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A COMPARISON OF NUMERICAL ODE SOLVERS BASED ON EULER METHODS

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Abstract-A class of nonlinear methods based on Euler's integration formula for the numerical solution of ordinary differential equations is presented. Numerical example involving stiff linear systems of first-order differential equations are given for test problem.

1.INTRODUCTION

The well-known trapezoidal formula for the numerical solution of the initial value problem $y' = f(x,y)$, $y(x_0)=y_0$ is given by

$$y_{n+1} - y_n = \frac{h}{2}\left[f_n + f_{n+1}\right] \tag{1.1}$$

where h is the mesh length in the x direction. The numerical formula (1.1) is a one-step implicit method that has the desirable features of performing well when applied to stiff problems and also being A-stable method [1].In approach by [2], the nonlinear equivalent of the trapezoidal formula (1.1) , referred to as the geometric mean (GM)Euler formula is of the form

$$y_{n+1} - y_n = h\sqrt{f_n\, f_{n+1}} \tag{1.2}$$

obtained by replacing the arithmetic mean (AM) of the function values f_n and f_{n+1} in (1.1) by the geometric mean (GM) averaging.

In this paper we will study the equivalent formulae in the geometric mean sense.

2.CLASS OF NONLINEAR METHODS BASED ON EULER FORMULAE

Consider the following two formulae of Euler

$$y_{n+1} = y_n + h f_n \tag{2.1}$$

$$y_{n+1} = y_n + h f_{n+1}. \tag{2.2}$$

The first formula being the forward Euler and the second is the backward Euler formula. Generalised form of weighted Euler formula is

$$y_{n+1} = y_n + h\left[\theta f_n + (1-\theta) f_{n+1}\right]. \tag{2.3}$$

By putting $\theta = 0$ in (2.3) we obtain the special case of backward Euler method , while putting $\theta = 1$ in (2.3) then we obtain the forward Euler method.

We shall refer to this formula as the GM Euler formula and can be similarly generalized by taking the powers of Equations (2.1) and (2.2) , i.e.,

$$(y_{n+1} - y_n)^{\theta} = (h f_n)^{\theta} \tag{2.4}$$

$$(y_{n+1} - y_n)^{1-\theta} = (h f_{n+1})^{1-\theta} \tag{2.5}$$

By multiplying (2.4) and (2.5) we obtain

$$y_{n+1} - y_n = h(f_n)^{\theta} (f_{n+1})^{1-\theta} \tag{2.6}$$

formula (2.6) is the equivalent nonlinear form of Equation (2.3) and contains as special cases Euler $(\theta = 1)$, Backward Euler $(\theta = 0)$ and GM Euler $(\theta = 1/2)$.

3.ACCURACY OF THE NONLINEAR GM EULER FORMULA

To establish the local truncation error for (1-2),we note from the Taylor series expansion of $y(x_{n+1})$ about x_n, an approximation of which is represented by y_{n+1} , the expansion for f_n and f_{n+1} can be written as

$$f_n = y_n' \tag{3.1}$$

$$f_{n+1} = y_{n+1}' = y_n' + h y_n'' + \frac{h^2}{2} y_n''' + \frac{h^3}{6} y_n'''' + \ldots \tag{3.2}$$

and

$$\sqrt{f_n f_{n+1}} = \sqrt{y_n' y_{n+1}'} = y_n' \left[1 + h \frac{y_n''}{y_n'} + h^2 \frac{y_n''}{2 y_n'} + \ldots \right]^{1/2}$$

$$= y_n' + \frac{h}{2} y_n'' + h^2 \left(\frac{y_n'''}{4} - \frac{y_n''^2}{8 y_n'} \right) + \ldots \tag{3.3}$$

Substituting (3.1) and (3.2) into (1.2) yields the expansion

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + h^3 \left(\frac{y_n'''}{4} - \frac{y_n''^2}{8y_n'} \right) + \ldots \tag{3.4}$$

The Taylor expansion of $y(x_{n+1})$ about x_n through terms of order h^3 is

$$y(x_{n+1}) = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n'' + \ldots \tag{3.5}$$

Subtracting (3.5) from (3.4) yields

$$y(x_{n+1}) - y_{n+1} = h^3 \left(-\frac{y_n'''}{12} + \frac{y_n''^2}{8y_n'} \right) + O(h^4). \tag{3.6}$$

Comparison of (3.4) and (3.5) shows that the formula (1.2) is of second order with principal error term given by

$$LTE = h^3 \left(-\frac{y_n'''}{12} + \frac{y_n''^2}{8y_n'} \right). \tag{3.7}$$

Thus, this shows that nonlinear GM Euler method has third order accuracy.

4.STABILITY ANALYSIS

To study the stability properties of the Geometric Mean (GM) formula (1.2) , we apply the test equation [4]

$$y' = \lambda y \quad , \quad y(0) = 1 \tag{4.1}$$

to the formula (1.2).For such a test , (1.2) can be seen to reduce to

$$y_{n+1} = y_n + h\lambda \sqrt{y_n y_{n+1}} \ . \tag{4.2}$$

Dividing both sides of (4.2) by y_n we obtain

$$\frac{y_{n+1}}{y_n} = 1 + h\lambda \sqrt{\frac{y_{n+1}}{y_n}} \ . \tag{4.3}$$

Thus we may write

$$y_{n+1} = y_n R(\mu) \quad \text{where} \quad \lambda h = \mu \ . \tag{4.4}$$

Then the amplification factor $R(\mu)$, is given by

$$R(\mu) = 1 + \mu\sqrt{R(\mu)}$$

and further writing $\sqrt{R(\mu)} = A_n$, we obtain the quadratic equation

$$A_n^2 = 1 + h\lambda\, A_n$$

or

$$A_n^2 - h\lambda\, A_n - 1 = 0 \quad . \tag{4.5}$$

Absolute stability requires that

$$\left|\frac{y_{n+1}}{y_n}\right| = \left|R(\mu)\right| = \left|A_n^2\right| \langle\, 1$$

or , similarly $\left| A_n \right| \langle\, 1$. From (4.5) we write the roots as

$$A_n = \frac{\mu \pm \sqrt{\mu^2 + 4}}{2}$$

Taking only the positive sign we have

$$\left|A_n\right| = \left|\frac{\mu + \sqrt{\mu^2 + 4}}{2}\right| \tag{4.6}$$

from the result $-1 \,\langle\, R(\mu) \,\langle\, 1$, the method (GM) is A-stable [1].But we have shown that this method is also L-stable.

Definition1.

A one-step method is said to be L-stable if it is A-stable and , in addition , when applied to the scalar test equation $y' = \lambda y$, λ is a complex number with $\mathrm{Re}\,\lambda \langle 0$, it yields $y_{n+1} = R(h\lambda)y_n$, where $\left|R(h\lambda)\right| \to 0$ as $\mathrm{Re}(h\lambda) \to -\infty$ [3].

Applying the Definition 1 to the formula (4.6) yields

$$\lim_{\mathrm{Re}(\mu)\to-\infty}\left|A_n\right| = \lim_{\mathrm{Re}(\mu)\to-\infty}\left|\frac{\mu + \sqrt{\mu^2 + 4}}{2}\right| = \lim_{\mathrm{Re}(\mu)\to-\infty}\left|\frac{\left(\mu + \sqrt{\mu^2 + 4}\right)\left(\mu - \sqrt{\mu^2 + 4}\right)}{2\left(\mu - \sqrt{\mu^2 + 4}\right)}\right|$$

$$= \lim_{Re(\mu)\to-\infty} \left| \frac{-2}{\left(\mu - \sqrt{\mu^2+4}\right)} \right| = 0$$

However we have shown that when coupled with (1-2) it produces a method which is A-stable and L-stable. Thus ,the magnification factors for the four methods are given

1)Euler's method $\quad , R(\mu) = 1 + \mu$

2)Backward Euler's method $\quad , R(\mu) = \dfrac{1}{1-\mu}$

3)Trapezoidal method (AM) $\quad , R(\mu) = \dfrac{1+\frac{\mu}{2}}{1-\frac{\mu}{2}}$

4)GM Euler's Method $\quad , R(\mu) = \dfrac{\mu + \sqrt{\mu^2+4}}{2}$

Thus , form the definition of A-stability , the Euler and Trapezoidal (AM) methods are A-stable but Backward Euler and G.M.Euler methods are L-stable.

5.NUMERICAL EXAMPLE

Problem 1:

$$\begin{aligned} y_1' &= y_1 + y_2; y_1(0) = 0 \\ y_2' &= -y_1 + y_2; y_2(0) = 1 \end{aligned} \qquad 0 \le x \le 1 \qquad (5.1)$$

This has an exact solution

$$y_1(x) = e^x \sin x \quad , \quad y_2(x) = e^x \cos x$$

and the absolute errors for both y_1 and y_2 are listed in Table 1.

Table 1 Absolute errors in y_1 and y_2 at x = 0.5

	Trapezoidal	Backward Euler	Geometric Euler	Forward Euler
h=0.2	y_1 = 1.76-03	y_1 = 2.26-03	y_1 = 1.77-03	y_1 = 2.25-03
	y_2 = 2.06-03	y_2 = -2.94-03	y_2 = 2.19-03	y_2 = -2.95-03
h=0.3	y_1 = 2.19-03	y_1 = 3.69-03	y_1 = 2.23-03	y_1 = 3.68-03
	y_2 = 6.57-03	y_2 = -8.43-03	y_2 = 7.27-03	y_2 = -8.44-03
h=0.5	y_1 = 2.40-03	y_1 = 0.72-02	y_1 = 0.62-03	y_1 = 0.70-02
	y_2 = 1.01-03	y_2 = 1.06-03	y_2 = 1.76-03	y_2 = 3.10-03

Problem 2:

$$y' = -y^2; y(0) = 1 , \qquad 0 \le x \le 1 \qquad (5.2)$$

This has an exact solution

$$y(x) = \frac{1}{1+x}$$

and the absolute errors at x=1 for Problem 2 are listed in Table 2.

Table 2 Absolute errors in y at x = 1

	Trapezoidal	Backward Euler	Geometric Euler	Forward Euler
h=0.2	y = 0.633 -03	y = 0.143-01	y = 0.141-01	y = 0.324-01
h=0.3	y = 0.213-02	y = 0.173-01	y = 0.304-01	y = 0.165-01
h=0.5	y = 0.875 -02	y = 0.208-01	y = 0.196-01	y = 0.476-01

Table 1and Table2 illustrates the errors obtained by using the various formulas in solving (5.1) and (5.2) for different stepsize. One can see that the improvement in accuracy is gained by using the nonlinear Trapezoidal (AM) formula by comparing the errors in each row.

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