I = 0$. Then the second equation $X_5 I = 0$ will be represented in the space of three independent solutions of the equation $X_4 I = 0$. Solving this equation yields two functionally independent solutions (invariants) and as in the previous case we can write the invariant solution as

$$\psi = e^{Bt}\Phi(\xi), \quad \xi = x^2 + z^2. \tag{8}$$

Substituting this into equation (7), we obtain

$$\xi\Phi'' + \Phi' + \frac{D}{4}\Phi = 0.$$

By the change of variable $\eta = \ln \xi$, the above equation is transformed to

$$\Phi''_{\eta\eta} + \frac{D}{4} e^{\eta} \Phi = 0.$$

The Liouville-Green approximation for the general solution of this equation (see for example [7]) is given by

$$\Phi(\eta) = A\left(-\frac{D}{4}e^{\eta}\right)^{\frac{-1}{4}} e^{\int\left(-\frac{D}{4}e^{\eta}\right)^{\frac{1}{2}} d\eta} + C\left(-\frac{D}{4}e^{\eta}\right)^{\frac{-1}{4}} e^{-\int\left(-\frac{D}{4}e^{\eta}\right)^{\frac{1}{2}} d\eta}$$

where A and C are arbitrary constants.
Consequently, equation (8) yields

$$\psi = e^{Bt}\left\{A\left[-\frac{D}{4}(x^2+z^2)\right]^{\frac{-1}{4}} e^{\{-D(x^2+z^2)\}^{\frac{1}{2}}} + C\left[-\frac{D}{4}(x^2+z^2)\right]^{\frac{-1}{4}} e^{-\{-D(x^2+z^2)\}^{\frac{1}{2}}}\right\}$$

which is an approximate invariant solution of equation (7).
We note that as a special case when $C = -A$, we obtain

$$\psi = 2Ae^{Bt}\left[-\frac{D}{4}(x^2+z^2)\right]^{\frac{-1}{4}} \sinh[-D(x^2+z^2)]^{\frac{1}{2}}.$$

We can in fact also obtain (non invariant) solutions of equation (7) of the form

$$\psi = f(t)\Phi(x)\psi(z)$$

provided $f' = Bf$ and $\frac{\Phi''}{\Phi} + \frac{\psi''}{\psi} = D$. If we let $\frac{\Phi''}{\Phi} = \alpha$ and $\frac{\psi''}{\psi} = \beta$, where α and β are real constants then we have $\alpha + \beta = D$ and

$$\Phi(x) = C_1 e^{\sqrt{\alpha}x} + C_2 e^{-\sqrt{\alpha}x}, \quad \psi(z) = C_3 e^{\sqrt{\beta}z} + C_4 e^{-\sqrt{\beta}z}$$

and $f(t) = C_5 e^{-Bt}$.

Particular case; $D = 0$.
If $D = 0, B \neq 0$, equation (1) has the form

$$\psi_t = \psi^{-\sigma}(\psi_{xx} + \psi_{zz}) - B\psi. \tag{9}$$

In this case there is a further extension of the principal Lie algebra by one operator, namely

$$X_6 = \sigma x \frac{\partial}{\partial x} + \sigma z \frac{\partial}{\partial z} - 2\psi \frac{\partial}{\partial \psi}.$$

We therefore have three further cases to discuss and construct invariant solutions by considering two operators at a time.

Case 1.
We first construct invariant solutions under the operators X_4 and X_5. The invariant solution in this case is again given by equation (8), but the differential equation satisfied by Φ is

$$\xi\Phi'' + \Phi' = 0.$$

The solution of this equation is given by

$$\Phi(\xi) = C_1 \ln \xi + C_2$$

and equation (8) yields

$$\psi = e^{Bt}\{C_1 \ln(x^2 + z^2) + C_2\}$$

as an invariant solution of equation (9).

Case 2.
We now construct invariant solutions under the operators X_4 and X_6. Repeating the calculations described above, we obtain the invariant solution

$$\psi = (x^2 + z^2)^{\frac{-1}{\sigma}}\Phi(t)$$

where Φ satisfies $\Phi' = \frac{4}{\sigma^2}\Phi^{1-\sigma} - B\Phi$.
Hence

$$t = \int \frac{d\Phi}{\Phi\left(\frac{4}{\sigma^2}\Phi^{-\sigma} - B\right)}.$$

For special cases the integral can be evaluated.
For example when $\sigma = 2$ and $B = -1$, we obtain

$$\Phi(t) = \sqrt{e^{2t} - 1}.$$

Case 3.
Finally we construct invariant solutions under the operators X_5 and X_6. In this case we obtain the invariant solution

$$\psi = z^{\frac{-2}{\sigma}} e^{Bt}\Phi(\xi), \quad \xi = \frac{x}{z}$$

where Φ satisfies

$$(1 + \xi^2)\Phi'' + \left(2 + \frac{4}{\sigma}\right)\xi\Phi' + \frac{2}{\sigma}\left(\frac{2}{\sigma} + 1\right)\Phi = 0.$$

The solution of this equation is given by (see for example [7])

$$\Phi(\xi) = C_1\Phi_1 + C_2\Phi_2$$

where C_1 and C_2 are arbitrary constants and

$$\Phi_1(\xi) = F\left(\frac{1}{\sigma}, \frac{1}{\sigma} + \frac{1}{2}; \frac{1}{2}; -\xi^2\right)$$

and

$$\Phi_2(\xi) = i\xi F\left(\frac{1}{\sigma} + \frac{1}{2}, \frac{1}{\sigma} + 1; \frac{3}{2}; -\xi^2\right).$$

Hence F is a hypergeometric function.
Hence the invariant solution of equation (9)is given by

$$\psi = z^{\frac{-2}{\sigma}} e^{Bt}\{C_1\Phi_1 + C_2\Phi_2\}.$$

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