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The Inertial Proximal Algorithm for the Split Variational Inclusion Problem

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Abstract

In this work, we propose an inertial proximal splitting algorithm for the split variational inclusion problem in real Hilbert space, in which a new stepsize rule is provided for avoiding the case that the original stepsize equals to zero, It also helps enlarge the value range of the stepsize parameter. A weak and strong convergence theorem is established under mild conditions. Additionally, our obtained result is extended to split feasibility problems and split minimization problems. Finally, some numerical experiments are conducted to illustrate and compare.

Keywords: split variational inclusion problem; splitting algorithm; inertial technique; strong convergence

MSC: 47J20; 65K05; 65K10; 47J25

1. Introduction

Let X be a real Hilbert space with induced norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $Y : X \to 2^X$ be a maximally monotone multi-valued operator. It is known that the classical split variational inclusion problem [1] (SVIP) is stated as

$$\text{find } p \in X_1 \text{ such that } 0 \in Y_1(p) \text{ and } 0 \in Y_2(Ap), \tag{1}$$

where X_1 and X_2 are both real Hilbert spaces, $Y_1 : X_1 \to 2^{X_1}$ and $Y_2 : X_2 \to 2^{X_2}$ are both monotone operators, and $A : X_1 \to X_2$ is a bounded linear operator. The solution set of SVIP is denoted by

$$S = \{p \in X_1 : 0 \in Y_1(p) \text{ and } 0 \in Y_2(Ap)\}. \tag{2}$$

The SVIP has many applications such as split variational inequalities, image restoration, Generative Adversarial Networks (GANs) and so on (see [2–8]). Some problems are usually used as a model in image recovery [9,10] and signal recovery [8], in which the signal recovery can be transformed to its special case; that is, the split feasibility problem (SFP) [11] is expressed as

$$\text{find } p \in C \text{ such that } Ap \in Q, \tag{3}$$

in which C and Q are nonempty, closed, and convex subsets of X_1 and X_2, respectively.

1.1. Related Algorithms

Due to its wide applications, many existing methods are used to deal with SVIP and its nonlinear programming problems (see the concrete introductions of [2–7,12–14]

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for reference). Additionally, a well-known approach to deal with the SVIP is Byrne's algorithm [15], whose frame is the following:

$$p_{n+1} = J_\lambda^{\Upsilon_1}\left(p_n - \lambda_n A^*\left(I - J_\lambda^{\Upsilon_2}\right)Ap_n\right),\ p_0 \in X_1, \tag{4}$$

where J_λ^{Υ} is defined in Section 2, $\lambda > 0$ is a parameter, $\lambda_n > 0$ is a fixed stepsize, and A^* is a adjoint operator of A. With the help of the fixed stepsize $\lambda_n \in \left(0, \frac{2}{\|A\|^2}\right)$, the weak convergence of the above algorithm was obtained. However, Byrne's algorithm is the first to calculate the norm $\|A\|$ in each iteration, which is somewhat difficult, if A is a dense matrix in practice. Regarding this consideration, this is usually avoided by an Armijo-linear search rule. Based on the rule, there are many proximal splitting algorithms emerging for solving the SVIP (detailed information can be found in [2,4–6,13,14] for reference). Since the Armijo-linear search rule generally makes the algorithm produce the inner iterations, the algorithm efficiency is affected. To overcome this, Lopéz [16] proposed a self-adaptive stepsize strategy below:

$$\lambda_n = \frac{\delta F(p_n)}{\|\nabla F(p_n)\|^2},$$

with $\inf \delta(4-\delta) > 0$ and $F(p) = \frac{1}{2}\|(I - P_Q)Ap\|^2$ and $\nabla F(p) = A^*(I - P_Q)Ap$, and P_Q can be expressed as the corresponding projections onto Q. For more details on this self-adaptive stepsize strategy, see [17–20]. Follow by this, the inertial extrapolation technique has received much more attention, and which is popularly used to increase the convergence (see, for instance, [21–28]). Among them, Nesterov [28] used the inertial technique to speed up the heavy ball method [18] in the following.

$$\begin{aligned} \Lambda_n &= p_n + \theta_n(p_n - p_{n-1}) \\ p_{n+1} &= \Lambda_n - \lambda_n F(\Lambda_n), n \geq 1, \end{aligned} \tag{5}$$

where $\theta_n \in [0, 1)$ and $\lambda_n > 0$. The term $\theta_n(p_n - p_{n-1})$ is the so-called inertial term. Recently, Thong [29] proposed a self-adaptive inertial subgradient extragradient method for solving non-monotone variational inequalities, and he obtained its weak convergence in real Hilbert space. Its iterative scheme is of the following form:

$$\begin{cases} \Lambda_n = p_n + \theta_n(p_n - p_{n-1}), \\ y_n = P_C(\Lambda_n - \lambda_n F\Lambda_n), \\ \Re_n = P_{T_n}(K_n - t\lambda_n\tau_n F y_n), \\ T_n = \{p \in X : \langle \Lambda_n - \lambda_n F\Lambda_n - y_n, p - y_n\rangle \leq 0\}, \\ p_{n+1} = P_{T_n}(\Lambda_n - \lambda_n F y_n), \end{cases} \tag{6}$$

where the stepsize λ_n is updated by

$$\lambda_{n+1} = \begin{cases} \min\left\{(p_n + r_n\delta)\frac{\|\Lambda_n - y_n\|}{\|F(\Lambda_n) - F(y_n)\|}, (1+\alpha_n)\lambda_n\right\}, & \text{if } F(\Lambda_n) - F(y_n) \neq 0, \\ (1+\alpha_n)\lambda_n, & \text{otherwise,} \end{cases}$$

where $\delta \in (0, 1)$, α_n is a sequence of nonnegative real numbers that satisfies $\sum_{n=1}^{\infty} \alpha_n < +\infty$, $\{r_n\} \subset [1, \infty)$, $\lim_{n\to\infty} r_n = 0$, $p_n \in [0, +\infty)$, and $\lim_{n\to\infty} p_n = 0$, $F : X \to X$ is a uniformly

continuous mapping on X. Based on Lopéz's stepsize thought [16], Cholamjiak et al. [30] suggested a linearized splitting algorithm for solving the SVIP, with the scheme

$$\begin{cases} \Lambda_n = p_n + \theta_n(p_n - p_{n-1}), \\ y_n = \Lambda_n - \lambda_n \nabla F(\Lambda_n), \\ z_n = J_\lambda^{Y_1}(2y_n - \Lambda_n), \\ p_{n+1} = \Lambda_n + \alpha(z_n - y_n), \end{cases} \tag{7}$$

where the stepsize λ_n is given by

$$\lambda_n = \begin{cases} \frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, & \text{if } \|\nabla F(\Lambda_n)\| \neq 0, \\ 0. & \text{otherwise,} \end{cases}$$

Here, both $F(p)$ and its gradient $\nabla F(p)$ are defined in Section 3. $\delta \in (0,2)$ and $\alpha \in (0,2)$, and $\theta_n \geq 0$ satisfies $\sum_{n=1}^{\infty} \theta_n < \infty$. Under these conditions, it is shown that the sequence $\{p_n\}$ created by the algorithm converges weakly to a point in S. Very recently, Xu et al. [31] introduced an inertial viscosity-type projection and contraction algorithm for addressing quasimonotone variational inequalities. The iterative form is described as follows:

$$\begin{cases} \Lambda_n = p_n + \theta_n(p_n - p_{n-1}), \\ K_n = \Lambda_n + J_n(\Lambda_n - \Lambda_{n-1}); \\ y_n = P_C(K_n - \mu\lambda_n F K_n); \\ z_n = P_{T_n}(K_n - \lambda_n \beta \tau_n F y_n); \\ T_n = \{p \in X | \langle K_n - \mu\lambda_n F K_n - y_n, p - y_n \rangle \leq 0\}, \\ \Delta_n = K_n - y_n - \mu\lambda_n(F K_n - F y_n); \\ p_{n+1} = \alpha_n f(p_n) + (1 - \alpha_n)[(1 - \eta_n)z_n + \eta_n T z_n], \end{cases} \tag{8}$$

where the stepsizes λ_n and τ_n are respectively defined as

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\delta\|K_n - y_n\|}{\|F K_n - F y_n\|}, \lambda_n + \hat{P}_n \right\}, & \text{if } F K_n - F y_n \neq 0, \\ \lambda_n + \hat{P}_n, & \text{otherwise,} \end{cases}$$

$$\tau_n = \begin{cases} \frac{\langle K_n - z_n, \Delta_n \rangle}{\|\Delta_n\|^2} & \text{if } \Delta_n \neq 0, \\ 0, & \text{if } \Delta_n = 0. \end{cases}$$

where $\delta \in (0,1)$, $\beta \in (0, \frac{2}{\delta})$, $\mu \in (\frac{\beta}{2}, \frac{1}{\delta})$, $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\eta_n \in (0,1)$, $\{\eta_n\} \subset (a, 1-\lambda)$ for some $a \in [0, 1-\lambda)$, $\lambda \in [0,1)$, and θ_n and J_n are the inertial parameters. $F : X \to X$ is a Lipschitz continuous and quasimonotone mapping, $f : X \to X$ is a contraction mapping with constant $\hat{\rho} \in (0,1)$, and $\sum_{n=1}^{\infty} \hat{P}_n < \infty$. The strong convergence of the algorithm (8) is discussed in real Hilbert spaces.

1.2. Motivation and Contributions

For the existing proximal gradient methods, e.g., Byrne's algorithm (4), the new iterative algorithms [5,32,33], and the inertial proximal algorithm [2,4,6,34–36], their convergence holds under assumptions on the inverse strongly monotonicity of ∇F, and this condition brings a small stepsize $\lambda_n \in \left(0, \frac{2}{\|A\|^2}\right)$, which may consume more computational time. On the other hand, it may produce many iterative numbers when the Armijo-linear

search in [21] aims to find a proper stepsize in the iterative process. Additionally, we observe that when $\|\nabla F(\Lambda_n)\| = 0$, the stepsize $\lambda_n = 0$ in Algorithm (7). These above considerations may affect the convergent rate of our listed algorithms. However, the weak and strong convergence of Algorithm (7) with the new stepsizes has yet to be investigated.

From these factors, we need to further improve the numerical results of Algorithm (7). Our contributions in this paper are the following.

- To design a new stepsize for avoiding the case that the original stepsize is equal to zero under $\|\nabla F(\Lambda_n)\| = 0$. This helps enhance the numerical efficiency of Algorithm (7).
- To modify the iterative term z_n to be a proximal point, which is suggested in [37]. This enlarges from $\delta \in (0, 2)$ to $\delta \in (0, 4)$, respectively, which helps improve the numerical results of Algorithm (7).
- To discuss a strong convergence theorem of our proposed algorithm by introducing the viscosity algorithm described in [31].
- To provide a theoretical application to split minimization problems and split feasibility problems, numerical experiments are conducted to demonstrate and compare.

1.3. Organization

This work is constructed as follows. Several theory results are stated in Section 2. We firstly introduce our algorithm and discuss its weak convergence in Section 3. The strong convergence analysis is presented in Section 4. In Section 5, we present an application. Some experiments are conducted to demonstrate the advantages of all compared algorithms in Section 6. Finally, we present a conclusion and discussion, respectively.

2. Preliminaries

Definition 1 ([4]). *Let X be a real Hilbert space. A single-value mapping $F : X \to X$ is named*

(a) nonexpansive if

$$\|Fp - Fy\| \leq \|p - y\|, \ \forall \ p, y \in X,$$

(b) firmly-nonexpansive if

$$\langle Fp - Fy, p - y\rangle \geq \|Fp - Fy\|^2, \ \forall \ p, y \in X.$$

Let $Y : X \to 2^X$ be a set-value maximal monotone operator. The resolvent operator $J_\lambda^Y : X \to X$ related to Y is defined by

$$J_\lambda^Y(p) = (I + \lambda Y)^{-1}(p), \ \forall \ p \in X,$$

where $\lambda > 0$. On the other hand, $Y^{-1}(0) = \{p \in X : 0 \in Yp\}$, the domain of F is defined by $D(F)$, and the fixed point set of F is denoted by $Fix(F)$; that is, $Fix(F) = \{p \in X : p = Fp\}$.

Lemma 1. *Let $\pi \in [0, 1]$, for all $(p, y) \in X \times X$. Then*

(a) $\|\pi p + (1 - \pi)y\|^2 = \pi\|p\|^2 + (1 - \pi)\|y\|^2 - \pi(1 - \pi)\|p - y\|^2$,

(b) $\|p + y\|^2 \leq \|p\|^2 + 2\langle y, p\rangle + \|y\|^2$.

Lemma 2 ([8]). *Let X_1 and X_2 be real Hilbert spaces, $A : X_1 \to X_2$ be a bounded linear operator with its adjoint A^*, and $\lambda > 0$. Let $Y_2 : X_2 \to 2^{X_2}$ be a set-valued maximal monotone mapping. Define a mapping $F : X_1 \to X_1$ by $Fp = A^*\left(I - J_\lambda^{Y_2}\right)Ap$ for all $p \in X_1$. Then, the following statements hold:*

(a) $\left\| \left(I - J_\lambda^{\Upsilon_2}\right) Ap - \left(I - J_\lambda^{\Upsilon_2}\right) Ay \right\|^2 \leq \langle \digamma p - \digamma y, p - y \rangle$, $\forall\, p, y \in X_1$;
(b) $\|\digamma p - \digamma y\|^2 \leq \|A\|^2 \langle \digamma p - \digamma y, p - y \rangle$, $\forall\, p, y \in X_1$.

Lemma 3 (**Demiclosedness principle** [32]). *Assume that* $\digamma : C \to C$ *is a nonexpansive mapping. Then we obtain the following:*

$$p_n \rightharpoonup p \in C \text{ and } \lim_{n\to\infty} \|p_n - \digamma p_n\| = 0 \Rightarrow p = \digamma p,$$

where C is described in (3).

Lemma 4 ([38,39]). *Let* $\Upsilon : X \to 2^X$ *be a set-valued maximal monotone mapping and* $\lambda > 0$. *Then the following properties hold:*

(a) J_λ^Υ *is a single-valued and firmly nonexpansive mapping;*
(b) $D\left(J_\lambda^\Upsilon\right) = X$ *and* $Fix\left(J_\lambda^\Upsilon\right) = \{p \in D(\Upsilon) : 0 \in \Upsilon p\}$;
(c) $\left\|p - J_\lambda^\Upsilon p\right\| \leq \left\|p - J_\gamma^\Upsilon p\right\|$ *for all* $0 < \lambda < \gamma$ *and* $p \in X$;
(d) $\left(I - J_\lambda^\Upsilon\right)$ *is a firmly nonexpansive mapping.*
(e) For $\Upsilon^{-1}(0) \neq \emptyset$, *one has* $\left\|p - J_\lambda^\Upsilon p\right\|^2 + \left\|J_\lambda^\Upsilon p - \tilde{p}\right\|^2 \leq \|p - \tilde{p}\|^2, \forall\ p \in X$, $\tilde{p} \in \Upsilon^{-1}(0)$ *and*

$$\left\langle p - J_\lambda^\Upsilon p, J_\lambda^\Upsilon p - \tilde{p} \right\rangle \geq 0, \forall\, p \in X, \tilde{p} \in \Upsilon^{-1}(0).$$

Lemma 5 ([8]). *Let* X_1 *and* X_2 *be the two real Hilbert spaces, and* $A : X_1 \to X_2$ *be a bounded linear operator with its adjoint operator* A^*. *Let* $\lambda > 0$, $\gamma > 0$, $\Upsilon_1 : X_1 \to 2^{X_1}$, *and* $\Upsilon_2 : X_2 \to 2^{X_2}$ *be set-valued maximal monotone mappings. For any* $\tilde{p} \in X_1$, *the following results hold:*

(a) Suppose that $\tilde{p}$ *is a solution of the SVIP. Then* $J_\lambda^{\Upsilon_1}\left(\tilde{p} - \gamma A^*\left(I - J_\lambda^{\Upsilon_2}\right) A\tilde{p}\right) = \tilde{p}$.

(b) If $J_\lambda^{\Upsilon_1}\left(\tilde{p} - \gamma A^*\left(I - J_\lambda^{\Upsilon_2}\right) A\tilde{p}\right) = \tilde{p}$ *and the solution set of the SVIP is nonempty, then* $\tilde{p}$ *is a solution of the SVIP.*

Lemma 6 ([40]). *Let* $\{\chi_n\}$ *be a sequence of nonnegative numbers fulfilling*

$$\chi_{n+1} \leq \vartheta_n \chi_n + \omega_n,\ \forall n \in \mathbb{N},$$

where $\{\vartheta_n\}$ *and* $\{\omega_n\}$ *are sequences of nonnegative numbers such that* $\{\vartheta_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty} (\vartheta_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty} \omega_n < \infty$. *Then* $\lim_{n\to\infty} \chi_n$ *exists.*

Lemma 7 ([30]). *Let* $C \subset X$ *be a non-empty set, and* $\{p_n\}$ *be a sequence in X that fulfills the conditions below:*

(a) $\lim_{n\to\infty} \|p_n - p\|$ *exists for all* $p \in C$;
(b) every sequential weak limit sequence of $\{p_n\}$ *is in C.*
Then $\{p_n\}$ *converges weakly to a point in C.*

Lemma 8 ([30]). *Let* $\{q_n\}, \{w_n\}$ *and* $\{e_n\}$ *be real positive sequences such that*

$$q_{n+1} \leq (1 + e_n) q_n + w_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} w_n < \infty$ *and* $\sum_{n=1}^{\infty} e_n < \infty$, *then* $\lim_{n\to\infty} q_n$ *exists.*

Lemma 9 ([30]). *Let* $\{q_n\}$ *and* $\{\hat{\jmath}_n\}$ *be real positive sequences such that*

$$q_{n+1} \leq (1 + \hat{\jmath}_n) q_n + \hat{\jmath}_n q_{n-1},\ n \geq 1.$$

Then $q_{n+1} \leq \kappa \prod_{i=1}^{n}(1+2\hat{f}_i)$, *where* $\kappa = \max\{q_1, q_2\}$. *Additionally, if* $\sum_{n=1}^{\infty} \hat{f}_n < \infty$, *then* q_n *is bounded.*

Let C be a nonempty closed and convex subset of X. Then the orthogonal projection of p onto C is defined by

$$P_C y = \operatorname{argmin}\{\|u - y\| \mid y \in C\}, \ \forall\, u \in X.$$

Lemma 10 ([19]). *Let C be a nonempty closed convex subset of a real Hilbert space X and let P_C be the metric projection from X onto C. Then the following statements hold:*

(a) $\langle u - P_C u, y - P_C u\rangle \leq 0$ *for all* $u \in X$ *and* $y \in C$;

(b) $\|P_C u - P_C y\| \leq \|u - y\|$ *for all* $u,\ y \in X$;

(c) $\|P_C u - P_C y\|^2 \leq \langle u - y, P_C u - P_C y\rangle$ *for all* $u,\ y \in X$.

Lemma 11 ([41]). *Suppose that* $\{q_n\}_{n=1}^{\infty}$ *is a sequence of nonnegative real numbers,* $\{e_n\} \subset (0,1)$ *such that* $\sum_{n=1}^{\infty} e_n = \infty$, *and let* $\{w_n\}$ *be a sequence of real numbers. Let*

$$q_{n+1} \leq (1 - e_n)q_n + e_n w_n, \ \forall\, n \geq 1.$$

In addition, suppose that for any subsequence $\{n_k\}$ *of integers fulfilling* $\liminf_{n\to\infty}(q_{n_k+1} - q_{n_k}) \geq 0$, *it holds that* $\limsup_{n\to\infty} w_{n_k} \leq 0$. *Then it is verified that* $\lim_{n\to\infty} q_n = 0$.

3. Weak Convergence

In this section, we propose an inertial proximal algorithm for solving Problem (1), and establish its weak convergence. First, we introduce the following assumptions for the convergence analysis before presenting our algorithmic frame.

(T1) Let the solution set of the split variational inclusion problem be nonempty, that is, $S \neq \emptyset$.

(T2) Let X_1 and X_2 be real Hilbert spaces, and let $A : X_1 \to X_2$ be a bounded linear operator with its adjoint A^*. Let $Y_1 : X_1 \to 2^{X_1}$ and $Y_2 : X_2 \to 2^{X_2}$ be set-valued maximal monotone operators.

(T3) Suppose that both X_1 and X_2 are real Hilbert spaces, and let $A : X_1 \to X_2$ be a bounded linear operator with its adjoint A^*. Now we define the following function

$$F(p) = \frac{1}{2}\left\|\left(I - J_\lambda^{Y_2}\right)Ap\right\|^2.$$

As shown in [23,42], its gradient is the following.

$$\nabla F(p) = A^*\left(I - J_\lambda^{Y_2}\right)Ap.$$

As described in [43], the gradient ∇F is an inverse strongly monotone operator, so it is Lipschitzian.

Now, our proposed algorithm reads as follows (Algorithm 1).

Algorithm 1: An inertial splitting algorithm for solving the SVIP

Step 0: Given $\lambda_1 > 0$, $\alpha \in (0,1)$,$\delta \in (0,\hat{\rho})$, $\hat{\rho} \in (0,4)$. Initialize $p_0,\ p_1 \in X_1$, $\lambda \in (0,\infty)$, $\{\vartheta_n\}$ satisfying Lemma 6.

Iterative steps: Update p_{n+1} via

Step 1: Given p_{n-1},p_n $(n \geq 1)$ satisfy

$$\sum_{n=1}^{\infty} \theta_n < \infty \text{ and } 0 \leq \theta_n. \tag{9}$$

Step 2: Compute

$$\begin{aligned} \Lambda_n &= p_n + \theta_n(p_n - p_{n-1}), \\ y_n &= \Lambda_n - \lambda_n \nabla F(\Lambda_n), \end{aligned} \tag{10}$$

Step 3: Update

$$z_n = 2J_\lambda^{Y_1}(y_n);\ \ p_{n+1} = (1-\alpha)\Lambda_n + \alpha(z_n - y_n). \tag{11}$$

Step 4: Update the stepsize λ_n via

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, \vartheta_n\lambda_n\right\}, & \text{if } \|\nabla F(\Lambda_n)\| \neq 0, \\ \vartheta_n\lambda_n, & \text{otherwise}, \end{cases} \tag{12}$$

Step 5: Set $n := n+1$ and go to Step 1.

Next, we illustrate that our given stepsize rule is well-defined.

Lemma 12. *Let the sequence $\{\lambda_n\}$ be created by Algorithm 1. Then we attain $\lim_{n\to\infty} \lambda_n = \hat{\lambda}$, $\hat{\lambda} \geq \min\left\{\frac{\delta}{2\|A\|^2}, \lambda_1\right\}$, where λ_1 is an initial stepsize.*

Proof. Since the mapping $J_\lambda^{Y_2}$ is firmly nonexpansive, if we suppose that $\nabla F(\Lambda_n) \neq 0$, we obtain

$$\begin{aligned} \|\nabla F(\Lambda_n)\|^2 &= \langle \nabla F(\Lambda_n), \nabla F(\Lambda_n)\rangle \\ &= \left\langle A^*\left(I - J_\lambda^{Y_2}\right)A\Lambda_n, A^*\left(I - J_\lambda^{Y_2}\right)A\Lambda_n \right\rangle \\ &= \left\langle \left(I - J_\lambda^{Y_2}\right)A\Lambda_n, AA^*\left(I - J_\lambda^{Y_2}\right)A\Lambda_n \right\rangle \\ &\leq \|A^*A\| \left\langle \left(I - J_\lambda^{Y_2}\right)A\Lambda_n, \left(I - J_\lambda^{Y_2}\right)A\Lambda_n \right\rangle \\ &\leq \|A^*A\| \left\| \left(I - J_\lambda^{Y_2}\right)A\Lambda_n \right\|^2 \\ &= 2\|A^*A\| F(\Lambda_n). \end{aligned}$$

After that arrangement, we see that

$$\frac{F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2} \geq \frac{1}{2\|A^*A\|},$$

which further implies that

$$\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2} \geq \frac{\delta}{2\|A\|^2}. \tag{13}$$

This together with $\vartheta_n \geq 1$ and (12), we obtain

$$\lambda_{n+1} = \min\left\{\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, \vartheta_n\lambda_n\right\} \geq \min\left\{\frac{\delta}{2\|A\|^2}, \lambda_n\right\}.$$

By induction, the sequence $\{\lambda_n\}$ is of a lower bound $\min\left\{\frac{\delta}{2\|A\|^2}, \lambda_1\right\}$. Using (12) again, we directly arrive at $\lambda_{n+1} \leq \vartheta_n \lambda_n$. From Lemma 6 with $\omega_n = 0$, the limit $\lim_{n\to\infty} \lambda_n$ exists and we denote by $\lim_{n\to\infty} \lambda_n = \hat{\lambda}$. Since $\min\left\{\frac{\delta}{2\|A\|^2}, \lambda_1\right\} > 0$, then $\hat{\lambda} > 0$. □

According to Lemma 12, we discuss the following weak convergence of our proposed algorithm.

Theorem 1. *Suppose that Assumptions (T1–T3) holds. Then the sequence $\{p_n\}$ created by Algorithm 1 converges weakly to a solution $p \in S$.*

Proof. Let $p \in S$. Then $p \in Y_1^{-1}(0)$ and $Ap \in Y_2^{-1}(0)$. Therefore, we attain $J_\lambda^{Y_2} Ap = Ap$. By Lemma 2(a), we see that

$$\begin{aligned}
\langle \Lambda_n - p, \nabla F(\Lambda_n)\rangle &= \left\langle \Lambda_n - p, A^*\left(I - J_\lambda^{Y_2}\right) A\Lambda_n \right\rangle \\
&= \left\langle A\Lambda_n - Ap, \left(I - J_\lambda^{Y_2}\right) A\Lambda_n \right\rangle \\
&= \left\langle A\Lambda_n - Ap, \left(I - J_\lambda^{Y_2}\right) A\Lambda_n - \left(I - J_\lambda^{Y_2}\right) Ap \right\rangle \\
&\geq \left\| J_\lambda^{Y_2} A\Lambda_n - A\Lambda_n \right\|^2 \\
&= 2F(\Lambda_n).
\end{aligned} \tag{14}$$

Now, from (11) and $y_n = \Lambda_n - \lambda_n \nabla F(\Lambda_n)$, we have

$$\begin{aligned}
p_{n+1} &= (1-\alpha)\Lambda_n + \alpha(z_n - y_n) \\
&= (1-\alpha)\Lambda_n + \alpha\left[2J_\lambda^{Y_1}(\Lambda_n - \lambda_n \nabla F(\Lambda_n)) - (\Lambda_n - \lambda_n \nabla F(\Lambda_n))\right] \\
&= (1-\alpha)\Lambda_n + 2\alpha J_\lambda^{Y_1} y_n - \alpha y_n \\
&= (1-\alpha)\Lambda_n + \alpha\left(2J_\lambda^{Y_1} y_n - y_n\right).
\end{aligned} \tag{15}$$

Using (12) and (14), we see that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\Lambda_n - \nabla F(\Lambda_n) - p\|^2 \\
&= \|\Lambda_n - p\|^2 + \lambda_n^2 \|\nabla F(\Lambda_n)\|^2 - 2\lambda_n \langle \Lambda_n - p, \nabla F(\Lambda_n)\rangle \\
&\leq \|\Lambda_n - p\|^2 + \lambda_n^2 \|\nabla F(\Lambda_n)\|^2 - 4\lambda_n F(\Lambda_n) \\
&= \|\Lambda_n - p\|^2 + \frac{\lambda_n^2}{\lambda_{n+1}} \lambda_{n+1} \|\nabla F(\Lambda_n)\|^2 - 4\lambda_n F(\Lambda_n) \\
&\leq \|\Lambda_n - p\|^2 + \frac{\lambda_n^2}{\lambda_{n+1}} \delta F(\Lambda_n) - 4\lambda_n F(\Lambda_n) \\
&= \|\Lambda_n - p\|^2 - \lambda_n\left(4 - \frac{\lambda_n}{\lambda_{n+1}}\delta\right) F(\Lambda_n).
\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(4 - \frac{\delta\lambda_n}{\lambda_{n+1}}\right) = 4 - \delta > 4 - \hat{\rho},$$

where $0 < \delta < \hat{\rho}$ and $\hat{\rho} \in (0, 4)$. Thus, $\exists N \geq 0$, $\forall n \geq N$, $4 - \frac{\delta\lambda_n}{\lambda_{n+1}} > 4 - \hat{\rho}$. Thereby, $\forall n \geq N$,

$$\|y_n - p\|^2 \leq \|\Lambda_n - p\|^2 - \lambda_n(4 - \hat{\rho}) F(\Lambda_n). \tag{16}$$

By a similar analysis of (14), we directly obtain

$$\left\langle y_n - J_\lambda^{Y_1} y_n, y_n - p \right\rangle \geq \left\| y_n - J_\lambda^{Y_1} y_n \right\|^2. \tag{17}$$

By applying (17), we show that

$$\begin{aligned}\left\|2J_\lambda^{\Upsilon_1}y_n - y_n - p\right\|^2 &= \left\|2J_\lambda^{\Upsilon_1}y_n - y_n + y_n - y_n - p\right\|^2\\ &= \left\|2\left(J_\lambda^{\Upsilon_1}y_n - y_n\right) + (y_n - p)\right\|^2\\ &= 4\left(\left\|J_\lambda^{\Upsilon_1}y_n - y_n\right\|^2 - \left\langle y_n - J_\lambda^{\Upsilon_1}y_n, y_n - p\right\rangle\right) + \|y_n - p\|^2\\ &\leq 4\left(\left\|y_n - J_\lambda^{\Upsilon_1}y_n\right\|^2 - \left\|y_n - J_\lambda^{\Upsilon_1}y_n\right\|^2\right) + \|y_n - p\|^2\\ &= \|y_n - p\|^2.\end{aligned} \tag{18}$$

This, working with (15), (16), (18) and Lemma 1(a), yields that $\forall n \geq N$,

$$\begin{aligned}\|p_{n+1} - p\|^2 &= \left\|(1-\alpha)K_n + \alpha\left(2J_\lambda^{\Upsilon_1}y_n - y_n\right) - p\right\|^2\\ &= (1-\alpha)\|\Lambda_n - p\|^2 + \alpha\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - p\right\|^2\\ &- \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - p - \Lambda_n + p\right\|^2\\ &\leq (1-\alpha)\|\Lambda_n - p\|^2 + \alpha\|y_n - p\|^2 - \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2\\ &\leq (1-\alpha)\|\Lambda_n - p\|^2 + \alpha(\|\Lambda_n - p\|^2 - \lambda_n(4-\hat{\rho})F(\Lambda_n))\\ &- \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2\\ &= (1-\alpha)\|\Lambda_n - p\|^2 + \alpha\|\Lambda_n - p\|^2 - \alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) - \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2\\ &\leq \|\Lambda_n - p\|^2 - \alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) - \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2.\end{aligned} \tag{19}$$

By (19), $0 < \alpha < 1$, $0 < \hat{\rho} < 4$ and the inertial step Λ_n (10), we further have

$$\begin{aligned}\|p_{n+1} - p\| &\leq \|\Lambda_n - p\|\\ &= \|p_n + \theta_n(p_n - p_{n-1}) - p\|\\ &\leq \|p_n - p\| + \theta_n\|p_n - p_{n-1}\|\\ &\leq \|p_n - p\| + \theta_n(\|p_n - p\| + \|p_{n-1} - p\|)\\ &= (1+\theta_n)\|p_n - p\| + \theta_n\|p_{n-1} - p\|.\end{aligned} \tag{20}$$

From Lemma 9, it yields that

$$\|p_{n+1} - p\| \leq \kappa\prod_{i=1}^{n}(1+2\theta_i), \tag{21}$$

where $\kappa = \max\{\|p_1 - p\|, \|p_2 - p\|\}$. Combining $\sum_{n=1}^{\infty}\theta_n < +\infty$ and Lemma 9, we deduce that the sequence $\{p_n - p\}$ is bounded. Thus, $\sum_{n=1}^{\infty}\theta_n\|p_n - p_{n-1}\| < +\infty$. Since $\sum_{n=1}^{\infty}\theta_n < +\infty$, $\sum_{n=1}^{\infty}\theta_n\|p_{n-1} - p\| < +\infty$ and by Lemma 8 and (20) again, we show that $\lim_{n\to\infty}\|p_n - p\|$ exists. From the definition of the inertial step Λ_n (10), we obtain

$$\begin{aligned}\|\Lambda_n - p\|^2 &= \|p_n + \theta_n(p_n - p_{n-1}) - p\|^2\\ &= \|p_n - p\|^2 + 2\theta_n\langle p_n - p, p_n - p_{n-1}\rangle + \theta_n^2\|p_n - p_{n-1}\|^2\\ &\leq \|p_n - p\|^2 + 2\theta_n\|p_n - p\|\|p_n - p_{n-1}\| + \theta_n^2\|p_n - p_{n-1}\|^2.\end{aligned} \tag{22}$$

After arrangement by substituting (22) into (19), it follows that $\forall n \geq N$,

$$\begin{aligned}\|p_{n+1} - p\|^2 \leq \|p_n - p\|^2 &+ 2\theta_n\|p_n - p\|\|p_n - p_{n-1}\| + \theta_n^2\|p_n - p_{n-1}\|^2\\ &- \alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) - \alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2.\end{aligned} \tag{23}$$

Since $\lim_{n\to\infty}\|p_n - p\|$ exists, $\sum_{n=1}^{\infty}\theta_n\|p_n - p_{n-1}\| < +\infty$, and by (23), one has

$$\lim_{n\to\infty}\alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) = 0 \text{ and } \lim_{n\to\infty}\alpha(1-\alpha)\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\|^2 = 0.$$

Since $\alpha \in (0,1)$, $4-\hat{\rho} > 0$ and $\lim_{n\to\infty}\lambda_n = \hat{\lambda} > 0$, we further show that

$$\lim_{n\to\infty}F(\Lambda_n) = 0 \text{ and } \lim_{n\to\infty}\left\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\right\| = 0. \tag{24}$$

Since $\|\nabla F(\Lambda_n)\|^2 \le 2\|A\|^2 F(\Lambda_n)$, it follows from (24) that

$$\lim_{n\to\infty}\|\nabla F(\Lambda_n)\| = 0.$$

Since $F(\Lambda_n) = \frac{1}{2}\|(I - J_\lambda^{\Upsilon_2})A\Lambda_n\|^2$, and $\lim_{n\to\infty}F(\Lambda_n) = 0$, we have

$$\lim_{n\to\infty}\left\|A\Lambda_n - J_\lambda^{\Upsilon_2}A\Lambda_n\right\| = 0. \tag{25}$$

By $\lim_{n\to\infty}\|\nabla F(\Lambda_n)\| = 0$, $\lim_{n\to\infty}\lambda_n = \hat{\lambda} > 0$ and $y_n = \Lambda_n - \lambda_n\nabla F(\Lambda_n)$, it easily obtains

$$\lim_{n\to\infty}\|y_n - \Lambda_n\| = 0. \tag{26}$$

Since $\sum_{n=1}^{\infty}\theta_n\|p_n - p_{n-1}\| < +\infty$, we have $\lim_{n\to\infty}\theta_n\|p_n - p_{n-1}\| = 0$, by the definition of Λ_n, we have $\|p_n - \Lambda_n\| = \theta_n\|p_n - p_{n-1}\|$, we further obtain

$$\lim_{n\to\infty}\|p_n - \Lambda_n\| = 0. \tag{27}$$

Using (26) and (27), we also see that

$$\begin{aligned}\|y_n - p_n\| &\le \|y_n - \Lambda_n\| + \|\Lambda_n - p_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{28}$$

Combining (24) and (26), one has

$$\begin{aligned}\left\|J_\lambda^{\Upsilon_1}y_n - y_n\right\| &= \left\|J_\lambda^{\Upsilon_1}(y_n) - \tfrac{y_n}{2} - \tfrac{\Lambda_n}{2} + \tfrac{\Lambda_n}{2} - \tfrac{y_n}{2}\right\| \\ &\le \left\|J_\lambda^{\Upsilon_1}y_n - \tfrac{y_n}{2} - \tfrac{\Lambda_n}{2}\right\| + \left\|\tfrac{\Lambda_n}{2} - \tfrac{y_n}{2}\right\| \\ &= \tfrac{1}{2}\left\|2J_\lambda^{\Upsilon_1}y_n - y_n - \Lambda_n\right\| + \tfrac{1}{2}\|\Lambda_n - y_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{29}$$

This, combining with (26), (28) and (29), deduces that

$$\begin{aligned}\left\|J_\lambda^{\Upsilon_1}\Lambda_n - p_n\right\| &= \left\|J_\lambda^{\Upsilon_1}\Lambda_n - J_\lambda^{\Upsilon_1}y_n + J_\lambda^{\Upsilon_1}y_n - p_n - y_n + y_n\right\| \\ &\le \left\|J_\lambda^{\Upsilon_1}\Lambda_n - J_\lambda^{\Upsilon_1}y_n\right\| + \left\|J_\lambda^{\Upsilon_1}y_n - y_n\right\| + \|y_n - p_n\| \\ &\le \|\Lambda_n - y_n\| + \left\|J_\lambda^{\Upsilon_1}y_n - y_n\right\| + \|y_n - p_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{30}$$

Combining this with (27), it yields that

$$\begin{aligned}\left\|J_\lambda^{\Upsilon_1}\Lambda_n - \Lambda_n\right\| &= \left\|\Lambda_n - p_n + p_n - J_\lambda^{\Upsilon_1}\Lambda_n\right\| \\ &= \|\Lambda_n - p_n\| + \left\|p_n - J_\lambda^{\Upsilon_1}\Lambda_n\right\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{31}$$

Using (25) and (27), we see that

$$\begin{aligned}\left\|Ap_n - J_\lambda^{\Upsilon_2}Ap_n\right\| &= \left\|Ap_n - J_\lambda^{\Upsilon_2}Ap_n - A\Lambda_n + A\Lambda_n - J_\lambda^{\Upsilon_2}A\Lambda_n + J_\lambda^{\Upsilon_2}A\Lambda_n\right\| \\ &\le \left\|\left(I - J_\lambda^{\Upsilon_2}\right)Ap_n - \left(I - J_\lambda^{\Upsilon_2}\right)A\Lambda_n\right\| + \left\|A\Lambda_n - J_\lambda^{\Upsilon_2}A\Lambda_n\right\| \\ &\le \|Ap_n - A\Lambda_n\| + \left\|A\Lambda_n - J_\lambda^{\Upsilon_2}A\Lambda_n\right\| \\ &\le \|A\|\|p_n - \Lambda_n\| + \left\|A\Lambda_n - J_\lambda^{\Upsilon_2}A\Lambda_n\right\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{32}$$

From (27) and (31), one sees that

$$\begin{aligned}\left\|p_n - J_\lambda^{\Upsilon_1}p_n\right\| &= \left\|p_n - \Lambda_n + \Lambda_n - J_\lambda^{\Upsilon_1}\Lambda_n + J_\lambda^{\Upsilon_1}\Lambda_n - J_\lambda^{\Upsilon_1}p_n\right\| \\ &\le \|p_n - \Lambda_n\| + \left\|\Lambda_n - J_\lambda^{\Upsilon_1}\Lambda_n\right\| + \left\|J_\lambda^{\Upsilon_1}\Lambda_n - J_\lambda^{\Upsilon_1}p_n\right\| \\ &\le 2\|p_n - \Lambda_n\| + \left\|\Lambda_n - J_\lambda^{\Upsilon_1}\Lambda_n\right\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{33}$$

Let p^* be a weak cluster point of the sequence $\{p_n\}$. That is, there is a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \rightharpoonup p^* \in X_1$. We know that A is a bounded linear operator, so we obtain $Ap_{n_i} \rightharpoonup Ap^*$. Using (32) and (33), Lemma 3 and Lemma 4(b), we show that $p^* \in S$. Using Lemma 7 again, we see that $p_n \rightharpoonup z \in S$. □

4. Strong Convergence

In this section, we propose a viscosity-type inertial splitting algorithm for solving the SVIP, and we establish its strong convergence in real Hilbert space. Now, we present the following assumptions for the convergence analysis.

(U_1) $f : X \to X$ is a contraction mapping with constant $\hat{\rho} \in (0,1)$.

(U_2) $\varrho_n \in (0,1)$, $\beta_n \in (0,1)$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$ and $\lim_{n\to\infty}\frac{\varrho_n}{\beta_n} = 0$. Our strongly convergent algorithm is summarized in the following.

Remark 1. *In Algorithm 2, the inertial parameter θ_n is selected as*

$$\theta_n = \begin{cases} \min\left\{\frac{\varrho_n}{\|p_n - p_{n-1}\|}, \beta\right\} & \text{if } p_n \neq p_{n-1}, \\ \beta, & \text{otherwise,} \end{cases}$$

Remark 2. *By Algorithm 2, it is verified that $\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|p_n - p_{n-1}\| = 0$. Indeed, we derive $\theta_n\|p_n - p_{n-1}\| \le \varrho_n$ for all $n \ge 1$, which, together with $\lim_{n\to\infty}\frac{\varrho_n}{\beta_n} = 0$, yields that*

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|p_n - p_{n-1}\| \le \lim_{n\to\infty}\frac{\varrho_n}{\beta_n} = 0.$$

Theorem 2. *Assume that Conditions (U_1)–(U_2) hold. Then the sequence $\{p_n\}$ created by Algorithm 2 converges strongly to a point $p^* \in S$, where $p^* = P_S o f(p^*)$.*

Algorithm 2: An inertial viscosity-type splitting algorithm for solving the SVIP

Step 0: Given $\lambda_1 > 0$, $\alpha \in (0,1)$,$\delta \in (0,\hat{\rho})$, $\hat{\rho} \in (0,4)$. Initialize $p_0,\ p_1 \in X_1$, $\lambda \in (0,\infty)$, $\{\vartheta_n\}$ satisfying Lemma 6.

Iterative steps: Update p_{n+1} via

Step 1: Given p_{n-1},p_n $(n \geq 1)$ satisfy

$$\sum_{n=1}^{\infty} \theta_n < \infty \text{ and } 0 \leq \theta_n. \tag{34}$$

Step 2: Compute

$$\begin{aligned} \Lambda_n &= p_n + \theta_n(p_n - p_{n-1}), \\ y_n &= \Lambda_n - \lambda_n \nabla F(\Lambda_n), \end{aligned} \tag{35}$$

Step 3: Update

$$\begin{aligned} z_n &= 2J_{\lambda}^{Y_1}(y_n); \\ d_n &= (1-\alpha)\Lambda_n + \alpha(z_n - y_n), \\ p_{n+1} &= \beta_n f(p_n) + (1-\beta_n)d_n. \end{aligned} \tag{36}$$

Step 4: Update the stepsize λ_n via

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, \vartheta_n\lambda_n\right\}, & \text{if } \|\nabla F(\Lambda_n)\| \neq 0, \\ \vartheta_n\lambda_n, & \text{otherwise}, \end{cases} \tag{37}$$

Step 5: Set $n := n+1$ and go to Step 1.

Proof. According to the expression (19), we directly obtain $\forall n \geq N$,

$$\|d_n - p^*\| \leq \|\Lambda_n - p^*\|. \tag{38}$$

By the definition of the inertial step Λ_n, one has

$$\begin{aligned} \|\Lambda_n - p^*\| &= \|p_n + \theta_n(p_n - p_{n-1}) - p^*\| \\ &\leq \|p_n - p^*\| + \beta_n \cdot \tfrac{\theta_n}{\beta_n}\|p_n - p_{n-1}\|. \end{aligned} \tag{39}$$

According to Remark 2, there is a constant $M > 0$ such that

$$\frac{\theta_n}{\beta_n}\|p_n - p_{n-1}\| \leq M,\ \forall n \geq 1. \tag{40}$$

Combining (38), (39) and (40), we show that $\forall n \geq N$,

$$\|d_n - p^*\| \leq \|\Lambda_n - p^*\| \leq \|p_n - p^*\| + \beta_n M \tag{41}$$

This verifies that $\forall n \geq N$,

$$\begin{aligned} \|p_{n+1} - p^*\| &= \|\beta_n f(p_n) + (1-\beta_n)d_n - p^*\| \\ &\leq \beta_n\|f(p_n) - f(p^*)\| + \beta_n\|f(p^*) - p^*\| + (1-\beta_n)\|d_n - p^*\| \\ &\leq [1 - \beta_n(1-\rho)]\|p_n - p^*\| + \beta_n(1-\rho)\tfrac{\|f(p^*)-p^*\|+M}{1-\rho} \\ &\leq \max\left\{\|p_n - p^*\|, \tfrac{\|f(p^*)-p^*\|+M}{1-\rho}\right\} \\ &\leq \cdots \leq \max\left\{\|p_n - p^*\|, \tfrac{\|f(p^*)-p^*\|+M}{1-\rho}\right\}. \end{aligned} \tag{42}$$

This result shows that the sequence $\{p_n\}$ is bounded. It is also shown that the sequences $\{\Lambda_n\}$ and $\{f(p_n)\}$ are bounded. Additionally, we have

$$\begin{aligned}\|\Lambda_n - p^*\|^2 &\leq (\|p_n - p^*\| + \beta_n M)^2 \\ &= \|p_n - p^*\|^2 + \beta_n(2M\|p_n - p^*\| + \beta_n M^2) \\ &\leq \|p_n - p^*\|^2 + \beta_n M_1,\end{aligned} \tag{43}$$

for some $M_1 > 0$, this, together with (19), further verifies that $\forall n \geq N$,

$$\begin{aligned}\|p_{n+1} - p^*\|^2 &= \|\beta_n(f(p_n) - p^*) + (1-\beta_n)(d_n - p^*)\|^2 \\ &\leq \beta_n\|f(p_n) - p^*\|^2 + (1-\beta_n)\|d_n - p^*\|^2 \\ &\leq \beta_n\|f(p_n) - p^*\|^2 + \|d_n - p^*\|^2 \\ &\leq \beta_n\|f(p_n) - p^*\|^2 + \|\Lambda_n - p^*\|^2 \\ &- \alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) - \alpha(1-\alpha)\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\|^2 \\ &\leq \beta_n\|f(p_n) - p^*\|^2 + \|p_n - p^*\|^2 + \beta_n M_1 \\ &- \alpha(4-\hat{\rho})\lambda_n F(\Lambda_n) - \alpha(1-\alpha)\|2J_\lambda^{\Upsilon_1}(y_n) - y_n - \Lambda_n\|^2\end{aligned} \tag{44}$$

On the other hand, by the inertial step Λ_n, we have

$$\begin{aligned}\|\Lambda_n - p^*\|^2 &= \|p_n + \theta_n(p_n - p_{n-1}) - p^*\|^2 \\ &= \|p_n - p^*\|^2 + 2\theta_n\langle p_n - p^*, p_n - p_{n-1}\rangle + \theta_n^2\|p_n - p_{n-1}\|^2 \\ &\leq \|p_n - p^*\|^2 + 3Q\theta_n\|p_n - p_{n-1}\|,\end{aligned} \tag{45}$$

where $Q := \sup_{n\in N}\{\|p_n - p^*\|, \theta_n\|p_n - p_{n-1}\|\} > 0$. This shows that $\forall n \geq N$,

$$\begin{aligned}\|p_{n+1} - p^*\|^2 &= \|\beta_n f(p_n) + (1-\beta_n)d_n - p^*\|^2 \\ &= \|\beta_n(f(p_n) - f(p^*)) + (1-\beta_n)(d_n - p^*) + \beta_n(f(p^*) - p^*)\|^2 \\ &\leq \|\beta_n(f(p_n) - f(p^*)) + (1-\beta_n)(d_n - p^*)\|^2 + 2\beta_n\langle f(p^*) - p^*, p_{n+1} - p^*\rangle \\ &\leq \beta_n\|f(p_n) - f(p^*)\|^2 + (1-\beta_n)\|d_n - p^*\|^2 + 2\beta_n\langle f(p^*) - p^*, p_{n+1} - p^*\rangle \\ &\leq \beta_n\|f(p_n) - f(p^*)\|^2 + (1-\beta_n)\|\Lambda_n - p^*\|^2 + 2\beta_n\langle f(p^*) - p^*, p_{n+1} - p^*\rangle \\ &\leq \beta_n\hat{\rho}\|p_n - p^*\|^2 + (1-\beta_n)\|p_n - p^*\|^2 + 2\beta_n\langle f(p^*) - p^*, p_{n+1} - p^*\rangle \\ &+ 3Q\theta_n\|p_n - p_{n-1}\| \\ &= [1 - (1-\hat{\rho})\beta_n]\|p_n - p^*\|^2 \\ &+ (1-\hat{\rho})\beta_n\left[\tfrac{2}{1-\hat{\rho}}\langle f(p^*) - p^*, p_{n+1} - p^*\rangle + \tfrac{3Q\theta_n}{(1-\hat{\rho})\beta_n}\|p_n - p_{n-1}\|\right].\end{aligned} \tag{46}$$

Next, we mainly prove that the sequence $\{\|p_n - p^*\|\}$ converges to zero. Actually, by Lemma 11, it suffices to verify that $\limsup_{k\to\infty}\langle f(p^*) - p^*, p_{n_k+1} - p^*\rangle \leq 0$ for every subsequence $\{\|p_{n_k} - p^*\|\}$ of $\{\|p_n - p^*\|\}$ such that

$$\liminf_{k\to\infty}(\|p_{n_k+1} - p^*\| - \|p_{n_k} - p^*\|) \geq 0. \tag{47}$$

Since $\lim_{n\to\infty}\beta_n = 0$ in (46), we have

$$\limsup_{n\to\infty}(\|p_{n+1} - p^*\| - \|p_n - p^*\|) \leq 0.$$

After arrangement of the inequality (47) by the above result, one has

$$\lim_{k\to\infty}(\|p_{n_k+1} - p^*\| - \|p_{n_k} - p^*\|) = 0. \tag{48}$$

Combining (44) and (48), we know that

$$\lim_{k\to\infty}(\alpha(4-\hat{\rho})\lambda_{n_k}F(\Lambda_{n_k})) = 0, \lim_{k\to\infty}(\alpha(1-\alpha)\|2J_\lambda^{\Upsilon_1}(y_{n_k}) - y_{n_k} - \Lambda_{n_k}\|^2) = 0. \tag{49}$$

Since $\alpha \in (0,1)$, $\hat{\rho} \in (0,4)$ and $\lim_{k\to\infty} \lambda_{n_k} = \hat{\lambda} > 0$, we further know that

$$\lim_{k\to\infty} F(\Lambda_{n_k}) = 0, \lim_{k\to\infty} \|2J_\lambda^{\Upsilon_1}(y_{n_k}) - y_{n_k} - \Lambda_{n_k}\|^2 = 0.$$

That is,

$$\lim_{k\to\infty} \|2J_\lambda^{\Upsilon_1}(y_{n_k}) - y_{n_k} - \Lambda_{n_k}\| = 0. \tag{50}$$

Since $\sum_{k=1}^{\infty} \theta_{n_k}\|p_{n_k} - p_{n_k-1}\| < +\infty$, we have $\lim_{k\to\infty} \theta_{n_k}\|p_{n_k} - p_{n_k-1}\| = 0$, and by the inertial step Λ_{n_k}, one has

$$\begin{aligned} \|\Lambda_{n_k} - P_{n_k}\| &= \|\theta_{n_k}(p_{n_k} - p_{n_k-1})\| \\ &= \theta_{n_k}\|p_{n_k} - p_{n_k-1}\| \\ &\to 0 \ \ as \ \ k \to \infty. \end{aligned} \tag{51}$$

Combining (50) and (51), it is verified that

$$\begin{aligned} \|d_{n_k} - p_{n_k}\| &= \|(1-\alpha)\Lambda_{n_k} + \alpha(z_{n_n} - y_{n_n}) - p_{n_k}\| \\ &= \|\Lambda_{n_k} - p_{n_k} + \alpha(z_{n_n} - y_{n_n} - \Lambda_{n_k})\| \\ &\le \|\Lambda_{n_k} - p_{n_k}\| + \alpha\|z_{n_n} - y_{n_n} - \Lambda_{n_k}\| \\ &= \|\Lambda_{n_k} - p_{n_k}\| + \alpha\|2J_\lambda^{\Upsilon_1}(y_{n_k}) - y_{n_k} - \Lambda_{n_k}\| \\ &\to 0 \ \ as \ \ k \to \infty, \end{aligned} \tag{52}$$

where $z_{n_k} = 2J_\lambda^{\Upsilon_1}(y_{n_k})$. Since $\lim_{k\to\infty} \beta_{n_k} = 0$ again, it follows that

$$\begin{aligned} \|p_{n_k+1} - p_{n_k}\| &= \|\beta_{n_k} f(p_{n_k}) + (1-\beta_{n_k})d_{n_k} - p_{n_k}\| \\ &= \|\beta_{n_k}(f(p_{n_k}) - p_{n_k}) + (1-\beta_{n_k})(d_{n_k} - p_{n_k})\| \\ &\le \beta_{n_k}\|f(p_{n_k}) - p_{n_k}\| + (1-\beta_{n_k})\|d_{n_k} - p_{n_k}\| \\ &\le \beta_{n_k}\|f(p_{n_k}) - p_{n_k}\| + \|d_{n_k} - p_{n_k}\| \\ &\to 0 \ \ as \ \ k \to \infty. \end{aligned} \tag{53}$$

Since the sequence $\{p_{n_k}\}$ is bounded, we suppose that $p_{n_k} \rightharpoonup z$, and thereby $p_{n_k+1} \rightharpoonup z$. According to (26), (27), (29) and Lemma 11, we know that $z \in S$. It needs to be shown that $p^* \in S$. We now use Lemma 11 to (46) and we derive that $\lim_{n\to\infty} \|p_n - p^*\| = 0$. □

5. Theoretical Application

This section mainly extends Algorithm 1 to solve split minimization problems.

5.1. Split Minimization Problem

We present the following *split minimization problem* (SMP):

$$\text{Find } \tilde{p} \in X_1 \text{ such that } \tilde{p} \in \operatorname*{argmin}_{x\in X_1} F(p) \text{ and } A\tilde{p} \in \operatorname*{argmin}_{y\in X_2} G(y),$$

where $F : X_1 \to \mathbb{R}$ and $G : X_2 \to \mathbb{R}$ are proper, lower semi-continuous convex functions (lsc). We also use S denote the solution set of the SMP. In a real Hilbert space X, we define the proximal operator of F by

$$\text{prox}_{\lambda,F}(y) = \operatorname*{argmin}_{u\in X_1}\left\{F(u) + \frac{1}{2\lambda}\|u - y\|^2\right\}, \lambda > 0, \ \forall u \in X.$$

Recall that

$$\text{prox}_{\lambda,F}(y) = (I + \lambda\partial F)^{-1}(y) = J_\lambda^{\partial F}(y),$$

where ∂F is the subdifferential of F given by

$$\partial F(y) = \{\tilde{p} \in X : F(y) + \langle u - y, \tilde{p} \rangle \leq F(u),\ \forall u \in X\}.$$

The reference [23] suggests that ∂F is a maximal monotone operator and $\text{prox}_{\lambda,F}$ is firmly nonexpansive.

Based on the above theory, we obtain the following corollary.

Corollary 1. *Suppose that $S \neq \emptyset$, $\lambda_1 > 0$, $\alpha \in (0,1)$, $\delta \in (0, \hat{\rho}) \subset (0,4)$, θ_n satisfies (9). The sequence $\{\vartheta_n\}$ satisfies Lemma 6. Let $p_0,\ p_1 \in X_1$ and $\{p_n\}$ be a sequence generated by*

$$\begin{cases} \Lambda_n = p_n + \theta_n(p_n - p_{n-1}), \\ y_n = \Lambda_n - \lambda_n A^*\Big(I - prox_{\lambda,G}\Big)A\Lambda_n, \\ z_n = 2prox_{\lambda,F}(y_n),\ p_{n+1} = (1-\alpha)\Lambda_n + \alpha(z_n - y_n), \end{cases}$$

where the stepsize λ_n is the following:

$$\lambda_{n+1} = \begin{cases} \min\Big\{\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, \vartheta_n\lambda_n\Big\}, & \text{if } \nabla F(\Lambda_n) \neq 0, \\ \vartheta_n\lambda_n, & \text{otherwise}, \end{cases}$$

then, the iterative sequence $\{p_n\}$ converges weakly to $p \in S$.

5.2. Split Feasibility Problem (SFP)

Let C and Q be non-empty closed convex subsets of real Hilbert spaces X_1 and X_2, respectively. Let $A : X_1 \to X_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is the following:

$$\text{Find } \tilde{p} \in C \text{ such that } A\tilde{p} \in Q$$

and its solution set is denoted by S. For the study of the problem (SFP), many related results can be found in existing references. Here, the reader can refer to [44,45] for more detail about it. Let $F : X \to (-\infty, \infty)$ be a proper lower semi-continuous (lsc) convex function. Then the subdifferential ∂F of F is defined by

$$\partial F(u) = \{\xi \in X : F(u) - F(y) \leq \langle \xi, u - y \rangle,\ \forall\, y \in X\},\ \forall\, u \in X.$$

Let C be non-empty closed convex subset of an infinite dimensional real Hilbert space X, and let δ_C be the indicator function of C, namely,

$$\delta_C(u) = \begin{cases} 0, & \text{if } u \in C, \\ \infty, & \text{otherwise.} \end{cases}$$

In addition, we define the normal cone $N_C(\tilde{\mu})$ of C at $\tilde{\mu}$ by

$$N_C(\tilde{\mu}) = \{\xi \in X : \langle \xi, \tilde{v} - \tilde{\mu} \rangle \leq 0,\ \forall\, \tilde{v} \in C\}.$$

Since δ_C is a proper, lsc and convex function on X, the subdifferential $\partial\delta_C$ of δ_C becomes a maximal monotone operator. Hence, we define the resolvent operator $J_\lambda^{\partial\delta_C}$ of $\partial\delta_C$ for an arbitrary $\lambda > 0$ by

$$J_\lambda^{\partial\delta_C}(u) = (I + \lambda\partial\delta_C)^{-1}u,\ \forall u \in X.$$

We further see that for all $u \in C$,

$$\begin{aligned}\partial\delta_C(u) &= \{\xi \in X : \delta_C(u) - \delta_C(y) \leq \langle \xi, u - y\rangle,\ \forall\, y \in H\}\\ &= \{\xi \in X : \langle \xi, y - u\rangle \leq 0,\ \forall\, y \in C\}\\ &= N_C(u).\end{aligned}$$

Thereby, for all $\lambda > 0$, we have

$$\begin{aligned}y = J_\lambda^{\partial\delta_C}(u) &\Leftrightarrow u \in y + \lambda\partial\delta_C(y)\\ &\Leftrightarrow u - y \in \lambda\partial\delta_C(y)\\ &\Leftrightarrow \langle u - y, p - y\rangle \leq 0\ \ \forall p \in C\\ &\Leftrightarrow y = P_C u\end{aligned}$$

By Theorem 1, we obtain the corollary as follows.

Corollary 2. *Suppose that $S \neq \emptyset$, $\lambda_1 > 0$, $\alpha \in (0,1)$, $\delta \in (0, \hat{\rho}) \subset (0,4)$, θ_n satisfies (9). The sequence $\{\vartheta_n\}$ satisfies Lemma 6. Let $p_0,\ p_1 \in X_1$ and $\{p_n\}$ be a sequence generated by*

$$\begin{cases}\Lambda_n = p_n + \theta_n(p_n - p_{n-1}),\\ y_n = \Lambda_n - \lambda_n A^*(I - P_Q)A\Lambda_n,\\ z_n = 2P_C(y_n),\ p_{n+1} = (1-\alpha)\Lambda_n + \alpha(z_n - y_n),\end{cases}$$

where the stepsize λ_n is the following:

$$\lambda_{n+1} = \begin{cases}\min\left\{\frac{\delta F(\Lambda_n)}{\|\nabla F(\Lambda_n)\|^2}, \vartheta_n\lambda_n\right\}, & \text{if } \nabla F(\Lambda_n) \neq 0,\\ \vartheta_n\lambda_n, & \text{otherwise},\end{cases}$$

then, the iterative sequence $\{p_n\}$ converges weakly to $p \in S$.

6. Numerical Experiments

The following numerical experiments are provided for illustrating the performance of Algorithm 1 for solving the SVIP (1). We compare Algorithm 1 with some weakly convergent algorithm such as Kesornprom et al.'s Algorithm 3.1 (henceforth referred to as Algorithm 3.1 [8]), Cholamjiak et al.'s Algorithm 1 (Algorithm 1 [30]) and Suantai et al.'s Algorithm 3.1 (Algorithm 3.1 [46]). Our experiments are conducted by using MATLAB R2018a installed on a PC Desktop Intel(R) Core(TM) i7-6700 CPU @ 3.40 GHZ computer with 8GB RAM.

Example 1 ([6,23]). *Let $A,\ A_1$ and A_2 both be $m \times m$ matrices, and satisfy a normal distribution with unit variance and mean zero. Let $Y_1,\ Y_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $Y_1(p) = A_1^*A_1p$ and $Y_2(y) = A_2^*A_2y$, respectively. We mainly seek a point $\tilde{p} = (\tilde{p}_1, \cdots, \tilde{p}_m)^T \in \mathbb{R}^m$ such that $Y_1(\tilde{p}) = (0, \cdots, 0)^T$ and $Y_2(A\tilde{p}) = (0, \cdots, 0)^T$. In fact, $\tilde{p}_1 = 0, \cdots, \tilde{p}_m = 0$. Let $\epsilon > 0$ and the stopping criterion satisfies $\|p_n - \tilde{p}\| < \epsilon$. We take $p_1 = p_0 = (1, 1, \cdots, 1) \in \mathbb{R}^m$ and $\lambda = 2$ for all algorithms.*

For Algorithm 1 [30], we adopt $\delta = 1.005$, $\alpha = 0.1$, $t_0 = 1$ and

$$\theta_n = \begin{cases}\frac{t_{n-1}-1}{t_n}, & \text{if } 1 \leq n \leq 1000,\\ 0, & \text{otherwise},\end{cases}$$

where $t_n = \frac{1+\sqrt{1+4t_{n-1}^2}}{2}$, which is suggested in [30].

As in [8], we set $\rho = 1$, $\delta = 0.5$, *the inertial parameter* $\theta = 0.3$, *where* ρ *is the parameter in the linear search rule for Algorithm 3.1 [8].*

For Algorithm 1, we select $\lambda_1 = 1$, $\delta = 0.99$, $\alpha = 0.9$, *and* $\vartheta_n = \frac{0.05}{(n+1)^{1.1}} + 1$. *Additionally,* θ_n *is taken as that in Algorithm 1 [30].*

For Algorithm 3.1 [46], we take $\delta = 0.9$, $\theta_n = \frac{1}{(n+1)^7}$, $J_n = \frac{1}{(n+100)^2}$, *and* $\gamma = 1.9$. *The corresponding results are reported in Table 1 and Figures 1–3.*

Remark 3. *In Figure 1, we take the case* $m = 500$, $\varepsilon = 10^{-5}$ *to test the effectiveness of Algorithm 1 by using different* α. *We obviously observe that the parameter* α *of our proposed algorithm (Algorithm 1) has an important impact on the numerical results. More specifically, the number of iterations* (n) *is the lowest when* $\alpha = 0.9$. *Additionally, when* $\alpha = 0.7$, *our proposed algorithm behaves well.*

Table 1. Comparison of Algorithm 3.1 [8], Algorithm 1 [30], Algorithm 3.1 [46] and Algorithm 1 for Example 1.

m	ϵ	Algorithm 3.1 [46]		Algorithm 1 [30]		Algorithm 1		Algorithm 3.1 [8]	
		Iter.	Time	Iter.	Time	iter.	Time	Iter.	Time
$m = 500$	10^{-5}	121	0.0326	574	0.0811	59	0.0202	76	0.0265
	10^{-6}	139	0.0551	720	0.1439	69	0.0326	101	0.0516
$m = 600$	10^{-5}	122	0.0463	495	0.1278	60	0.0408	86	0.0449
	10^{-6}	142	0.0527	697	0.1502	69	0.0340	105	0.0510

Remark 4. *As shown in Table 1, as well as Figures 2 and 3, Algorithm 1, Algorithm 3.1 [8], Algorithm 1 [30] and Algorithm 3.1 [46] are Alg.1, Alg.3.1 [8], Alg.1 [30] and Alg.3.1 [46], respectively. All algorithms have a fast convergence rate. Especially, our proposed algorithm (Algorithm 1) needs fewer iterative numbers and less time than Algorithm 3.1 [8], Algorithm 1 [30] and Algorithm 3.1 [46] under different m and ε.*

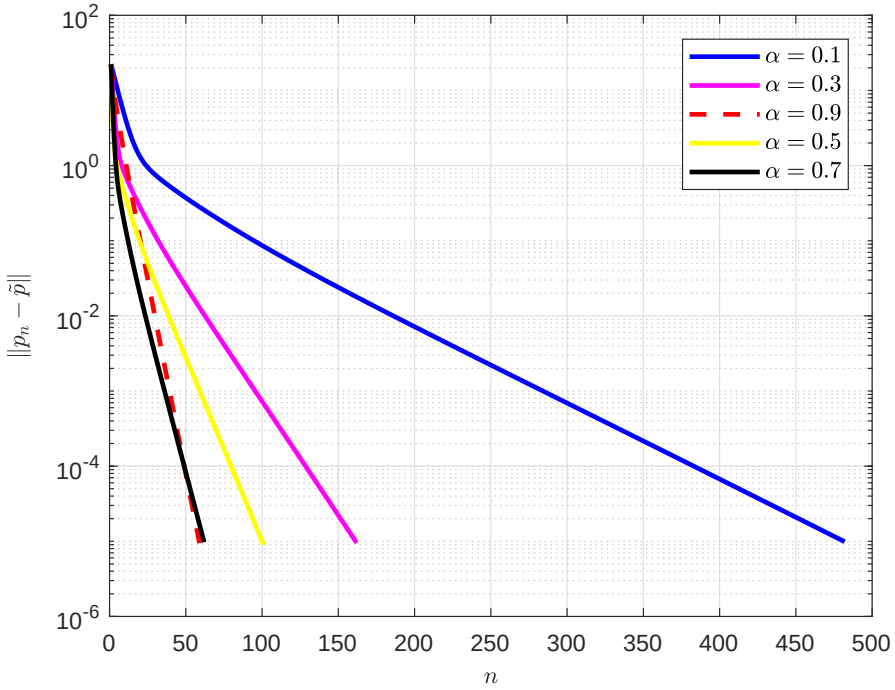


Figure 1. The sensitivity analysis of α for Algorithm 1 and number of iterations (n) for Example 1.

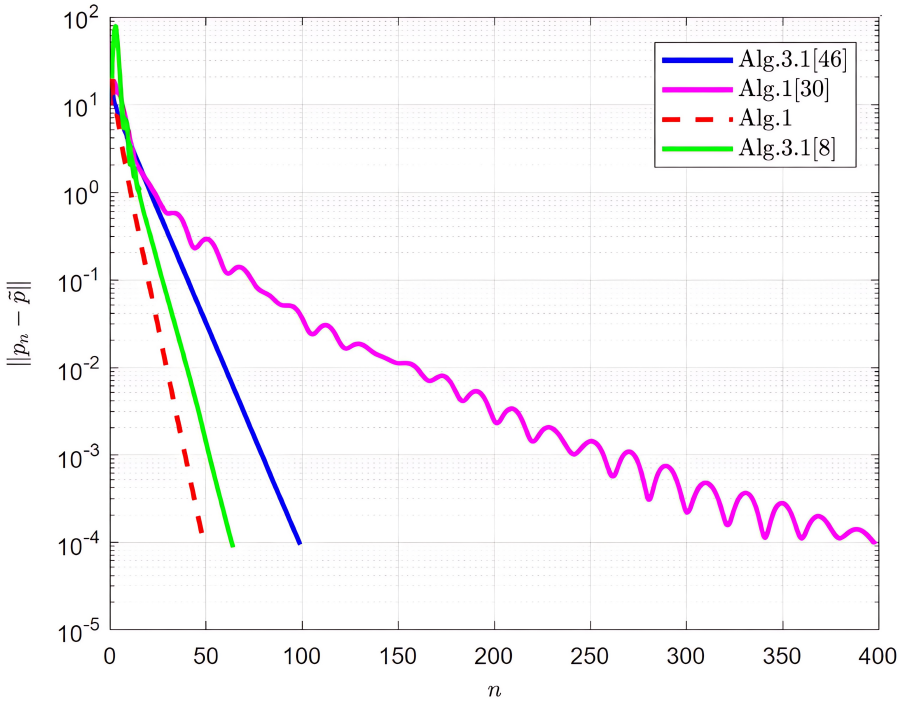


Figure 2. Number of iterations (n) for Example 1 ($m = 400$, $\varepsilon = 10^{-4}$).

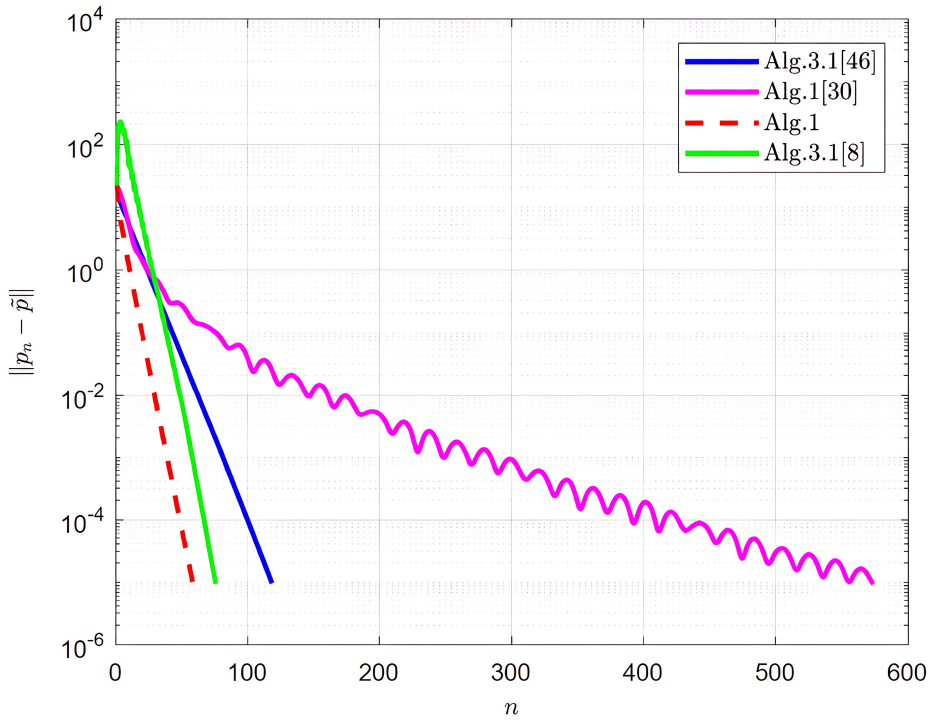


Figure 3. Number of iterations (n) for Example 1 ($m = 500$, $\varepsilon = 10^{-5}$).

Example 2. *LASSO problem [7,23]. In this example, we introduce the following LASSO problem.*

$$\min\left\{\frac{1}{2}\|Ap - V\|^2 : p \in \mathbb{R}^N, \|p\|_1 \leq \kappa\right\}, \tag{54}$$

where $A \in \mathbb{R}^{M\times N}$, $M < N$, $V \in \mathbb{R}^M$ *and* $\kappa > 0$. *This problem is devoted to finding a sparse solution of the SFP. The sampling matrice* A *is produced by a standard normal distribution with*

mean zero and unit variance. The truly sparse signal p^* is built by distributing W nonzero elements uniformly at random in the interval $[-2, 2]$ while the rest is kept zero. The sample data $V = Ap^*$.

In consideration of the SFP, we define $C = \{\tilde{p} | \|\tilde{p}\|_1 \leq \kappa\}$, $\kappa = W$, and $Q = \{V\}$. We suggest the subgradient projection, because of the fact there has no closed-form expression for the projection of a closed convex set C. Let $c(\tilde{p}) = \|\tilde{p}\|_1 - \kappa$ and

$$C_n = \{\tilde{p} : c(y_n) + \langle \varepsilon_n, \tilde{p} - y_n \rangle \leq 0\},$$

where $\varepsilon_n \in \partial c(y_n)$. Additionally, the orthogonal projection of a point $\tilde{p} \in \mathbb{R}^N$ onto C_n can be achieved by

$$P_{C_n}(\tilde{p}) = \begin{cases} p, & \text{if } c(y_n) + \langle \varepsilon_n, \tilde{p} - y_n \rangle \leq 0, \\ \tilde{p} - \frac{c(y_n) + \langle \varepsilon_n, \tilde{p} - y_n \rangle}{\|\varepsilon_n\|^2} \varepsilon_n, & \text{otherwise}, \end{cases}$$

The subdifferential ∂c at y_n is

$$\partial c(y_n) = \begin{cases} 1, & \text{if } y_n > 0, \\ [-1, 1], & \text{if } y_n = 0, \\ -1, & \text{if } y_n < 0. \end{cases}$$

Additionally, we initialize the algorithms at the original and define

$$E_n = \frac{\|p_n - p^*\|}{\max\{1, \|p_n\|\}}.$$

We test all methods under the same iteration error E_n for different M, N, and W and limit the number of iterations to 5000, and the corresponding results are reported in Table 2.

The second problem is the recovery of the signal p^* when $M = 1200$, $N = 5120$, $W = 150$, $M \times N$ matrix A is randomly obtained with independent samples of standard Gaussian distribution. The original signal p^* contains 150 randomly placed ± 1 spikes. The initial point $p_0 = 0$, we adopt the following mean square error for measuring the restoration accuracy:

$$MSE = \frac{1}{N} \|p_n - p^*\|^2.$$

For Algorithm 1 [30], we adopt $\delta = 1.3$, $\alpha = 0.9$, $t_0 = 1$ and

$$\theta_n = \begin{cases} \frac{t_{n-1} - 1}{t_n}, & \text{if } 1 \leq n \leq 10000, \\ 0, & \text{otherwise}, \end{cases}$$

where $t_n = \frac{1 + \sqrt{1 + 4t_{n-1}^2}}{2}$, which is suggested in [30].

As in [8], we set $\rho = 2$, $\delta = 0.5$, the inertial parameter $\theta = 0.3$, where ρ is the parameter in the linesearch for Algorithm 3.1 [8].

For Algorithm 1, we select $\lambda_1 = 0.7$, $\delta = 0.9$, $\alpha = 0.6$, and $\vartheta_n = \frac{0.6}{(n+1)^{1.1}} + 1$. Additionally, θ_n is defined as in Algorithm 1 [30].

For Algorithm 3.1 [46], we take $\delta = 0.9$, $\theta_n = \frac{1}{(1+n)^7}$, $J_n = \frac{1}{(1+n)^2}$, and $\gamma = 1.9$.

The corresponding results are reported in Table 2 and Figure 4, and Figure 4 provides numerical results we achieved for the original signal, MSE, and CPU time.

Remark 5. *In Figure 4, Algorithm 1, Algorithm 3.1 [8], Algorithm 1 [30] and Algorithm 3.1 [46] are Alg.1, Alg.3.1 [8], Alg.1 [30] and Alg.3.1 [46], respectively. From Table 2 and Figure 4, we see that all algorithms solve LASSO problems efficiently. In detail, our proposed algorithm (Algorithm 1) shows that the value of E_n is smaller than the other compared algorithms. For the running time of*

all algorithms, Algorithm 1 behaves better than Algorithm 3.1 [8] and Algorithm 3.1 [46]; however, its time somewhat runs longer than Algorithm 1 [30] by testing the different cases.

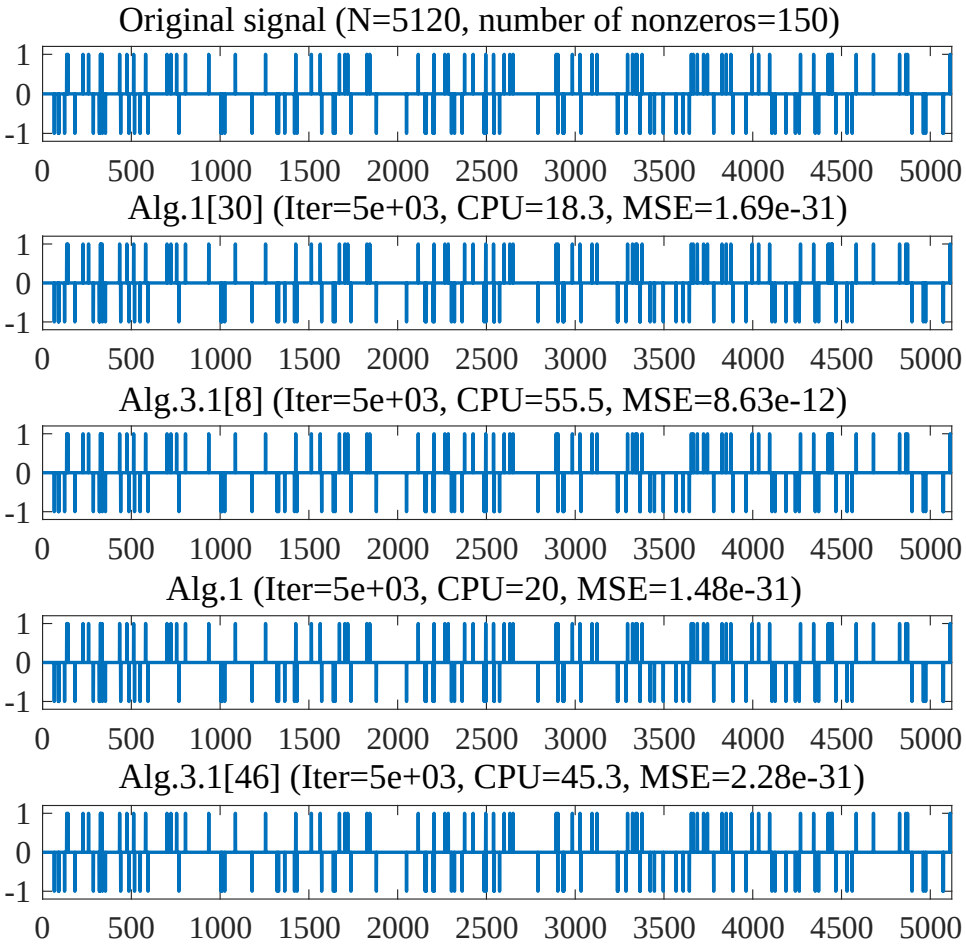


Figure 4. Comparison of signal processing.

Table 2. Comparison of Algorithm 3.1 [8], Algorithm 1 [30], Algorithm 3.1 [46], and Algorithm 1 for Example 2.

(M, N, k)	**Algorithm 1 [30]**		**Algorithm 3.1 [8]**		**Algorithm 1**		**Algorithm 3.1 [46]**	
	E_n	**Time**	E_n	**Time**	E_n	**Time**	E_n	**Time**
(240, 1024, 30)	2.2065×10^{-15}	0.2962	0.0039	0.7729	2.0945×10^{-15}	0.3297	3.5102×10^{-16}	0.7262
(480, 2048, 60)	1.3495×10^{-13}	2.9787	3.0280×10^{-5}	6.7701	1.3471×10^{-14}	2.3525	2.8004×10^{-15}	5.2247
(720, 3072, 90)	2.4382×10^{-15}	5.1924	2.4423×10^{-5}	17.0075	2.2733×10^{-15}	5.5532	2.6143×10^{-15}	13.2987
(960, 4096, 120)	2.7018×10^{-15}	12.1214	1.9863×10^{-5}	32.1827	2.5447×10^{-15}	11.0499	2.9284×10^{-15}	29.4340
(1200, 5120, 150)	2.4027×10^{-15}	18.3007	1.7162×10^{-5}	55.4954	2.2513×10^{-15}	20.0427	2.7921×10^{-15}	45.3349

7. Conclusions

We combine the inertial technique, new stepsize rules, the viscosity algorithm with linearized splitting algorithm for solving split variational inclusion problems, and we obtain its weak and strong convergence in real Hilbert spaces. Further, we extend our obtained results to split minimization problems and split feasibility problems. Finally, we make several experiments, which shows the advantages of our given algorithm.

8. Discussion

Compared with the linear splitting method in [30], our work has three important improvements. First, the new adaptive stepsize avoids zero-value stagnation and guarantees a positive lower bound. Second, the modified proximal term expands the feasible range of parameters from $(0, 2)$ to $(0, 4)$ for better numerical stability. Third, we introduce the viscosity algorithm to discuss the strong convergence in real Hilbert space. Further, we try to apply our proposed algorithm to the large-scale engineering optimization.

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References

1. Moudafi, A. Split monotone variational inclusions. *J. Optim. Theory Appl.* **2011**, *150*, 275–283. [CrossRef]
2. Kazmi, K.R.; Rizvi, S.H. An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping. *Optim. Lett.* **2014**, *8*, 1113–1124. [CrossRef]
3. Arfat, Y.; Iyiola, O.; Wickramasinghe, M.; Kumam, P.; Khan, M.A.A. An inertia based viscosity type regularized splitting scheme for finding common solutions of split monotone variational inclusion and fixed point problems. *Ann. Oper. Res.* **2026**. [CrossRef]
4. Chuang, C.S. Hybrid inertial proximal algorithm for the split variational inclusion problem in Hilbert spaces with applications. *Optimization* **2017**, *66*, 777–792. [CrossRef]
5. Sitthithakerngkiet, K.; Deepho, J.; Martinez-Moreno, J.; Kuman, P. Convergence analysis of a general iterative algorithm for finding a common solution of split variational inclusion and optimization problems. *Numer. Algorithms* **2018**, *79*, 801–824. [CrossRef]
6. Thong, D.; Dung, V.; Cho, Y. A new strong convergence for solving split variational inclusion problems. *Numer. Algorithms* **2020**, *86*, 565–591. [CrossRef]
7. Tibshirani, R. Regression shrinkage and selection via the Lasso. *J. R. Stat. Soc. Ser. B* **1996**, *58*, 267–288. [CrossRef]
8. Kesornprom, S.; Cholamjiak, P. Proximal type algorithms involving linesearch and inertial technique for split variational inclusion problem in hilbert spaces with applications. *Optimization* **2019**, *68*, 2369–2395. [CrossRef]
9. Censor, Y.; Bortfeld, T.; Martin, B.; Trofmov, A. A unifed approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **2006**, *51*, 2353–2365. [CrossRef] [PubMed]
10. Combettes, P.L. The convex feasibility problem in image recovery. In *Advances in Imaging and Electron Physics*; Hawkes, P., Ed.; Academic Press: New York, NY, USA, 1996; pp. 155–270.
11. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projection in a product space. *Numer. Algorithms* **1994**, *8*, 221–239. [CrossRef]
12. Wang, Z.; Lei, Z.; Long, X.; Chen, Z. A modified Tseng splitting method with double inertial steps for solving monotone inclusion problems. *J. Sci. Comput.* **2023**, *96*, 92. [CrossRef]
13. Zhao, Y.; Tian, P.; Wu, D. An iterative method for solving split variational inclusion problem and variational inequality problem constrained by a common fixed point problem. *J. Inequal. Appl.* **2026**, *2026*, 71. [CrossRef]
14. Arfat, Y.; Grad, S.; Tammer, C.; Kumam, P.; Khan, M.A.A. Solving split monotone variational inclusion problems via a relaxed-inertial self-adaptive regularized splitting algorithm. *Numer. Algorithms* **2026**. [CrossRef]
15. Byrne, C.; Censor, Y.; Gibali, A. Weak and strong convergence of algorithms for the split common null point problem. *J. Nonlinear Convex Anal.* **2011**, *13*, 759–775.
16. López, G.; Martin, V.; Wang, F.; Xu, H.K. Solving the split feasibility problem without prior knowledge of matrix norms. *Inverse Probl.* **2012**, *28*, 085004. [CrossRef]

17. Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. *SIAM J. Optim.* **2004**, *14*, 773–782. [CrossRef]
18. Polyak, B.T. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **1964**, *4*, 1–17. [CrossRef]
19. Ma, X.; Liu, H.; Li, X. The iterative method for solving the proximal split feasibility problem with an application to LASSO problem. *Comp. Appl. Math.* **2022**, *41*, 5.
20. Khairoowala, M.U.; Husain, S.; Furkan, M. An iterative algorithm for split variational inclusion, a system of variational inequalities and fixed point problems. *J. Anal.* **2025**, *33*, 3027–3049. [CrossRef]
21. Li, H.; Wang, X.; Wang, F. Projection and contraction method with double inertial steps for quasi-monotone variational inequalities. *Optimization* **2025**, *74*, 1643–1674. [CrossRef]
22. Yao, Y.; Iyiola, O.S.; Shehu, Y. Subgradient extragradient method with double inertial steps for variational inequalities. *J. Sci. Comput.* **2022**, *90*, 71. [CrossRef]
23. Ma, X.; Liu, H.; Li, X. Two optimization approaches for solving split variational inclusion problems with applications. *J. Sci. Comput.* **2022**, *91*, 58. [CrossRef]
24. Chang, X.; Yang, J. A golden ratio primal-dual algorithm for structured convex optimization. *J. Sci. Comput.* **2021**, *87*, 47. [CrossRef]
25. Chang, X.; Yang, J.; Zhang, H. Golden ratio primal-dual algorithm with linesearch. *SIAM J. Optim.* **2022**, *32*, 1584–1613. [CrossRef]
26. Rao, N.; Farid, M.; Jha, N.K. Szász-integral operators linking general-Appell polynomials and approximation. *AIMS Math.* **2025**, *10*, 13836–13854. [CrossRef]
27. Dong, Q.L.; Cho, Y.J.; Rassias, T.M. General inertial Mann algorithms and their convergence analysis for nonexpansive mappings. *Appl. Nonlinear Anal.* **2018**, *134*, 175–191. [CrossRef]
28. Nesterov, Y. A method for solving the convex programming problem with convergence rate $O(1/k^2)$. *Dokl. Akad. Nauk. SSSR* **1983**, *269*, 543–547.
29. Thong, D.V. Convergence analysis of a self-adaptive inertial subgradient extragradient method for non-monotone variational inequalities with non-Lipschitz continuous mappings. *Numer. Algorithms* **2026**. [CrossRef]
30. Cholamjiak, P.; Suantai, S.; Kesornprom, S. Linearized splitting algorithm with inertial extrapolation for solving split variational inclusion problem and applications. *Optimization* **2025**, *75*, 1007–1029. [CrossRef]
31. Xu, S.; Li, S.; Yao, J.C. Four double inertial algorithms for variational inequalities with fixed point constraints: Theory and signal processing applications. *Numer. Algorithms* **2026**. [CrossRef]
32. Wen, D.; Li, Q.; Zhang, H. A new proximal-type regularization algorithm for split monotone variational inclusions with applications. *Comp. Appl. Math.* **2026**, *45*, 373. [CrossRef]
33. Dey, S.; Izuchukwu, C.; Taiwo, A.; Reich, S. New iterative algorithms for solving split variational inclusions. *J. Glob. Optim.* **2025**, *91*, 587–609. [CrossRef]
34. Chai, Z.; Li, Y.; Han, Y.; Zhu, S. Recommendation System Based on Singular Value Decomposition and Multi-Objective Immune Optimization. *IEEE Access* **2019**, *7*, 6060–6071. [CrossRef]
35. Tian, C.; Chen, Y. image segmentation and denoising algorithm based on partial differential equations. *IEEE Sens. J.* **2020**, *20*, 11935–11942. [CrossRef]
36. Tian, C.; Chang, K.; Chen, J. Application of hyperbolic partial differential equations in global optimal scheduling of UAV. *Alex. Eng.* **2020**, *59*, 2283–2289. [CrossRef]
37. Oikonomidis, K.A.; Bodard, A.; Laude, E.; Patrinos, P. Global convergence analysis of the power proximal point and augmented Lagrangian method. *Comput. Optim. Appl.* **2025**, *93*, 617–649. [CrossRef]
38. Marino, G.; Xu, H.K. Convergence of generalized proximal point algorithm. *Commun. Pure Appl. Anal.* **2004**, *3*, 791–808. [CrossRef]
39. Takahashi, W. *Nonlinear Functional Analysis-Fixed Point Theory and Its Applications*; Yokohama Publishers: Yokohama, Japan, 2000.
40. Osilike, M.O.; Aniagbosor, S.C. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Model.* **2000**, *32*, 1181–1191. [CrossRef]
41. Saejung, S.; Yotkaew, P. Approximation of zeros of inverse strongly monotone operators in Banach spaces. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 742–750. [CrossRef]
42. Tang, Y.; Gibali, A. New self-adaptive step size algorithms for solving split variational inclusion problems and its applications. *Numer. Algorithms* **2020**, *83*, 305–331. [CrossRef]
43. Takahashi, W.; Xu, H.K.; Yao, J.C. Iterative methods for generalized split feasibility problem in Hilbert spaces. *Set. Valued Var. Anal.* **2015**, *23*, 205–221. [CrossRef]
44. Ceng, L.; Ansari, Q.; Yao, J. An extragradient method for solving split feasibility and fixed point problems. *Comput. Math. Appl.* **2012**, *64*, 633–642. [CrossRef]

45. Long, L.; Thong, D.; Dng, V. New algorithms for the split variational inclusion problems and application to split feasibility problems. *Optimization* **2019**, *68*, 2339–2367. [CrossRef]
46. Suantai, S.; Cholamjiak, P.; Inkrong, P.; Kesornprom, S. A fast contraction algorithm using two inertial extrapolations for variational inclusion problem and data classification. *Carpath. J. Math.* **2024**, *40*, 737–752. [CrossRef]

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