Mathematical and Computational Applications

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Common Attractor for Hutchinson θ-Contractive Operators in Partial Metric Spaces

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Abstract: This paper investigates the existence of common attractors for generalized θ-Hutchinson operators within the framework of partial metric spaces. Utilizing a finite iterated function system composed of θ-contractive mappings, we establish theoretical results on common attractors, generalizing numerous existing results in the literature. Additionally, to enhance understanding, we present intuitive and easily comprehensible examples in one-, two-, and three-dimensional Euclidean spaces. These examples are accompanied by graphical representations of attractor images for various iterated function systems. As a practical application, we demonstrate how our findings contribute to solving a functional equation arising in a dynamical system, emphasizing the broader implications of the proposed approach.

Keywords: common attractor; generalized θ-contraction; θ-Hutchinson operator; iterated function system; partial metric spaces

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Received: 8 February 2025
Revised: 11 March 2025
Accepted: 13 March 2025
Published: 14 March 2025

Citation: Shabir, N.; Raza, A.; De la Sen, M.; Abbas, M.; Ahmad, S. Common Attractor for Hutchinson θ-Contractive Operators in Partial Metric Spaces. *Math. Comput. Appl.* **2025**, *30*, 27. https://doi.org/10.3390/mca30020027

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1. Introduction and Preliminaries

The iterated function system (IFS) originates from the mathematical foundations laid by Hutchinson, as cited in [1]. Hutchinson demonstrated that the operator, consisting of a finite set of contraction mappings on $\mathbb{R}^n$ and referred to as the Hutchinson operator, has a closed and bounded subset of $\mathbb{R}^n$ as its fixed point set. This fixed point is known as the attractor [2]. In this context, fixed point theory plays a pivotal role in facilitating the creation of fractals. Figure 1 are the two most common examples of such structures mentioned in [2]:

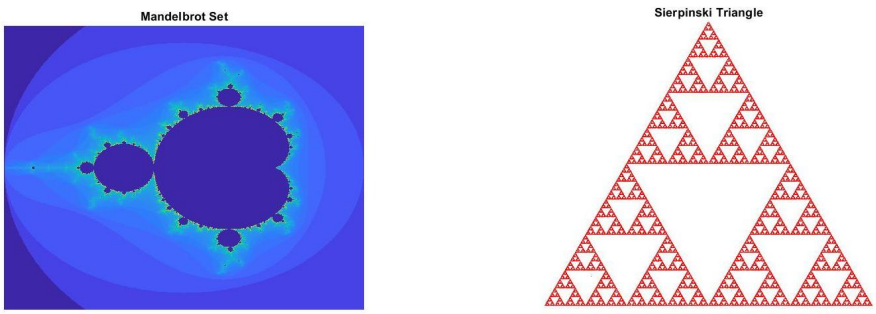


Figure 1. Mandelbrot set and Serpinski triangle.

Many significant results on attractors have been established in the literature. Recently, Iqbal et al. in [3] studied the existence of common attractors for generalized Hutchinson–Wardowski contractive operators, proving that such systems admit a unique common attractor for various classes of these operators. Additionally, Nazir et al. in [4] investigated generalized F-iterated function systems in G-metric spaces, establishing several results on common attractors using generalized F-Hutchinson operators. For a deeper understanding of fractals and attractors, one may refer to [5–9], along with the references therein.

On the other hand, numerous generalizations of the Banach contraction principle have been developed, as many authors have worked on extending this principle either by modifying the underlying metric space or by generalizing the contraction condition through different techniques. In this context, Jleli and Samet [10] introduced the concept of θ-contraction and provided a generalization of the Banach contraction principle. Following this remarkable extension, several researchers further advanced and expanded the results within this framework. Notable contributions to θ-contractions can be found in [11–13].

In this paper, we combine the idea of θ-contraction as introduced by Jleli and Samet in [10] with the Hutchinson operator and extends the results of common attractors as derived in [14] for θ-contraction over partial metric space. So, in this regard, our results generalized the concept of common attractors and so the results in [14,15] become the special cases of our main results. Furthermore, we also justify our results with suitable examples and provide their corresponding fractals images. Additionally, we summarize our results by presenting an application derived from our main findings.

Considering a partial metric space (Υ, p), we shall denote by $C^p(\Upsilon)$ a collection of all non-empty compact subsets of Υ. For any $\Omega, \mho \in C^p(\Upsilon)$ and $v \in \Upsilon$, we define

$$p(v, \Omega) = \inf\{p(v, \mu) : \mu \in \Omega\}$$

and

$$\delta_p(\Omega, \mho) = \sup\{p(v, \mho) : v \in \Omega\}.$$

Thus,

$$\mathcal{H}_p(\Omega, \mho) = \max\{\delta_p(\Omega, \mho), \delta_p(\mho, \Omega)\}.$$

Proposition 1 ([16]). *Consider a partial metric space (Υ, p). Then, for all $\Omega, \mho, C \in C^p(\Upsilon)$,*

(i) $\mathcal{H}_p(\Omega, \Omega) \leq \mathcal{H}_p(\Omega, \mho)$.
(ii) $\mathcal{H}_p(\Omega, \mho) = \mathcal{H}_p(\mho, \Omega)$.
(iii) $\mathcal{H}_p(\Omega, \mho) \leq \mathcal{H}_p(\Omega, C) + \mathcal{H}_p(C, \mho) - \inf_{\alpha \in C} p(\alpha, \alpha)$.

Corollary 1 ([16]). *Consider a partial metric space (Υ, p). Then, for any $\Omega, \mho \in C^p(\Upsilon)$, we have*

$$\mathcal{H}_p(\Omega, \mho) = 0 \text{ implies } \Omega = \mho.$$

Thus, by following Proposition 1 and Corollary 1, we shall refer to the mapping $\mathcal{H}_p : C^p(\Upsilon) \times C^p(\Upsilon) \to [0, \infty)$ as a partial Hausdorff metric induced by p.

Lemma 1 ([14]). *Let (Υ, p) be a partial metric space. Then, for any $\Omega, \mho, C, D \in C^p(\Upsilon)$, the following holds:*

(i) *If $\mho \subset C$, then* $\sup_{a \in \Omega} p(a, C) \leq \sup_{a \in \Omega} p(a, \mho)$,
(ii) $\sup_{t \in \Omega \cup \mho} p(t, C) = \max\{\sup_{a \in \Omega} p(a, C), \sup_{b \in \mho} p(b, C)\}$,
(iii) $\mathcal{H}_p(\Omega \cup \mho, C \cup D) \leq \max\{\mathcal{H}_p(\Omega, C), \mathcal{H}_p(\mho, D)\}$.

Definition 1. *Consider Θ^* to be the collection of all such functions $\theta : (0,\infty) \to (1,\infty)$ which satisfy the following:*

(θ_1) *θ is non-decreasing.*
(θ_2) *For any sequence $\{x_n\} \subset R^+$, $\lim_{n\to\infty} \Theta(x_n) = 1$ iff $\lim_{n\to\infty}(x_n) = 0$.*
(θ_3) *There exist $0 < h < 1$ and $l \in (0,\infty]$ such that $\lim_{\alpha\to 0^+} \frac{\theta(\alpha)-1}{\alpha^h} = l$.*

Moreover, the class Θ is the collection of all such $\theta \in \Theta^*$ which are continuous as well.

In the next section, we present our key findings. The organization of this paper is as follows: In Section 1, we provide a necessary literature review and preliminaries. Section 2 contains the main results of the article, where we establish the existence of a common attractor in our generalized setting. Section 3 is dedicated to illustrating our main results through examples in one-, two-, and three-dimensional Euclidean spaces. A particularly interesting aspect of this section is the inclusion of graphical representations, which offer a clearer understanding of our claims regarding common attractors. In Section 4, we discuss an application from dynamical systems, demonstrating the relevance of our results in other fields. Finally, we conclude this manuscript with the conclusion in Section 5.

2. Main Results

Definition 2. *A pair of self-mappings (h,g) defined on a partial metric space (Y,p) is said to be a generalized θ-contraction if there exist $\theta \in \Theta$ and $k \in (0,1)$ such that for all $\varrho,\varsigma \in \mathrm{Y}$, we have*

$$p(h\varrho, g\varsigma) > 0 \implies \theta\left(p(h\varrho, g\varsigma)\right) \leq \left(\theta(p(\varrho,\varsigma))\right)^k.$$

Theorem 1. *Consider a partial metric space (Y,p) and $h,g : \mathrm{Y} \to \mathrm{Y}$ to be two self-mappings which are continuous as well. If the pair (h,g) is a generalized θ-contraction on (Y,p), then the pair (h,g) is a generalized θ-contraction on $(C^p(\mathrm{Y}),\mathcal{H}_p)$, where for any $\Omega \in C^p(\mathrm{Y})$, the mappings $h,g : C^p(\mathrm{Y}) \to C^p(\mathrm{Y})$ are defined as $h(\Omega) = \{h(\varrho) : \varrho \in \Omega\}$ and $g(\Omega) = \{g(\varrho) : \varrho \in \Omega\}$.*

Proof. Since h,g are continuous mappings, for any $\Omega \in C^p(\mathrm{Y})$, we have $h(\Omega)$, $g(\Omega) \in C^p(\mathrm{Y})$. Let $\Omega,\mho \in C^p(\mathrm{Y})$, and for all ϱ, $\varsigma \in \mathrm{Y}$, there exists $k \in (0,1)$, such that

$$\theta(p(h(\varrho),g(\varsigma))) \leq [\theta(p(\varrho,\varsigma))]^k < \theta(p(\varrho,\varsigma)).$$

Since $\theta \in \Theta$ is non-decreasing and $p(\varrho,\varsigma) \geq 0$, we obtain $p(h(\varrho),g(\varsigma)) < p(\varrho,\varsigma)$. Thus,

$$p(h(\varrho),g(\mho)) = \inf_{\varsigma\in\mho} p(h(\varrho),g(\varsigma)) < \inf_{\varsigma\in\mho} p(\varrho,\varsigma) = p(\varrho,\mho)$$

and

$$p(g(\varsigma),h(\Omega)) = \inf_{\varrho\in\Omega} p(g(\varsigma),h(\varrho)) < \inf_{\varrho\in\Omega} p(\varsigma,\varrho) = p(\varsigma,\Omega).$$

Hence,

$$\begin{aligned}\mathcal{H}_p(h(\Omega),g(\mho)) &= \max\{\sup_{\varrho\in\Omega} p(h(\varrho),g(\mho)), \sup_{\varsigma\in\mho} p(g(\varsigma),h(\Omega))\}\\ &< \max\{\sup_{\varrho\in\Omega} p(\varrho,\mho), \sup_{\varsigma\in\mho} p(\varsigma,\Omega)\} = \mathcal{H}_p(\Omega,\mho).\end{aligned}$$

By applying any $\theta \in \Theta$ on both sides, we obtain

$$\theta(\mathcal{H}_p(h(\Omega),g(\mho))) < \theta(\mathcal{H}_p(\Omega,\mho)),$$

so, there exists $k \in (0,1)$ such that

$$\theta(\mathcal{H}_p(h(\Omega), g(\mho))) \leq [\theta(\mathcal{H}_p(\Omega, \mho))]^k.$$

Hence, (h, g) is a generalized θ-contraction on $(C^p(\mathrm{Y}), \mathcal{H}_p)$. □

Theorem 2. *Consider a partial metric space (Y, p) and $h_i, g_i : \mathrm{Y} \to \mathrm{Y}$ for $i = 1, 2, 3, \cdots, r$ are continuous and each pair (h_i, g_i) is a generalized θ-contraction with a contractive constant $k_i \in (0,1)$, respectively; then, for any $\Omega, \mho \in C^p(\mathrm{Y})$, the mappings $\psi, \phi : C^p(\mathrm{Y}) \to C^p(\mathrm{Y})$ defined as*

$$\psi(\Omega) = h_1(\Omega) \cup h_2(\Omega) \cup, \ldots, \cup h_r(\Omega) = \cup_{i=1}^{r} h_i(\Omega), \tag{1}$$

$$\phi(\mho) = g_1(\mho) \cup g_2(\mho) \cup, \ldots, \cup g_r(\mho) = \cup_{i=1}^{r} g_i(\mho), \tag{2}$$

satisfy

$$\theta(\mathcal{H}_p(\psi(\Omega), \phi(\mho))) \leq [\theta(\mathcal{H}_p(\Omega, \mho))]^{\hat{k}}, \tag{3}$$

where $\hat{k} = \max\{k_1, k_2, \ldots, k_r\}$. Moreover, the pair (ψ, ϕ) is called a generalized θ-contraction on $C^p(\mathrm{Y})$.

Proof. For any $\Omega, \mho \in C^p(\mathrm{Y})$, by using Lemma 1, we obtain the following for $r = 2$,

$$\begin{aligned}\mathcal{H}_p(\psi(\Omega), \phi(\mho)) &= \mathcal{H}_p\big(h_1(\Omega) \cup h_2(\Omega), g_1(\mho) \cup g_2(\mho)\big) \\ &\leq \max\{\mathcal{H}_p(h_1(\Omega), g_1(\mho)), \mathcal{H}_p(h_2(\Omega), g_2(\mho))\},\end{aligned}$$

since θ is non-decreasing; thus, by applying it on both sides, we obtain

$$\theta\big(\mathcal{H}_p(\psi(\Omega), \phi(\mho))\big) \leq \theta\big(\max\{\mathcal{H}_p(h_1(\Omega), g_1(\mho)), \mathcal{H}_p(h_2(\Omega), g_2(\mho))\}\big).$$

If $\max\{\mathcal{H}_p(h_1(\Omega), g_1(\mho)), \mathcal{H}_p(h_2(\Omega), g_2(\mho))\} = \mathcal{H}_p(h_1(\Omega), g_1(\mho))$, then we have

$$\theta\big(\mathcal{H}_p(\psi(\Omega), \phi(\mho))\big) \leq \theta\big(\{\mathcal{H}_p(h_1(\Omega), g_1(\mho))\big) \leq \theta(\mathcal{H}_p(\Omega, \mho))^{k_1} \leq \theta(\mathcal{H}_p(\Omega, \mho))^{\max\{k_1, k_2\}},$$

because $\theta : (0, \infty) \to (1, \infty)$ and $k, k_2 \in (0,1)$. Further, if

$$\max\{\mathcal{H}_p(h_1(\Omega), g_1(\mho)), \mathcal{H}_p(h_2(\Omega), g_2(\mho))\} = \mathcal{H}_p(h_2(\Omega), g_2(\mho)),$$

then we have

$$\theta\big(\mathcal{H}_p(\psi(\Omega), \phi(\mho))\big) \leq \theta(\mathcal{H}_p(\Omega, \mho))^{k_2} \leq \theta(\mathcal{H}_p(\Omega, \mho))^{\max\{k_1, k_2\}}.$$

Now, for $r = 3$, by using the above expression for $r = 2$ and Lemma 1, we obtain

$$\begin{aligned}\mathcal{H}_p(\psi_3(\Omega), \phi_3(\mho)) &= \mathcal{H}_p\big(\psi_2(\Omega) \cup h_3(\Omega), \phi_2(\mho) \cup g_3(\mho)\big) \\ &\leq \max\{\mathcal{H}_p(\psi_2(\Omega), \phi_2(\mho)), \mathcal{H}_p(h_3(\Omega), g_3(\mho))\}.\end{aligned}$$

Here, we use the notation $\psi_r(\Omega) = \cup_{i=1}^{r} h_i(\Omega)$ and $\phi_r(\mho) = \cup_{i=1}^{r} g_i(\mho)$; further, by applying θ on both sides and considering both cases of maximum, we obtain

$$\mathcal{H}_p(\psi_3(\Omega), \phi_3(\mho)) \leq \theta(\mathcal{H}_p(\Omega, \mho))^{\max\{k_1, k_2, k_3\}}.$$

Following a similar pattern, i.e., by using the result for $r = 3$, Lemma 1, and the fact that θ is non-decreasing, we obtain the following result for $r = 4$:

$$\mathcal{H}_p(\psi_4(\Omega), \phi_4(\mho)) \leq \theta(\mathcal{H}_p(\Omega, \mho))^{\max\{k_1,k_2,k_3,k_4\}}.$$

Consequently, following a similar pattern, we can obtain the required relation for any $1 \leq i \leq r$. □

Definition 3. *A pair of mappings ψ, ϕ defined on $C^p(\Upsilon)$ is said to be a generalized Hutchinson θ-contractive operator if there exists a constant $k \in (0,1)$ such that for any $\Omega, \mho \in C^p(\Upsilon)$, the following holds:*

$$\theta(\mathcal{H}_p(\psi(\Omega), \phi(\mho))) \leq [\theta(M_{\psi,\phi}(\Omega, \mho))]^k, \tag{4}$$

where

$$M_{\psi,\phi}(\Omega, \mho) = \max\Big\{\mathcal{H}_p(\Omega, \mho), \mathcal{H}_p(\Omega, \psi(\Omega)), \mathcal{H}_p(\mho, \phi(\mho)), \frac{\mathcal{H}_p(\Omega, \phi(\mho)) + \mathcal{H}_p(\mho, \psi(\Omega))}{2}\Big\}. \tag{5}$$

Definition 4. *A pair of mappings ψ, ϕ defined on $C^p(\Upsilon)$ is said to be a generalized rational Hutchinson θ-contractive operator if there exists a constant $k \in (0,1)$ such that for any $\Omega, \mho \in C^p(\Upsilon)$, the following holds:*

$$\theta(\mathcal{H}_p(\psi(\Omega), \phi(\mho))) \leq [\theta(R_{\psi,\phi}(\Omega, \mho))]^k, \tag{6}$$

where

$$R_{\psi,\phi}(\Omega, \mho) = \max\Big\{\frac{\mathcal{H}_p(\Omega, \phi(\mho))(1 + \mathcal{H}_p(\Omega, \psi(\Omega)))}{2(1 + \mathcal{H}_p(\Omega, \mho))}, \frac{\mathcal{H}_p(\mho, \phi(\mho))(1 + \mathcal{H}_p(\Omega, \psi(\Omega))}{1 + \mathcal{H}_p(\Omega, \mho)}, \frac{\mathcal{H}_p(\Omega, \mho)(1 + \mathcal{H}_p(\Omega, \psi(\Omega)))}{1 + \mathcal{H}_p(\Omega, \mho)}\Big\}. \tag{7}$$

Definition 5. *If we consider (h_i, g_i), $i = 1, 2, \cdots r$, to be a pair of continuous self-mappings defined on a partial metric space Υ, where both mappings are generalized θ-contractions as well, then*

$$\{\Upsilon; (h_i, g_i),\ i = 1, 2, 3, \ldots, r\}$$

is called a generalized θ-contractive iterative function system.

Definition 6. *Let $\Omega \subseteq \Upsilon$ be a non-empty compact set. Then, Ω is said to be a common attractor of a generalized θ-contractive iterative function system, if*

(i) $\psi(\Omega) = \Omega = \phi(\Omega)$.

(ii) *There exists an open set $O \subseteq \Upsilon$ such that $\Omega \subseteq O$ and*

$$\lim_{n \to +\infty} \psi^n(\mho) = \Omega = \lim_{n \to +\infty} \phi^n(\mho),$$

for any compact set $\mho \subseteq O$.

Theorem 3. *Let (Υ, p) be a complete partial metric space and*

$$\{\Upsilon; (h_i, g_i),\ i = 1, 2, 3, \ldots, r\}$$

be the a generalized θ-contractive IFS, and the pair (ψ, ϕ) is a generalized Hutchinson θ-contractive operator; then, ψ and ϕ have a unique common attractor. Moreover, for any $\Omega_0 \in C^p(Y)$, the sequence

$$\{\Omega_0, \psi(\Omega_0), \phi\psi(\Omega_0), \psi\phi\psi(\Omega_0), \ldots\}$$

converges to the common attractor of ψ and ϕ.

Proof. Choose $\Omega_0 \in C^p(Y)$ and define a sequence (Ω_i) in $C^p(Y)$ as follows:

$$\Omega_1 = \psi(\Omega_0),\ \Omega_3 = \psi(\Omega_2),\ \ldots, \Omega_{2i+1} = \psi(\Omega_{2i}),$$

$$\Omega_2 = \phi(\Omega_1),\ \Omega_4 = \phi(\Omega_3),\ \ldots, \Omega_{2i+2} = \phi(\Omega_{2i+1}),$$

for $i \in \{0, 1, 2, \ldots, \}$. Since each pair (h_i, g_i) is a generalized θ-contraction, thus (ψ, ϕ) is a Hutchinson θ-contractive operator. Assuming that no three consecutive terms in the sequence are identical, we have

$$\theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})) \leq [\theta(M_{\psi,\phi}(\Omega_{2i}, \Omega_{2i+1}))]^k, \tag{8}$$

where

$$\begin{aligned} M_{\psi,\phi}(\Omega_{2i}, \Omega_{2i+1})) &= \max\{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i}, \psi(\Omega_{2i})), \mathcal{H}_p(\Omega_{2i+1}, \phi(\Omega_{2i+1})), \\ &\qquad \frac{\mathcal{H}_p(\Omega_{2i}, \phi(\Omega_{2i+1})) + \mathcal{H}_p(\Omega_{2i+1}, \psi(\Omega_{2i}))}{2}\}, \\ &= \max\{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2}), \\ &\qquad \frac{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+2}) + \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+1})}{2}\}. \end{aligned}$$

By triangular inequality and Lemma 1, we have

$$\theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})) \leq [\theta(\max\{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})\})]^k. \tag{9}$$

Since $\max\{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})\} = \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})$ is not possible, we have

$$\theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})) \leq [\theta(\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}))]^k.$$

Also, again by definition of a generalized Hutchinson θ-contractive operator and Lemma 1, we obtain

$$\theta(\mathcal{H}_p(\Omega_{2i+2}, \Omega_{2i+3})) \leq \theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2}))^k.$$

Thus, for any $i \in \{0, 1, 2, \ldots\}$, we have

$$\theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) \leq \theta(\mathcal{H}_p(\Omega_{i-1}, \Omega_i))^k \leq \cdots \leq \theta(\mathcal{H}_p(\Omega_0, \Omega_1))^{k^i}.$$

Hence,

$$\theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) \leq \theta(\mathcal{H}_p(\Omega_0, \Omega_1))^{k^i}, \tag{10}$$

by taking limit as $i \to \infty$, $k^i \to 0$, so

$$\lim_{i\to\infty} \theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) \leq 1,$$

and since the range of θ is $(1, \infty)$,

$$\lim_{i\to\infty} \theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) = 1.$$

By the θ_2 condition, we have

$$\lim_{i\to\infty}(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))=0.$$

Also, from θ_3, there exist $0<h<1$ and $l\in(0,\infty]$, such that

$$\lim_{i\to\infty}\frac{\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1}{(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h}=l.$$

So, when $l<\infty$, choose $\beta=\frac{l}{2}>0$. By the definition of the limit, there exists N_1 such that

$$\left|\frac{\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1}{(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h}-l\right|\le\beta,\quad\forall i\ge N_1,$$

$$-\beta\le\frac{\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1}{(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h}-l\le\beta,\quad\forall i\ge N_1,$$

or

$$\beta=\frac{l}{2}=l-\beta\le\frac{\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1}{(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h},\quad\forall i\ge N_1.$$

Thus,

$$(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h\le\frac{1}{\beta}[\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1],$$

and by multiplying i on both sides and taking $\frac{1}{\beta}=\alpha$, we obtain

$$i[\mathcal{H}(\Omega_i,\Omega_{i+1})]^h\le i\alpha[\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1].\tag{11}$$

If $l=\infty$, let $\beta>0$ be an arbitrary large number; thus, by the definition of the limit, there exists N_2 such that

$$\frac{\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1})-1}{(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))^h}\ge\beta,\qquad\forall i\ge N_2,$$

so

$$i[\mathcal{H}(\Omega_i,\Omega_{i+1})]^h\le\frac{1}{\beta}\,i\,[\theta(\mathcal{H}_p(\Omega_i,\Omega_{i+1}))-1],\quad\forall i\ge N_2.\tag{12}$$

Moreover, for all $i\ge N=\max\{N_1,N_2\}$ and by relations (10)–(12) and taking $\lim_{i\to\infty}$, we obtain

$$\lim_{i\to\infty}i[\mathcal{H}_p(\Omega_i,\Omega_{i+1})]^h\le\lim_{i\to\infty}\alpha i[\theta[\mathcal{H}_p(\Omega_i,\Omega_{i+1})]^{k^i}-1]=0.$$

Hence,

$$\lim_{i\to\infty}i[\mathcal{H}_p(\Omega_i,\Omega_{i+1})]^h=0,$$

and thus, by definition, there exists N^*, such that

$$\mathcal{H}_p(\Omega_i,\Omega_{i+1})\le\frac{1}{i^{\frac{1}{h}}},\qquad\forall\, i\ge N^*.\tag{13}$$

Now, for any natural numbers m,n with $m\ge n\ge N^*$ and by following (13), we obtain

$$\mathcal{H}_p(\Omega_n,\Omega_m)\le\sum_{i=n}^{m-1}\frac{1}{i^{\frac{1}{h}}}\le\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{h}}}.$$

Since $h \in (0,1)$, the series on right is convergent; thus, by taking the limits $n, m \to \infty$, we obtain

$$\lim_{n,m\to\infty} \mathcal{H}_p(\Omega_n, \Omega_m) \leq \lim_{n,m\to\infty} \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}} = 0.$$

Thus, $\{\Omega_n\}$ is a Cauchy sequence in $C^p(\mathrm{Y})$. Since $(C^p(\mathrm{Y}), \mathcal{H}_p)$ is complete, there exists $\pounds \in C^p(\mathrm{Y})$ such that $\lim_{i\to\infty} \Omega_i = \pounds$, that is,

$$\lim_{i\to\infty} \mathcal{H}_p(\Omega_i, \pounds) = \lim_{i\to\infty} \mathcal{H}_p(\Omega_i, \Omega_{i+1}) = \mathcal{H}_p(\pounds, \pounds),$$

so we have

$$\lim_{i\to\infty} \mathcal{H}_p(\Omega_n, \pounds) = 0 \text{ and } \mathcal{H}_p(\pounds, \pounds) = 0.$$

Now, to show that $\psi(\pounds) = \pounds$, we assume $\psi(\pounds) \neq \pounds$ and $\mathcal{H}_p(\psi(\pounds), \pounds) \neq 0$; thus, by using the fact that (ψ, ϕ) is a generalized Hutchinson θ-contractive operator and θ is continuous, we obtain

$$\begin{aligned}
\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) &= \theta(\lim_{i\to\infty} \mathcal{H}_p(\psi(\pounds), \Omega_{2i+2})), \\
&= \lim_{i\to\infty} \theta(\mathcal{H}_p(\psi(\pounds), \phi(\Omega_{2i+1}))) \\
&\leq \lim_{i\to\infty} [\theta(M_{\psi,\phi}(\pounds, \Omega_{2i+1}))]^k,
\end{aligned}$$

where

$$\begin{aligned}
M_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \max\{&\mathcal{H}_p(\pounds, \Omega_{2i+1}), \mathcal{H}_p(\pounds, \psi(\pounds)), \mathcal{H}_p(\Omega_{2i+1}, \phi(\Omega_{2i+1})), \\
&\frac{\mathcal{H}_p(\pounds, \phi(\Omega_{2i+1})) + \mathcal{H}_p(\Omega_{2i+1}, \psi(\pounds))}{2}\}.
\end{aligned}$$

If $M_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \mathcal{H}_p(\pounds, \Omega_{2i+1})$, then

$$\begin{aligned}
\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) &\leq \lim_{i\to\infty} [\theta(\mathcal{H}_p(\pounds, \Omega_{2i+1}))]^k \\
&= [\theta(\lim_{i\to\infty} \mathcal{H}_p(\pounds, \Omega_{2i+1}))]^k \\
&= \theta(\mathcal{H}_p(\pounds, \pounds))^k < \theta(\mathcal{H}_p(\pounds, \pounds)),
\end{aligned}$$

which is a contradiction, because by Lemma 1, we have $\mathcal{H}_p(\pounds, \pounds) \leq \mathcal{H}_p(\psi(\pounds), \pounds)$, and θ is non-decreasing.

If $M_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \mathcal{H}_p(\pounds, \psi(\pounds))$, then $\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) < \theta(\mathcal{H}_p(\pounds, \psi(\pounds)))$, which is again a contradiction.

If $M_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \mathcal{H}_p(\Omega_{2i+1}, \phi(\Omega_{2i+1}))$, then $\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) < \theta(\mathcal{H}_p(\pounds, \pounds))$, which is a contradiction.

If

$$M_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \frac{\mathcal{H}_p(\pounds, \phi(\Omega_{2i+1})) + \mathcal{H}_p(\Omega_{2i+1}, \psi(\pounds))}{2},$$

then

$$\begin{aligned}
\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) &\leq \lim_{i\to\infty} \left[\theta\left(\frac{(\mathcal{H}_p(\pounds, \phi(\Omega_{2i+1}))) + \mathcal{H}_p(\Omega_{2i+1}, \psi(\pounds))}{2}\right)\right]^k \\
&\leq [\theta(\mathcal{H}_p(\pounds, \psi(\pounds)))]^k < \theta(\mathcal{H}_p(\pounds, \psi(\pounds)),
\end{aligned}$$

which gives a contradiction.

Consequently, in all the cases above, we get a contradiction; thus,

$$\mathcal{H}_p(\psi(\pounds),\pounds) = 0 \implies \pounds = \psi(\pounds).$$

Also, to show that $\phi(\pounds) = \pounds$, for the sake of contradiction, we assume that $\mathcal{H}_p(\pounds,\phi(\pounds)) \neq 0$; then, we have

$$\begin{aligned}\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) &= \theta(\lim_{i\to\infty}\mathcal{H}_p(\Omega_{2i+1},\phi(\pounds)))\\ &= \lim_{i\to\infty}\theta(\mathcal{H}_p(\psi(\Omega_{2i}),\phi(\pounds)))\\ &\leq \lim_{i\to\infty}[\theta(M_{\psi,\phi}(\Omega_{2i},\pounds))]^k, \end{aligned} \tag{14}$$

where

$$\begin{aligned}M_{\psi,\phi}(\Omega_{2i},\pounds) = \max\Big\{&\mathcal{H}_p(\Omega_{2i},\pounds),\mathcal{H}_p(\Omega_{2i},\psi(\Omega_{2i})),\mathcal{H}_p(\pounds,\phi(\pounds)),\\ &\frac{\mathcal{H}_p(\Omega_{2i},\phi(\pounds))+\mathcal{H}_p(\pounds,\psi(\Omega_{2i}))}{2}\Big\},\\ = \max\Big\{&\mathcal{H}_p(\Omega_{2i},\pounds),\mathcal{H}_p(\Omega_{2i},\Omega_{2i+1}),\mathcal{H}_p(\pounds,\phi(\pounds)),\\ &\frac{\mathcal{H}_p(\Omega_{2i},\phi(\pounds))+\mathcal{H}_p(\pounds,\Omega_{2i+1})}{2}\Big\}.\end{aligned}$$

Now, we discuss the following possible cases:

If $M_{\psi,\phi}(\Omega_{2i},\pounds) = \mathcal{H}_p(\Omega_{2i},\pounds)$, then, from (14), we have

$$\begin{aligned}\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) &\leq \lim_{i\to\infty}[\theta(\mathcal{H}_p(\Omega_{2i},\pounds))]^k\\ &= [\theta(\lim_{i\to\infty}\mathcal{H}_p(\Omega_{2i},\pounds))]^k,\\ &= \theta(\mathcal{H}_p(\pounds,\pounds))^k < \theta(\mathcal{H}_p(\pounds,\pounds)),\end{aligned}$$

which is a contradiction because θ is non-decreasing.

If $M_{\psi,\phi}(\Omega_{2i},\pounds) = \mathcal{H}_p(\Omega_{2i},\Omega_{2i+1})$, then from (14), we obtain $\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) < \theta(\mathcal{H}_p(\pounds,\pounds))$, thus presenting a contradiction.

If $M_{\psi,\phi}(\Omega_{2i},\pounds) = \mathcal{H}_p(\pounds,\phi(\pounds))$, then from (14), we obtain $\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) < \theta(\mathcal{H}_p(\pounds,\phi(\pounds)))$, a contradiction.

If

$$M_{\psi,\phi}(\Omega_{2i},\pounds) = \frac{\mathcal{H}_p(\Omega_{2i},\phi(\pounds))+\mathcal{H}_p(\pounds,\Omega_{2i+1})}{2},$$

then from (14), we obtain

$$\begin{aligned}\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) &\leq \lim_{i\to\infty}\left[\frac{\theta(\mathcal{H}_p(\Omega_{2i},\phi(\pounds)))+\mathcal{H}_p(\pounds,\Omega_{2i+1})}{2}\right]^k,\\ &\leq [\theta(\mathcal{H}_p(\pounds,\phi(\pounds))]^k < \theta(\mathcal{H}_p(\pounds,\phi(\pounds)).\end{aligned}$$

a contradiction.

Hence, from all the above cases, we conclude that

$$\mathcal{H}_p(\pounds,\phi(\pounds)) = 0 \implies \pounds = \phi(\pounds);$$

thus, $\psi(\pounds) = \pounds = \phi(\pounds)$. Hence, $\pounds$ is the common attractor for ψ and ϕ. Now, to show that $\pounds$ is unique for the sake of contradiction, we suppose that V is another common attractor for ψ and ϕ, so we have

$$\theta(\mathcal{H}_p(\pounds, V)) = \theta(\mathcal{H}_p(\psi(\pounds), \phi(V))) \leq [\theta(M_{\psi,\phi}(\pounds, V))]^k < \theta(\mathcal{H}_p(\pounds, V)),$$

a contradiction. □

Remark 1. *If we take metric space (Υ, d) instead of a partial metric and $h_i = g_i$ for all $i = 1, 2, 3, \cdots r$, in a θ-generalized IFS, then Theorem 2.7 of [15] becomes special cases of our main result Theorem 3.*

Remark 2. *Let $S_p(\Upsilon)$ be the collection of all singleton subsets of a partial metric space (Υ, p); then, $S_p(\Upsilon) \subseteq C^p(\Upsilon)$. Moreover, if we consider $h_k = h$ and $g_k = g$ for every k, then the pair of operators (ψ, ϕ) becomes*

$$(\psi(\{\varrho_1\}), \phi(\{\varrho_2\})) = (h(\varrho_1), g(\varrho_2)))$$

Thus, with these settings, we can obtain the following result:

Corollary 2. *Let (Υ, p) be a complete partial metric space, and*

$$\{\Upsilon; (h_k, g_k), k = 1, 2, \ldots, r\}$$

is a generalized IFS, and define a pair of mappings $h, g : \Upsilon \to \Upsilon$ as in Remark 2. If some $k \in [0, 1)$ and $\theta \in \Theta$ exist such that for any $\varrho_1, \varrho_2 \in \Upsilon$, the following condition holds:

$$\theta(p(h(\varrho_1), g(\varrho_2)) \leq \theta(Z_{h,g}(\varrho_1, \varrho_2))^k,$$

where

$$Z_{h,g}(\varrho_1, \varrho_2) = \max\{p(\varrho_1, \varrho_2), p(\varrho_1, h(\varrho_1)), p(\varrho_2, g(\varrho_2)), \frac{p(\varrho_1, g(\varrho_2)) + p(\varrho_2, h(\varrho_1))}{2}\}.$$

Then, h and g have a unique common fixed point $u_0 \in \Upsilon$. Furthermore, for any $u \in \Upsilon$, the sequence $\{u_0, hu_0, ghu_0, hghu_0, \ldots\}$ converges to the common fixed point of h and g.

Theorem 4. *Let (Υ, p) be a complete partial metric space and*

$$\{\Upsilon; (h_i, g_i), i = 1, 2, \cdots, r\}$$

be the generalized θ-contractive IFS, and the pair (ψ, ϕ) is a generalized rational Hutchinson θ-contractive operator; then, ψ and ϕ have a unique common attractor. Moreover, for any $\Omega_0 \in C^p(\Upsilon)$, the sequence

$$\{\Omega_0, \psi(\Omega_0), \phi\psi(\Omega_0), \ldots, \}$$

converges to the common attractor of ψ and ϕ.

Proof. Choose $\Omega_0 \in C^p(\Upsilon)$ and define a sequence as follows:

$$\Omega_1 = \psi(\Omega_0), \Omega_3 = \psi(\Omega_2), \ldots, \Omega_{2i+1} = \psi(\Omega_{2i}),$$

$$\Omega_2 = \phi(\Omega_1), \Omega_4 = \phi(\Omega_3), \ldots, \Omega_{2i+2} = \psi(\Omega_{2i+1}),$$

for any $i \in \{0, 1, 2, 3, \dots, \}$. Assuming that no three consecutive terms in the sequence are identical, then by using the fact that (ψ, ϕ) is a generalized rational Hutchinson θ-contractive operator, we have

$$\theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i})) \leq [\theta(R_{\psi,\phi}(\Omega_{2i}, \Omega_{2i+1}))]^k,$$

where

$$\begin{aligned} R_{\psi,\phi}(\Omega_{2i}, \Omega_{2i+1}) = \max \Big\{ & \frac{\mathcal{H}_p(\Omega_{2i}, \phi(\Omega_{2i+1}))[1 + \mathcal{H}_p(\Omega_{2i}, \psi(\Omega_{2i}))]}{2(1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}))}, \\ & \frac{\mathcal{H}_p(\Omega_{2i+1}, \phi(\Omega_{2i+1}))[1 + \mathcal{H}_p(\Omega_{2i}, \psi(\Omega_{2i}))]}{1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})}, \\ & \frac{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})[1 + \mathcal{H}_p(\Omega_{2i}, \psi(\Omega_{2i}))]}{1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})} \Big\} \\ = \max & \Big\{ \frac{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+2})}{2}, \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2}), \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}) \Big\} \\ \leq \max & \Big\{ \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2}) \Big\}, \end{aligned}$$

and since $\max\{\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}), \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})\} = \mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})$ is not possible, we have

$$\theta(\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})) \leq [\theta(\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1}))]^k,$$

and similarly

$$\theta(\mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})) \leq [\theta(\mathcal{H}_p(\Omega_{2i-1}, \Omega_{2i})]^k.$$

Thus, for any $i \in \{0, 1, 2, 3, \dots\}$, we have

$$\theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) \leq [\theta(\mathcal{H}_p(\Omega_{i-1}, \Omega_i))]^k.$$

Furthermore,

$$\theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) \leq [\theta(\mathcal{H}_p(\Omega_0, \Omega_1))]^{k^i}. \tag{15}$$

By taking limit $i \to \infty$ on both sides, we obtain

$$\lim_{i \to \infty} \theta(\mathcal{H}_p(\Omega_i, \Omega_{i+1})) = 1,$$

so, by θ_2 condition, we obtain

$$\lim_{i \to \infty} \mathcal{H}_p(\Omega_i, \Omega_{i+1}) = 0.$$

Moreover, by θ_3 condition, there exists $h \in (0, 1)$ such that

$$\lim_{i \to \infty} i\mathcal{H}_p(\Omega_i, \Omega_{i+1})^h = 0.$$

By the definition of the limit, there exists N_1, such that

$$\mathcal{H}_p(\Omega_i, \Omega_{i+1}) \leq \frac{1}{i^{\frac{1}{h}}}, \qquad \forall i \geq N_1.$$

Now, for natural numbers n, m with $m > n \geq N_1$, we have

$$\lim_{n,m \to \infty} \mathcal{H}_p(\Omega_n, \Omega_m) \leq \lim_{n,m \to \infty} \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{h}}} = 0$$

Therefore, $\{\Omega_i\}$ is a Cauchy sequence. Since $(C^p(\mathrm{Y}), \mathcal{H}_p)$ is complete, there exists $\pounds \in C^p(\mathrm{Y})$ such that (Ω_i) converges to $\pounds$. Thus,

$$\lim_{i\to+\infty} \mathcal{H}_p(\Omega_i, \pounds) = \lim_{i\to+\infty} \mathcal{H}_p(\Omega_i, \Omega_{i+1}) = \mathcal{H}_p(\pounds, \pounds), \tag{16}$$

and thus,

$$\lim_{i\to+\infty} \mathcal{H}_p(\Omega_i, \pounds) = 0 \text{ and } \mathcal{H}_p(\pounds, \pounds) = 0.$$

Now, first we show that $\psi(\pounds) = \pounds$; for this, we let $\mathcal{H}_p(\psi(\pounds), \pounds) \neq 0$. Now, by using the fact that (ψ, ϕ) is a generalized rational Hutchinson θ-contractive operator and θ is continuous, we have

$$\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) = \lim_{i\to\infty} \theta(\mathcal{H}_p(\psi(\pounds), \phi(\Omega_{2i+1}))) \leq \lim_{i\to\infty} [\theta(R_{\psi,\phi}(\pounds, \Omega_{2i+1}))]^k, \tag{17}$$

where

$$\begin{aligned} R_{\psi,\phi}(\pounds, \Omega_{2i+1}) = \max \Big\{ & \frac{\mathcal{H}_p(\pounds, \phi(\Omega_{2i+1}))[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{2(1 + \mathcal{H}_p(\pounds, \Omega_{2i+1}))}, \\ & \frac{\mathcal{H}_p(\Omega_{2i+1}, \phi(\Omega_{2i+1}))[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{1 + \mathcal{H}_p(\pounds, \Omega_{2i+1})}, \\ & \frac{\mathcal{H}_p(\pounds, \Omega_{2i+1})[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{1 + \mathcal{H}_p(\pounds, \Omega_{2i+1})} \Big\} \\ = \max \Big\{ & \frac{\mathcal{H}_p(\pounds, \Omega_{2i+2})[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{2(1 + \mathcal{H}_p(\pounds, \Omega_{2i+1}))}, \\ & \frac{\mathcal{H}_p(\Omega_{2i+1}, \Omega_{2i+2})[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{1 + \mathcal{H}_p(\pounds, \Omega_{2i+1})}, \\ & \frac{\mathcal{H}_p(\pounds, \Omega_{2i+1})[1 + \mathcal{H}_p(\pounds, \psi(\pounds))]}{1 + \mathcal{H}_p(\pounds, \Omega_{2i+1})} \Big\}. \end{aligned}$$

For all three possible cases of $R_{\psi,\phi}(\pounds, \Omega_{2i+1})$, we obtain $\theta(\mathcal{H}_p(\psi(\pounds), \pounds)) < \theta(0)$, but since θ is non-decreasing, we have $\mathcal{H}_p(\psi(\pounds), \pounds) < 0$, which is not possible. Hence, we reach a contradiction. So our supposition is wrong, and thus, we have $\mathcal{H}_p(\psi(\pounds), \pounds) = 0$; hence, $\psi(\pounds) = \pounds$. In a similar way, to show that $\phi(\pounds) = \pounds$, for the sake of contradiction, we assume that $\mathcal{H}_p(\pounds, \phi(\pounds)) \neq 0$, so by continuity of θ, we have

$$\theta(\mathcal{H}_p(\pounds, \phi(\pounds)) = \lim_{i\to\infty} \theta(\mathcal{H}_p(\psi(\Omega_{2i}), \phi(\pounds))) \leq \lim_{i\to\infty} [\theta(R_{\psi,\phi}(\Omega_{2i}, \pounds))]^k, \tag{18}$$

where

$$\begin{aligned} R_{\psi,\phi}(\Omega_{2i}, \pounds) = \max \Big\{ & \frac{\mathcal{H}_p(\Omega_{2i}, \phi(\pounds))[1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})]}{2(1 + \mathcal{H}_p(\Omega_{2i}, \pounds))}, \\ & \frac{\mathcal{H}_p(\pounds, \phi(\pounds))[1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})]}{1 + \mathcal{H}_p(\Omega_{2i}, \pounds)}, \\ & \frac{\mathcal{H}_p(\Omega_{2i}, \pounds)[1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})]}{1 + \mathcal{H}_p(\Omega_{2i}, \pounds)} \Big\}. \end{aligned}$$

Now, we discuss all possible cases for $R_{\psi,\phi}$.

If

$$R_{\psi,\phi}(\Omega_{2i}, \pounds)) = \frac{\mathcal{H}_p(\Omega_{2i}, \phi(\pounds))[1 + \mathcal{H}_p(\Omega_{2i}, \Omega_{2i+1})]}{2(1 + \mathcal{H}_p(\Omega_{2i}, \pounds))},$$

or

$$R_{\psi,\phi}(\Omega_{2i},\pounds) = \frac{\mathcal{H}_p(\Omega_{2i},\pounds)[1+\mathcal{H}_p(\Omega_{2i},\Omega_{2i+1})]}{1+\mathcal{H}_p(\Omega_{2i},\pounds)},$$

then we have $\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) < \theta(0)$, which is not possible. Moreover, if

$$R_{\psi,\phi}(\Omega_{2i},\pounds)) = \frac{\mathcal{H}_p(\pounds,\phi(\pounds))[1+\mathcal{H}_p(\Omega_{2i},\Omega_{2i+1})]}{1+\mathcal{H}_p(\Omega_{2i},\pounds)},$$

then by (18), we obtain $\theta(\mathcal{H}_p(\pounds,\phi(\pounds))) < \theta(\mathcal{H}_p(\pounds,\phi(\pounds)))$, which is again not possible. Hence, in all the possible cases, we get a contradiction, so our supposition is wrong; thus, we have $\mathcal{H}_p(\pounds,\phi(\pounds)) = 0$, and hence, $\phi(\pounds) = \pounds$. Consequently, we have $\psi(\pounds) = \pounds = \phi(\pounds)$, and hence, $\pounds$ is the common attractor for ψ and ϕ. Now, we show that $\pounds$ is unique. For this, for the sake of contradiction, assume that V is another common attractor for ψ and ϕ. Thus,

$$\theta(\mathcal{H}_p(\pounds,V)) = \theta(\mathcal{H}_p(\psi(\pounds),\phi(V))) \leq [\theta(R_{\psi,\phi}(\pounds,V))]^k < \mathcal{H}_p(\pounds,V),$$

a contradiction. □

Remark 3. *Since the function $\theta(t) = e^{\sqrt{t}}$ satisfies all the required conditions to be part of the class Θ, it follows that for this particular choice of θ in Θ, Theorems 3.1 and 3.7 of [14] are special cases of our Theorem 3 and Theorem 4, respectively.*

3. Examples

Example 1. *Let (Υ, p) be a partial metric space with $\Upsilon = [0,1]$, and for any $\varrho, \varsigma \in \Upsilon$, $p : \Upsilon \times \Upsilon \to \mathbb{R}$ is defined as follows:*

$$p(\varrho,\varsigma) = \max\{\varrho,\varsigma\}.$$

Moreover, we define $h_i, g_i : \Upsilon \to \Upsilon$, $i = 1,2,3$, as follows:

$$h_1(\varrho) = \frac{\varrho}{2}, \quad h_2(\varrho) = \frac{\varrho}{3}, \quad h_3(\varrho) = \frac{\varrho}{5}, \quad g_1(\varrho) = \frac{\varrho}{4}, \quad g_2(\varrho) = \frac{\varrho}{6}, \quad g_3(\varrho) = \frac{\varrho}{10}.$$

Thus, for $\theta(t) = e^t$, we have $\theta((p(h_1(\varrho),g_1(\varsigma))) = e^{(p(h_1(\varrho),g_1(\varsigma)))} = e^{\max\{\frac{\varrho}{2},\frac{\varsigma}{4}\}} = (e^{\max\{\varrho,\frac{\varsigma}{2}\}})^{\frac{1}{2}} \leq (e^{\max\{\varrho,\varsigma\}})^{\frac{1}{2}} = \theta(p(\varrho,\varsigma))^{\frac{1}{2}}$. Moreover, $\theta(p(h_2(\varrho),g_2(\varsigma))) \leq \theta(p(\varrho,\varsigma))^{\frac{1}{3}}$ and $\theta(p(h_3(\varrho),g_3(\varsigma))) \leq \theta(p(\varrho,\varsigma))^{\frac{1}{5}}$. Hence, (h_i,g_i) is a pair of generalized θ-contractive mappings; thus, by Theorem 2, for any $\Omega, \mho \in C^p(\Upsilon)$, we have

$$\theta(\mathcal{H}(\psi(\Omega),\phi(\mho))) \leq \theta(\mathcal{H}(\Omega,\mho))^k$$

where $\psi(\Omega) = h_1(\Omega) \cup h_2(\Omega) \cup h_3(\Omega), \phi(\mho) = g_1(\mho) \cup g_2(\mho) \cup g_3(\mho)$, and $k = \max\{1/2, 1/3, 1/5\} = 1/2$. Since θ is non-decreasing, we have $\theta(\mathcal{H}(\Omega,\mho)) \leq \theta(M_{\psi,\phi}(\Omega,\mho))$. Thus, we have

$$\theta(\mathcal{H}(\psi(\Omega),\phi(\mho))) \leq \theta(M_{\psi,\phi}(\Omega,\mho))^k.$$

Hence, (ψ,ϕ) is a generalized θ-contractive Hutchinson operator, so by Theorem 3, ψ,ϕ have a unique common attractor, which is actually at $\pounds = \{0\}$. In Figure 2, Figure 2a–c represent the values of $h_i(\Omega), i = 1,2,3$ for different iterations. The first horizontal line corresponds to $\Omega = [0,1]$, and the second, third, and fourth lines parallel to the horizontal axis represent the images of $h_i^2(\Omega), h_i^3(\Omega)$, and $h_i^4(\Omega)$, respectively. Moreover, the parallel lines in Figure 2d correspond to $\Omega, \psi(\Omega), \psi^2(\Omega), \psi^3(\Omega)$, and $\psi^4(\Omega)$, respectively. Also, in Figure 3, Figure 3a–c represent the values of $g_i(\Omega), i = 1,2,3$ for different iterations. The first horizontal line corresponds to $\Omega = [0,1]$, and the second, third, and fourth lines parallel to the horizontal axis represent the images

of $g_i^2(\Omega), g_i^3(\Omega)$, *and* $g_i^4(\Omega)$, *respectively. Moreover, the parallel lines in Figure 3d correspond to* $\Omega, \phi(\Omega), \phi^2(\Omega), \phi^3(\Omega)$, *and* $\phi^4(\Omega)$, *respectively.*

Example 2. *Let* $\Upsilon = [0,1] \times [0,1]$ *with a partial metric* $p : \Upsilon \times \Upsilon \to \mathbb{R}$ *be defined for any* $\varrho = (\varrho_1, \varsigma_1), \varsigma = (\varrho_2, \varsigma_2) \in \Upsilon$ *as follows:*

$$p(\varrho, \varsigma) = \sqrt{(\varrho_1 - \varrho_2)^2 + (\varsigma_1 - \varsigma_2)^2} + b,$$

where $b \geq 0$. *Let* $h_i, g_i : \Upsilon \to \Upsilon$, *for* $i = 1, 2$, *be self-mappings defined as*

$$h_1(\varrho, \varsigma) = \left(\frac{sin\varrho}{1 + sin\varrho}, \frac{1}{1 + \varsigma}\right), \quad g_1(\varrho, \varsigma) = \left(\frac{\varrho}{1 + \varrho}, \frac{1}{1 + \varsigma}\right)$$

and

$$h_2(\varrho, \varsigma) = \left(\frac{sin\varrho}{1 + sin\varrho}, \frac{1}{1 + sin\varsigma}\right), \quad g_2(\varrho, \varsigma) = \left(\frac{\varrho}{1 + \varrho}, \frac{1}{1 + sin\varsigma}\right).$$

Thus,

$$\begin{aligned} p(h_1(\varrho), g_1(\varsigma)) &= p(h_1(\varrho_1, \varsigma_1), g_1(\varrho_2, \varsigma_2)) \\ &= \sqrt{\left(\frac{sin\varrho_1}{1 + sin\varrho_1} - \frac{\varrho_2}{1 + \varrho_2}\right)^2 + \left(\frac{1}{1 + sin\varsigma_1} - \frac{1}{1 + \varsigma_2}\right)^2} + b \\ &= \sqrt{\frac{(sin\varrho_1 - \varrho_2)^2}{(1 + sin\varrho_1)^2(1 + \varrho_2)^2} + \frac{(\varsigma_1 - \varsigma_2)^2}{(1 + \varsigma_2)^2(1 + sin\varsigma_1)^2}} + b \\ &< \sqrt{(sin\varrho_1 - \varrho_2)^2 + (\varsigma_1 - \varsigma_2)^2} + b \\ &\leq \sqrt{(\varrho_1 - \varrho_2)^2 + (\varsigma_1 - \varsigma_2)^2} + b = p(\varrho, \varsigma). \end{aligned}$$

Similarly, $p(h_2(\varrho), g_2(\varsigma)) < p(\varrho, \varsigma)$; *hence, for any* $\theta \in \Theta$ *and* $i = 1, 2$, *we have* $\theta(p(h_i(\varrho), g_i(\varsigma))) < \theta(p(\varrho, \varsigma))$, *so there exist* k_i *such that* $\theta(p(h_i(\varrho), g_i(\varsigma))) \leq \theta(p(\varrho, \varsigma))^{k_i}$, *and thus* (h_i, g_i) *is a generalized* θ*-contraction. Thus, by Theorem 2,* (ψ, ϕ) *is a generalized* θ*-contractive Hutchinson operator, so by Theorem 3,* ψ *and* ϕ *have a unique common attractor, which is actually at* $\pounds = \{(0,0)\}$. *Figures 4 and 5 show the images for different iterations of* h_1, h_2, ψ *and* g_1, g_2, ϕ, *respectively.*

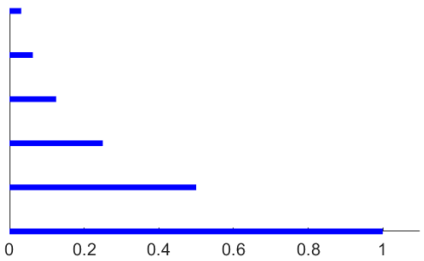
Example 3. *Let* $\Upsilon = [0,1] \times [0,1] \times [0,1]$ *with a partial metric* $p : \Upsilon \times \Upsilon \to \mathbb{R}$ *be defined for any* $\varrho = (\varrho_1, \varsigma_1, z_1), \varsigma = (\varrho_2, \varsigma_2, z_2) \in \Upsilon$ *as follows:*

$$p(\varrho, \varsigma) = \sqrt{(\varrho_1 - \varrho_2)^2 + (\varsigma_1 - \varsigma_2)^2 + (z_1 - z_2)^2} + b,$$

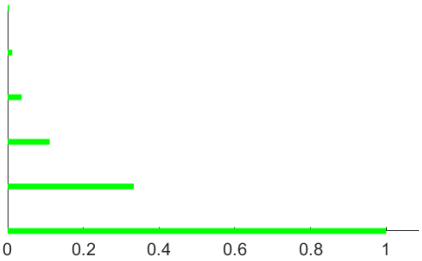
where $b \geq 0$. *Let* $h, g : \Upsilon \to \Upsilon$ *be self-mappings defined as* $h(\varrho_1, \varsigma_1, z_1) = (\frac{\varrho_1}{6}, \frac{\varsigma_1}{27}, \frac{z_1}{12})$ *and* $g(\varrho_2, \varsigma_2, z_2) = (\frac{\varrho_2}{18}, \frac{\varsigma_2}{9}, \frac{z_2}{36})$; *then,*

$$\begin{aligned} p(h(\varrho), g(\varsigma)) &= \sqrt{\left(\frac{\varrho_1}{6} - \frac{\varrho_2}{18}\right)^2 + \left(\frac{\varsigma_1}{27} - \frac{\varsigma_2}{9}\right)^2 + \left(\frac{z_1}{12} - \frac{z_2}{36}\right)^2} + b \\ &\leq \frac{1}{6}\sqrt{\left(\varrho_1 - \frac{\varrho_2}{3}\right)^2 + \left(\varsigma_1 - \frac{\varsigma_2}{3}\right)^2 + \left(z_1 - \frac{z_2}{3}\right)^2} + b \\ &\leq \sqrt{(\varrho_1 - \varrho_2)^2 + (\varsigma_1 - \varsigma_2)^2 + (z_1 - z_2)^2} + b = p(\varrho, \varsigma). \end{aligned}$$

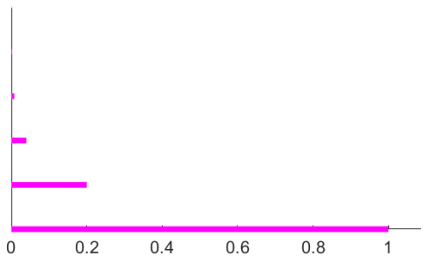
Thus, for any $\theta \in \Theta$, we have $\theta(p(h(\varrho), g(\varsigma))) \leq \theta(\varrho, \varsigma)$; hence, (h, g) is a generalized θ-contractive mapping, and thus, by Theorem 2, (ψ, ϕ) is a generalized θ-Hutchinson contractive operator. Here, for any $\Omega, \mho \in C^p(\mathrm{Y})$, $\psi(\Omega) = h(\Omega) = \{h(\varrho); \varrho \in \Omega\}$ and $\phi(\mho) = g(\mho) = \{g(\varrho); \varrho \in \mho\}$. Hence, by Theorem 3, ψ and ϕ have a unique common fixed attractor, which is actually at $\{(0,0,0)\}$. In Figure 6, the images show different iterations for ψ, while the images for different iterations of ϕ are similar as shown in Figure 6, so that is why we show the different iterations of ϕ under the influence of ψ in Figure 7; furthermore, Figure 8 provides the different iterations as in the previous two figures for the settings of the common attractors for ψ and ϕ.



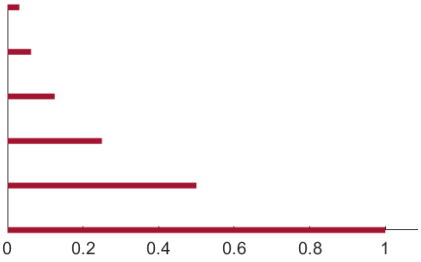
(a) $h_1(\Omega) = \{h_1(\varrho) : \varrho \in \Omega\}$



(b) $h_2(\Omega) = \{h_2(\varrho) : \varrho \in \Omega\}$

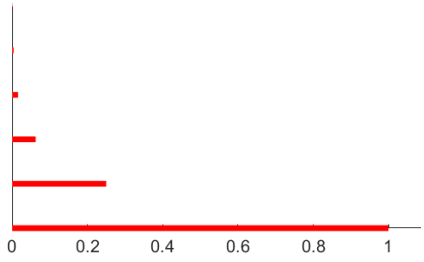


(c) $h_3(\Omega) = \{h_3(\varrho) : \varrho \in \Omega\}$

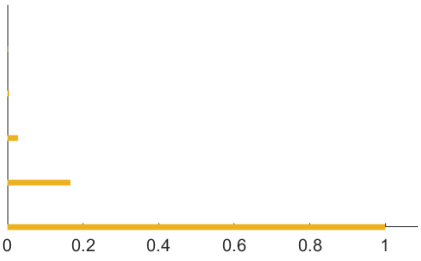


(d) $\psi(\Omega) = h_1(\Omega) \cup h_2(\Omega) \cup h_3(\Omega)$

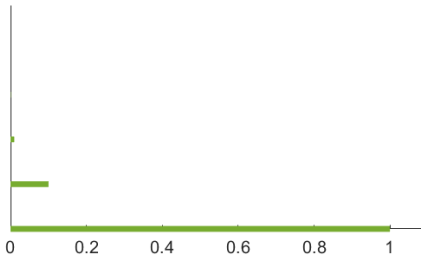
Figure 2. Iterations for ψ.



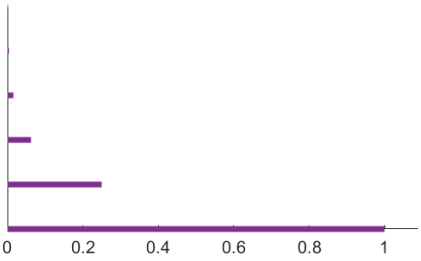
(a) $g_1(\Omega) = \{g_1(\varrho) : \varrho \in \Omega\}$



(b) $g_2(\Omega) = \{g_2(\varrho) : \varrho \in \Omega\}$



(c) $g_3(\Omega) = \{g_3(\varrho) : \varrho \in \Omega\}$



(d) $\phi(\Omega) = g_1(\Omega) \cup g_2(\Omega) \cup g_3(\Omega)$

Figure 3. Iterations for ϕ.

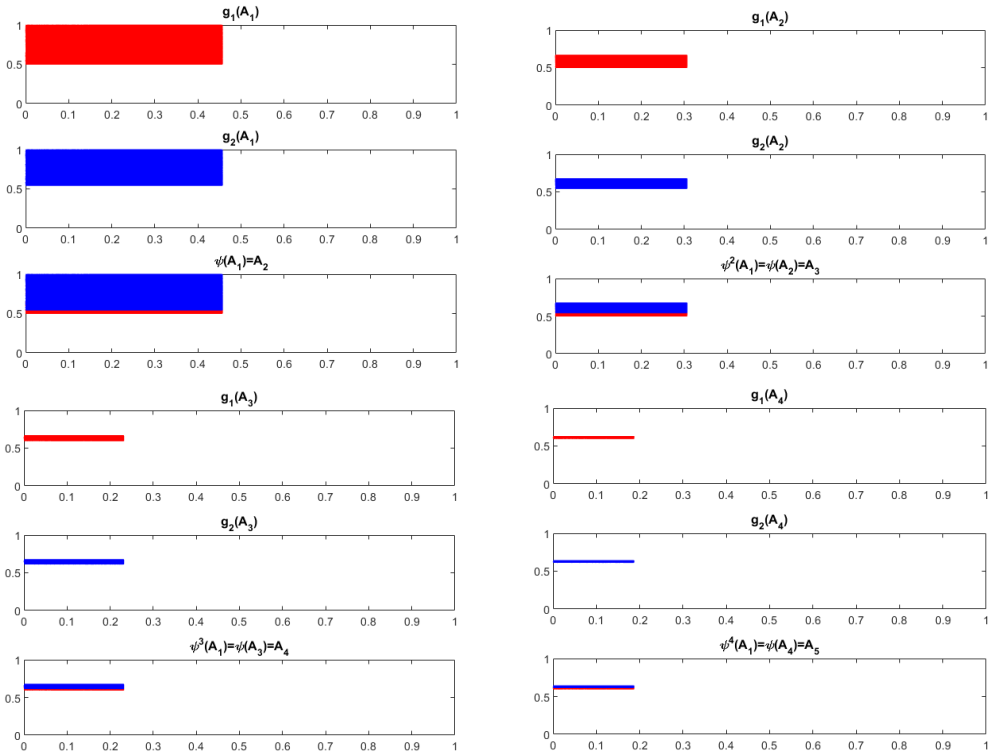


Figure 4. Iterations for ψ.

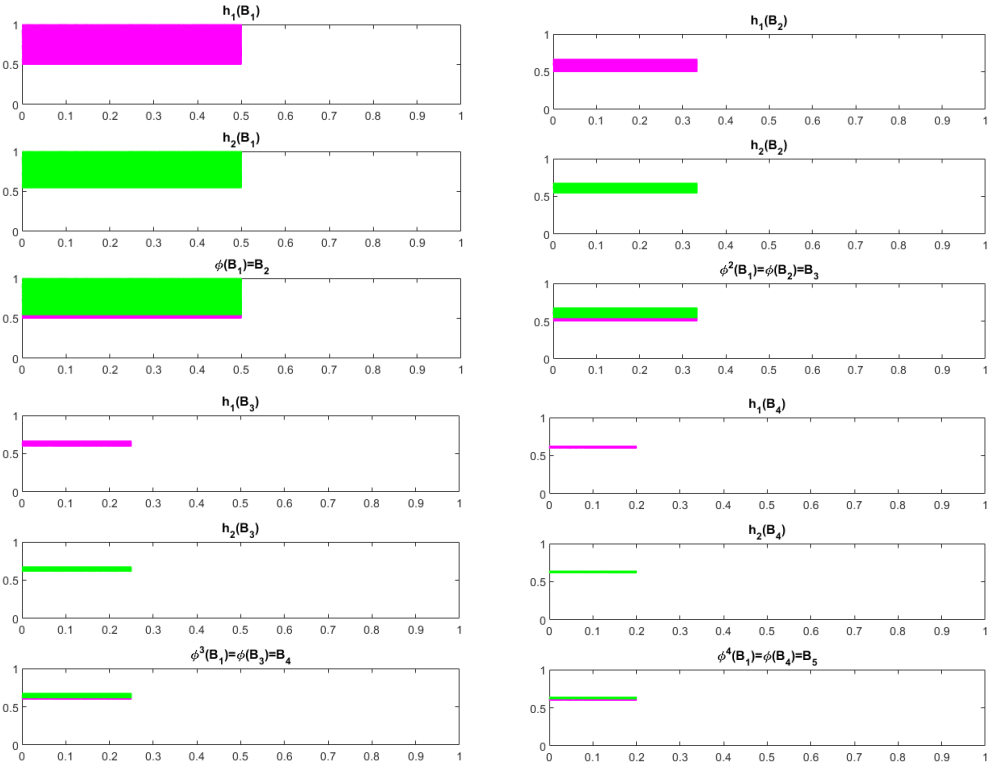


Figure 5. Iterations for ϕ.

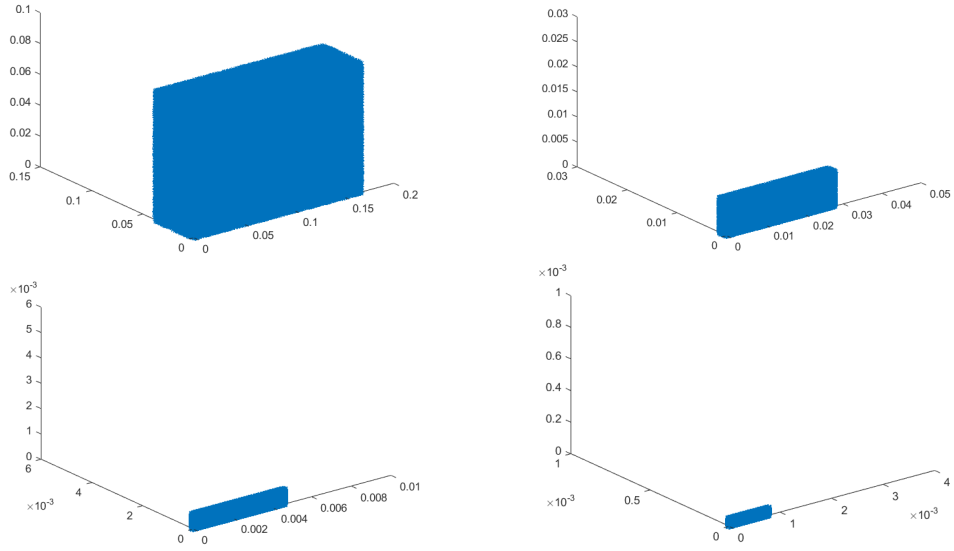


Figure 6. Iterations for ψ.

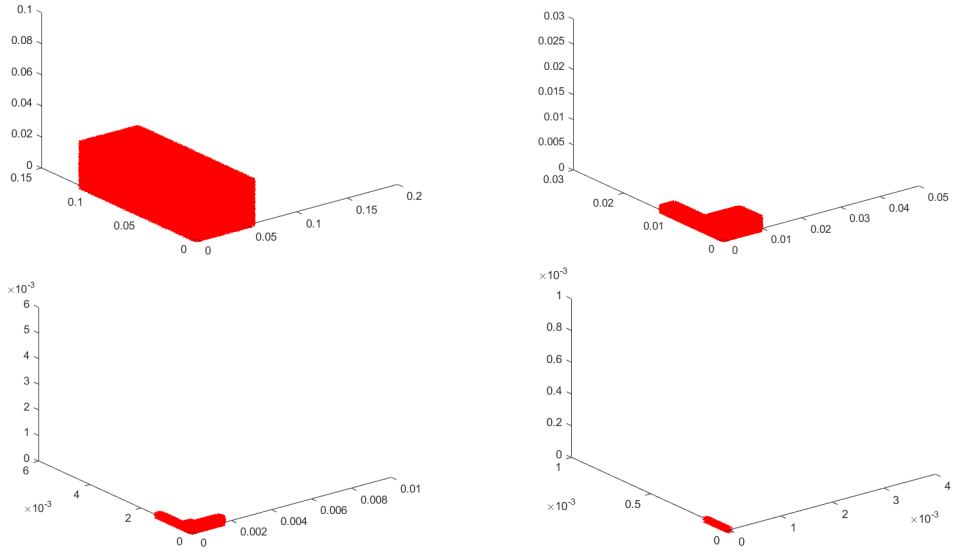


Figure 7. Iterations for ϕ under influence of ψ.

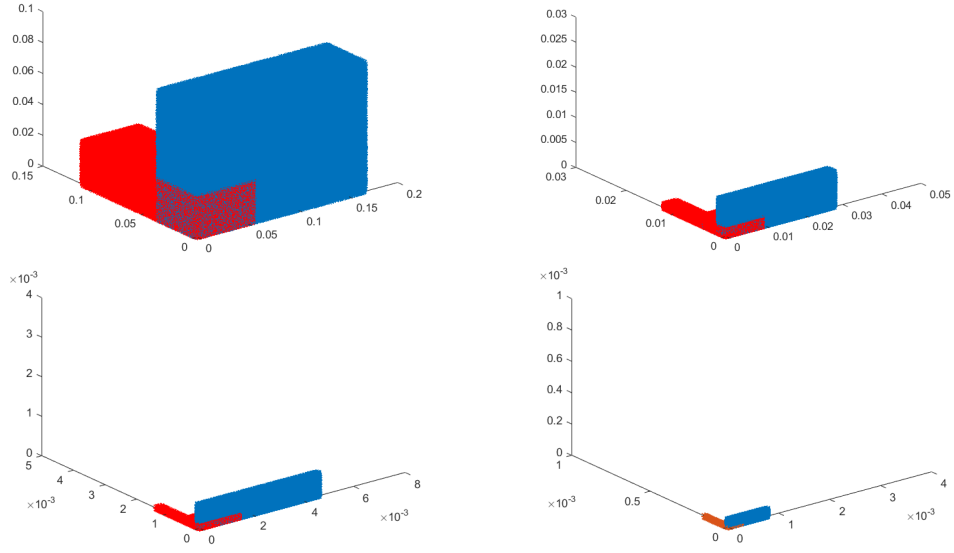


Figure 8. Iterations for understanding of common attractor.

4. Application to Functional Equation

In this section, we present an application of our main result by solving a functional equation that arises in dynamic programming. For a deeper understanding of functional equations in dynamic programming, one may refer to [17,18] and the references therein. Consider W_1, W_2 to be Banach spaces with V_1, V_2 as their sub spaces, respectively. Suppose $k : V_1 \times V_2 \to V_1$, $g_1, g_2 : V_1 \times V_2 \to R$, and $h_1, h_2 : V_1 \times V_2 \times R \to R$. If we consider V_1 and V_2 as the state and decision spaces, respectively, then the problem of dynamic programming reduces to the problem of solving the functional equations:

$$q_1(\varrho) = \sup_{\varsigma \in V_2} \{g_1(\varrho, \varsigma) + h_1(\varrho, \varsigma, q_1(k(\varrho, \varsigma)))\}, \quad \varrho \in V_1,$$

$$q_2(\varrho) = \sup_{\varsigma \in V_2} \{g_1(\varrho, \varsigma) + h_2(\varrho, \varsigma, q_2(k(\varrho, \varsigma)))\}, \quad \varrho \in V_1.$$

The above equations can be reformulated as

$$q_1(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_1(\varrho, \varsigma, q_1(k(\varrho, \varsigma)))\} - b, \quad \varrho \in V_1, \tag{19}$$

$$q_2(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_2(\varrho, \varsigma, q_2(k(\varrho, \varsigma)))\} - b, \ \varrho \in V_1, \tag{20}$$

where $b > 0$. Let $\mho(V_1)$ be the collection of bounded real valued functions which are defined on V_1, and for any $\tau \in \mho(V_1)$, the norm $\|.\|$ on $\mho(V_1)$ is defined as follows:

$$\|\tau\| = \sup_{t \in V_1} |\tau(t)|.$$

Thus, $\mho(V_1)$ along with the above norm is a Banach space. Moreover, the following is the partial metric on $\mho(V_1)$:

$$p_{\mho}(\tau, \xi) = \sup_{t \in V_1} | \tau(t) - \xi(t) | + b,$$

where $\tau, \xi \in \mho(V_1)$ and $b \geq 0$. Assume that

(C_1) g_1, g_2, h_1, and h_2 are bounded and continuous.
(C_2) For $\varrho \in V_1, \tau \in \mho(V_1)$ and $b > 0$, take $\vartheta, \phi : \mho(V_1) \to \mho(V_1)$ as

$$\vartheta_\tau(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_1(\varrho, \varsigma, \tau(k(\varrho, \varsigma)))\} - b, \quad \varrho \in V_1 \tag{21}$$

$$\phi_\tau(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_2(\varrho, \varsigma, \tau(k(\varrho, \varsigma)))\} - b, \quad \varrho \in V_1. \tag{22}$$

Moreover, for every $(\varrho, \varsigma) \in V_1 \times V_2, \tau, \xi \in \mho(V_1)$ and $t \in V_1$, the following is implied:

$$\theta(p(h_1(\tau), h_2(\vartheta)) + b) \leq \theta(Z_{\vartheta, \phi(\tau(t), \xi(t))})^k, \tag{23}$$

where

$$Z_{\vartheta, \phi(\tau(t), \xi(t))} = \max\{p_{\mho}(\tau(t), \xi(t)), p_{\mho}(\tau(t), \vartheta_\tau(t)), p_{\mho}(\xi(t), \phi(\xi(t)), \frac{p_{\mho}(\tau(t), \phi\xi(t)) + p_{\mho}(\xi(t), \vartheta\tau(t))}{2}\}. \tag{24}$$

Theorem 5. *Assume that the conditions* (C_1) *and* (C_2) *hold. Then, the functional equations*

$$q_1(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_1(\varrho, \varsigma, q_1(k(\varrho, \varsigma)))\} - b, \quad \varrho \in V_1,$$

$$q_2(\varrho) = \sup_{\varsigma \in V_2} \{g_2(\varrho, \varsigma) + h_2(\varrho, \varsigma, q_2(k(\varrho, \varsigma)))\} - b, \ \varrho \in V_1,$$

have a unique common and bounded solution in $\mho(V_1)$.

Proof. Since $(\mho(V_1), p_{\mho})$ is a complete partial metric space, by (C_1) condition, ϑ and ϕ are self-mappings of $\mho(V_1)$. Moreover, by the definition of the supremum and the C_2 condition, it follows that for any $\tau, \xi \in \mho(V_1)$ and $b > 0$, we can choose $\varrho \in V_1$ and $\varsigma_1, \varsigma_2 \in V_2$ such that

$$\vartheta_\tau < g_2(\varrho, \varsigma_1) + h_1(\varrho, \varsigma_1, \tau(k(\varrho, \varsigma_1))) \tag{25}$$

$$\phi_\xi < g_2(\varrho, \varsigma_2) + h_2(\varrho, \varsigma_2, \tau(k(\varrho, \varsigma_2))), \tag{26}$$

which further implies that

$$\vartheta_\tau \geq g_2(\varrho, \varsigma_2) + h_1(\varrho, \varsigma_2, \tau(k(\varrho, \varsigma_2))) - b, \tag{27}$$

$$\phi_\xi \geq g_2(\varrho, \varsigma_1) + h_2(\varrho, \varsigma_1, \tau(k(\varrho, \varsigma_1))) - b. \tag{28}$$

From (25) and (28), we obtain

$$\begin{aligned} \vartheta_\tau(t) - \phi_\xi(t) &< h_1(\varrho, \varsigma_1, \tau(k(\varrho, \varsigma_1))) - h_2(\varrho, \varsigma_1), \xi(k(\varrho, \varsigma_1)) + b \\ &\leq \mid h_1(\varrho, \varsigma_1, \tau(k(\varrho, \varsigma_1))) - h_2(\varrho, \varsigma_1), \xi(k(\varrho, \varsigma_1)) \mid + b \\ &= p(h_1(\tau), h_2(\vartheta)). \end{aligned} \tag{29}$$

From (26) and (27), we obtain

$$\begin{aligned} \vartheta_\tau(t) - \phi_\xi(t) &< h_2(\varrho, \varsigma_2, \xi(k(\varrho, \varsigma_2))) - h_1(\varrho, \varsigma_2), \tau(k(\varrho, \varsigma_2)) + b \\ &\leq \mid h_1(\varrho, \varsigma_2, \tau(k(\varrho, \varsigma_2))) - h_2(\varrho, \varsigma_2), \xi(k(\varrho, \varsigma_2)) \mid + b \\ &= p(h_1(\tau), h_2(\vartheta)). \end{aligned} \tag{30}$$

From (29) and (30), we obtain

$$-p(h_1(\tau), h_2(\vartheta)) \leq \vartheta_\tau(t) - \phi_\xi(t) \leq p(h_1(\tau), h_2(\vartheta)).$$

Hence,

$$\mid \vartheta_\tau(t) - \phi_\xi(t) \mid \leq p(h_1(\tau), h_2(\vartheta)),$$

or

$$\mid \vartheta_\tau(t) - \phi_\xi(t) \mid + b \leq p(h_1(\tau), h_2(\vartheta)) + b,$$

and thus

$$p(\vartheta_\tau, \phi_\xi) \leq p(h_1(\tau), h_2(\vartheta)) + b.$$

Since θ is non-decreasing, we have

$$\theta(p(\vartheta_\tau, \phi_\xi)) \leq \theta(p(h_1(\tau), h_2(\vartheta)) + b).$$

Hence, by applying C_2, we obtain

$$\theta(p(\vartheta_\tau, \phi_\xi)) \leq \theta(p(h_1(\tau), h_2(\vartheta)) + b) \leq \theta(Z_{\vartheta,\phi}(\tau(t), \xi(t))^k,$$

where $Z_{\vartheta,\phi(\tau(t),\xi(t))}$ is the same as defined in (24). Thus, all requirements of Corollary 2 are satisfied. Hence, there exists a common fixed point of ϑ and ϕ that is $\tau^* \in \mho(V_1)$, where $\tau^*(t)$ is a common solution of functional Equations (19) and (20). □

5. Conclusions

This work focuses on a generalized framework for the study of attractors, which are among the most fascinating aspects of mathematics due to their intricate and visually appealing patterns. The primary results of this study are Theorems 3 and 4. In Theorem 3, we investigate the existence and uniqueness of common attractors for generalized Hutchinson θ-contractive operators. In Theorem 4, we extend these results to a more generalized setting by considering cases where the Hausdorff metric appears in the denominator. Specifically, we establish the existence and uniqueness of common attractors for generalized rational Hutchinson θ-contractive operators. Consequently, our findings generalize numerous existing results in the literature, as highlighted in Remarks 2 and 3.

A particularly interesting aspect of this work is the illustration of these theoretical results with intuitive and easily comprehensible examples in one-, two-, and three-dimensional Euclidean spaces. We not only justify our theoretical claims but also provide graphical representations of these examples in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. The attractors manifest as lines in the one-dimensional case, strips in the two-dimensional case, and cubes in the three-dimensional case. Through iterative computations, we verify the existence of common attractors in each scenario, distinguishing them using different colors for better visualization.

In the final section of this manuscript, we further validate our main results by applying them to a problem in dynamical systems. Specifically, we demonstrate the effectiveness of our approach by solving a functional equation arising in dynamic programming. This application reinforces the practical significance and broad applicability of our findings.

For future research, this framework can be extended by exploring different types of contraction mappings and visualizing their effects on attractor formation. Additionally, instead of relying solely on metric structures, one can investigate more generalized topological or algebraic frameworks to analyze the validity of such results. Another promising direction is the construction of more complex and innovative examples, as this study primarily focuses on cases that are easily understandable for readers. Furthermore, the applicability of these theoretical findings in other mathematical fields can be explored, leading to new results and deeper insights.

Overall, this study contributes to a deeper understanding of attractors and their mathematical properties, providing both theoretical advancements and practical insights into their structure and behavior.

Author Contributions: Conceptualization, A.R. and M.A.; methodology, N.S.; software, N.S. and S.A.; validation, N.S., A.R. and M.D.l.S.; formal analysis, N.S. and M.A.; investigation, M.D.l.S.; writing—original draft preparation, N.S. and A.R.; writing—review and editing, M.A. and M.D.l.S.; funding acquisition, M.D.l.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Basque Government for funding this research work through Grant IT1555-22. They also thank MICIU/AEI/10.13039/501100011033 and FEDER/UE for partially funding their research work through Grants PID2021-123543OB-C21 and PID2021-123543OB-C22.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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