Mathematical & Computational Applications, Vol. 3, No. 3, pp. 139-151, 1998

BOUNDARY ELEMENT FORMULATIONS IN ELASTOPLASTIC STRESS ANALYSIS

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Abstract: This paper presents a review of different elasto-plastic Boundary Element (BE) formulations with particular emphasis on two main approaches; the initial strain displacement- gradient approach with its modeling of the partial or full interior domain, and the particular integral approach which can be applied exclusively to the surface avoiding any modeling interior. The initial strain formulation is implemented in a computer program using two-dimensional isoparametric quatratic elements to discretise either the complete interior domain or only the part associated with the plastic region. The BE solutions are shown to be in good agreement with analytical and Finite Element (FE) solutions.

1. INTRODUCTION

The Boundary Element (BE) method is well established as an accurate numerical tool particularly well suited for linear elastic problems. Its extension to non-linear analysis such as elasto-plasticity, however, is not widespread and in many formulation the interior of the solution domain has to be discretized, thus losing the main BE advantage of surface-only modeling.

An outline of different elasto-plastic BE formulation is presented in this paper with particular emphasis on two main approaches; (i) the initial strain displacement-gradient approach in which interior discretization is required either for whole domain or only in the region of expected plastic behaviour, and (ii) the particular integral approach in which interior modeling is not required and can be applied exclusively to the surface. The analytical and numerical implementation of both approaches are presented.

Details of a quadratic BE formulation for the displacement-gradient approach are presented for two-dimensional elasto-plastic problems in which three-node isoparametric quadratic elements are used to model boundary and eight-node isoparametric quadrilateral quadratic elements are used to model the interior domain. The values of stress and strain rates at interior nodes are calculated via the numerical differentiation of the displacement rates in an element-wise manner; an approach similar to that used in FE formulation.

Details of the numerical implementation algorithm which uses load incrementation and an iterative procedure are presented. To asses the accuracy of the BE formulation, the initial strain displacement-gradient BE formulation is implemented in a computer program and applied to some practical problems. The problems include a square subjected to uniform tension, and a thick cylinder under internal pressure. The BE solutions are compared with the corresponding FE solutions and exact or experimental solutions.

2. REVIEW OF ELASTOPLASTIC BE FORMULATION

The first elasto-plastic BE formulation presented by Swedlow and Cruse [1] was based on a direct analytical formulation. Riccardella [2] presented the initial strain formulation based on a constant plastic strain over each internal cell with a non-iterative procedure. Mendelson [3] provided a review of the BE formulations which is based on indirect, semi-direct and direct approaches in two and three-dimensional problems. Mukherjee [4] presented a correct direct BIE formulation in plane strain analysis. Telles and Brebbia [5] presented a direct BE formulation based on the initial strain approach with corrections for the internal stresses and a semi-analytical approach for the efficient evaluation of the strongly singular integrals appearing in the domain kernels by using linear elements.The indirect BE approach in elasto-plasticity was developed by some others such as Banerjee and Mustoe [6], Kobayashi and Nishimura [7], Morjaria and Mukherjee [8], but the direct approach has been widely developed.

Faria *et al* [9] performed the singular integrals in a manner similar to that of Telles and Brebbia [10] by using quadratic elements. The elasto-plastic BE formulation was discussed in detail by Lee [11] who presented an accelerated convergence procedure using an initial strain approach and quadratic elements. Some authors, such as Tan and Lee [12] and Lee and Fenner [13], used this approach to analyse practical problems such as fracture problems.

Other non-linear BE formulations have been developed for viscoplasticity and time-dependent problems (see, e.g. Kumar and Mukherjee [14], Telles and Brebbia [15], Banerjee and Davies [16] and Ahmad and Banerjee [17]).

One of the difficulties encountered in all non-linear BE analysis is the evaluation of the singular integrals, defined only in the cauchy principal values sense, arising in the solution domain. Henry and Banerjee[18] presented a particular integral approach to circumvent the singular volume integrals. Okada *et al* [19] presented another approach, which handles geometric and material non-linearity problems, based on the interpolation of the basic variables to be computed in solution domain. Banerjee and Ravendra [20] presented a direct approach to evaluate the strongly singular integrals by excluding a small sphere, where load point is located, from the integration of volume cell. Banerjee *et al* [21] presented an indirect approach, initial stress expansion technique, which is based on the admissible stress states for the evaluation of the strongly singular integrals.

Lu and Ye [22] presented a direct technique to evaluate the strongly singular integrals by use of coordinate transformation and a form of Stokes' theorem with numerical examples, whereas Guiggiani and Gigante [23] used a taylor series expansion and local polar coordinates. The study of Guiggiani *et al*[24] provide a genaral algorithm in order to treat numerically the hyper-singular integrals arising in BE formulations.Dallner and Kuhn[25] presented a direct approach for the efficient evaluation of the strongly singular integrals with tree-dimensional examples by using a regularised formulation based on the Gauss theorem. This approach is capable of handling viscoplasticity and large deformation problems.

3. THE INITIAL STRAIN BE APPROACH

3.1. Analytical Formulation

Unlike the FE method which requires whole body discretization, the BE method reduces the dimensionalty by one by transforming the variables from volume to surface values, hence only the boundary of the domain requires discretization. The basis of the BE formulation is a boundary integral identity for displacements, relating the displacement at an interior point P to the displacements and tractions at boundary point Q, over the surface S, as follows (e.g. Brebbia *et al* [26], Becker [27]and Banerjee[28]):

$$u_i(P) + \int_S T_{ij}(P,Q)\, u_j(Q)\, dS(Q) = \int_S U_{ij}(P,Q)\, t_j(Q)\, dS(Q) \qquad (1)$$

Where u_i and t_i are the displacement and traction vectors respectively, and U_{ij} and T_{ij} are the displacement and traction kernels, respectively, which are functions of position and material properties. The above equation ignores the effects of body forces and plasticity.

To include the effect of the elastoplastic material behaviour, an additional volume term based on the work done by the strain rate, $\dot{\varepsilon}^p_{ij}$ multiplied by the stresses at the load point can be written as follows (see e.g. Lee[11]):

$$\int_A W_{kij}(P,q)\, \dot{\varepsilon}^p_{ij}(q)dA \qquad (2)$$

where A is the area of the domain and q is an interior point. The dot above the strain indicates the rate of the chance of the strain with respect to time. The kernel W_{kij} can be interpreted as the stress at point q due to a unit load at the load point P in the k[th] direction. This approach is referred to as initial strain method. Alternatively, the initial stress, $\dot{\sigma}^p_{ij}$, can be used as the primary unknown. Hence, by using the given relationship between the initial stress increment(rates) and the initial strain increments, in quasi-static behaviour, the plastic integral term can be written as follows:

$$\int_A V_{kij}(P,q)\, \dot{\sigma}^p_{ij}(q)dA \qquad (3)$$

where the kernel V_{kij} can be interpreted as the strain at point q due to a unit load at the load point p in the k[th] direction.

To calculate the plastic strain rates, the Von Mises rule is used. The following expressions can be used for plastic strain increments in terms of the total strain increments

$$\dot{\varepsilon}^p_{ij} = \frac{3}{2}\left(\frac{\dot{S}_{kl}\ \dot{\varepsilon}_{kl}}{1+H/3\mu}\right)\frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \qquad (4)$$

where $\dot{S}_{kl}$ and $\dot{\sigma}_{eq}$ are current deviatoric and equivalent stresses respectively, and H is the slope of the uniaxial plastic stress-strain curve. Alternatively the plastic strain increments can be expressed in terms of the stress increments as follow:

$$\dot{\varepsilon}^p_{ij} = \frac{9}{4}\left(\frac{\dot{S}_{kl}\ \dot{\sigma}_{kl}}{1+H}\right)\frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \qquad (5)$$

3.2. Numerical Implementation

It is obvious from the elasto-plastic BE formulation discussed previously that both boundary elements and domain cells (internal cells) are necessary in order to perform the integrals arising in the BE formulation. Both the boundary elements and the domain cells are used in two-dimensional elasto-plastic BE analysis are illustrated in Figure 1. In a manner similar to the elasto-static BE analysis, the boundary is represented as a collection of boundary elements. The zones, where the plastic deformation is expected in solution domain, is discretised into domain cells in order to perform domain integrals. The elastoplastic BE equation in the initial strain approach (without considering body forces) in discretized form, can be written as follows:

$$C_{ij}\dot{u}_i(P) + \sum_{m=1}^{M}\sum_{c=1}^{3}\dot{u}_j(Q)\int_{-1}^{+1}T_{ij}(P,Q)N_c(\xi)J(\xi)d\xi = \sum_{m=1}^{M}\sum_{c=1}^{3}\dot{t}_j(Q)\int_{-1}^{+1}U_{ij}(P,Q)N_c(\xi)J(\xi)d\xi$$

$$+\sum_{m=1}^{D}\sum_{c=1}^{8}\dot{\varepsilon}_{ij}^{p}(Q)\int_{-1}^{+1}\int_{-1}^{+1}W_{ijk}(P,q)N_c(\xi_1,\xi_2)J(\xi_1,\xi_2)d\xi_1 d\xi_2 \qquad (6)$$

where P denotes the node where the integration is performed, Q indicates the c^{th} node of the boundary element and q indicates the c^{th} node of a domain cell. $N_c(\xi)$ is the quadratic shape function and $J(\xi)$ is the jacobian of transformation.

3.2.1. Integration Scheme

The integrals appearing in equation (6) have to be calculated in order to obtain the coefficients. The kernels functions contain singularities of the order of $1/r$ or $1/r^2$ where r is the distance between the load point P and the field point Q or the interior point q. Therefore, the integrals become singular when P coincides with either Q or q. It is important the examine the numerical evaluation of the integrals in such cases.

When P and Q or q are in different elements, there is no singularity, and the gaussian quadrature formulae can be used. When P is a node of the element in which Q or q are located, there are two situations to be considered. When P and Q are different nodes in the same element, the Gaussian quadrature formulae can also be used. However, for the case when P coincides with either Q or q, special integration schemes have to be devised. For the boundary element, when P coincedes with Q, the integrals and the free-term coefficients, $C_{ij}(P)$ can be calculated using rigid body motion(see e.g. Becker[27]). For the domain integrals, when P coincides with q, the domain cell must be divided into sub-elements in order to performing integration, as shown in Figure 2. In this scheme, the quadrilateral elements is divided into or three triangular sub-elements (for details, see Lee[11], and Becker[27]).

3.2.2. The Displacement Gradient Approach

The main advantage of the displacement gradient approach is that it is possible to circumvent the strongly singular integrals by differentiating the displacement rates via the shape functions in order to obtain the strain and the stress rates.

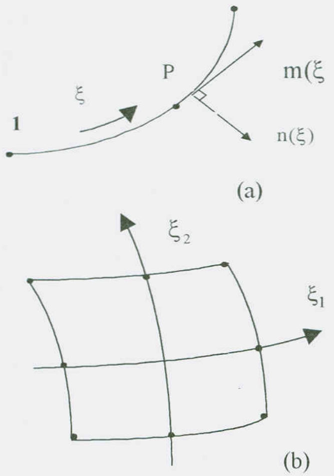

Figure 1: Typical 3-node boundary element and 8-node domain cell

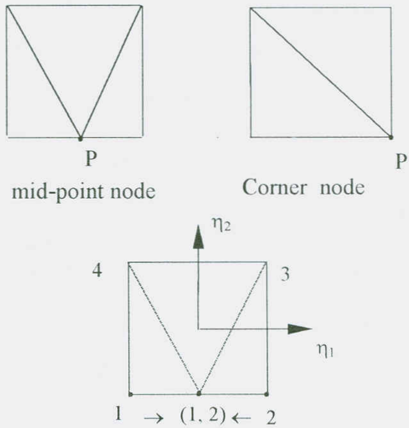

Figure 2: Sub-division of the domain cell into triangles in singular integration

By differentiating the interior displacements via the shape functions over each domain cell, and using the jacobian of transformation, the displacement differentials with respect to the local coordinates ξ_1 and ξ_2 can be obtained. To obtain the total strain rates, the strain-displacement relationships can be used as follows:

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) \tag{7}$$

In plane stress problems the total strain rate in the third direction can be obtained by using $\sigma_{33} = 0$, i.e.

$$\dot{\varepsilon}_{zz} = \frac{-\upsilon}{1-\nu}(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}) - \frac{1-2\nu}{1-\nu}(\dot{\varepsilon}^p_{xx} + \varepsilon^p_{yy}) \tag{8}$$

Using Hook's law, the cell stress rates can be written as follows:

$$\dot{\sigma}_{ij} = 2\mu(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij}) + \frac{2\mu\nu}{1-2\nu}\delta_{ij}\dot{\varepsilon}_{kk} \tag{9}$$

Therefore, the internal stress rates can be calculated without having to deal with an integral identity for which contains strongly singular integrals. Full details of the evaluation of stress and strain rates at boundary and interior nodes are presented by Gun [29].

4. THE PARTICULAR INTEGRAL BE APPROACH

4.1. Analytical Formulation

It is well known that the solution of partial differential equations (e.g. [30]) can be obtained using a complementary function (C.F.) and a particular integral (P.I.). Therefore, the displacement in the governing differential equations can be defined as a combination of a complementary function and a particular integral, as follows:

$$u_i = (u_i)^{CF} + (u_i)^{PF} \tag{10}$$

Similarly the tractions, stresses and strains can be written in terms of complementary and particular integral components as follows:

$$\begin{aligned} t_i &= (t_i)^{CF} + (t_i)^{PF} \\ \sigma_i &= (\sigma_i)^{CF} + (\sigma_i)^{PF} \\ \varepsilon_i &= (\varepsilon_i)^{CF} + (\varepsilon_i)^{PF} \end{aligned} \tag{11}$$

The elastic solution given by the boundary integral equation (equation 1) is the complementary solution, i.e.

$$\dot{u}_i^{CF}(p) + \int_s T_{ij}(P,Q)\dot{u}_j^{CF}(Q)dS = \int_S U_{ij}(P,Q)\dot{t}_j^{CF}(Q)dS \tag{12}$$

The following expressions can be written for the particular solutions for the stresses.

$$\dot{\sigma}_{ij} = 2\mu(\dot{\varepsilon}_{ij}^{CF} + \dot{\varepsilon}_{ij}^{PF}) + \frac{2\mu}{1-2\nu}(\dot{\varepsilon}_{kk}^{CF} + \dot{\varepsilon}_{kk}^{PI}) - \dot{\sigma}_{ij}^{i} \tag{13}$$

Where $\dot{\sigma}_{ij}^{i}$ is the initial stress which can be expressed in terms of a global shape function, $K(Q,P)$, as follows(see Henry and Banerjee[18]):

$$\dot{\sigma}_{ij}^{i}(Q) = \sum_{m=1}^{M} K(Q,P_m)\ \dot{\phi}_{ij}(P_m) \tag{14}$$

where P_m represents the boundary nodes and internal (fictitious) nodes in the solution domain, and $\dot{\phi}_{ij}(P_m)$ is a fictitious tensor density. It should be noted that fictitious interior nodes have to be consistent with the boundary discretization of the domain to be solved. M is the total number of known can be expressed for two-dimensional problems as a fourth-order differential as follows (see Henry [31]):

$$K(Q,P_m) = \frac{\partial^4 C(Q,P_m)}{\partial^2 x_m \partial^2 x_n} \tag{15}$$

where $C(Q,P_m)$ is given by

$$C(Q,P_m) = C_n^4(r^4 - b_n r^5) \tag{16}$$

where C_0 is characteristic length related to the solution domain, which can be chosen as the largest distance between the nodes in the solution domain, and $r(Q,P_m)$ is the distance between the field point Q and load point P_m. The parameter b_n is chosen to minimize the solution error which may be caused by arbitrary ordering of the nodes, by scaling down each column of the matrix $K(Q,P_m)$ such that the lowest value is forced to be zero in order to optimize the solution matrix.

For two-dimensional problems, the particular integral for the displace ment rates can be expressed as follows(see Henrry[31] and Kane[32]):

$$\dot{u}_i^{PI}(P,Q) = \sum_{m=1}^{M} D_{iml}^{PI}(Q,P_m)\ \dot{\phi}_{lm}(P_m) \tag{17}$$

The particular solution for stress can be obtained by differentiating the displacement functions to obtain the strain functions and then substituting in equation (13). The particular solution for the traction rate is obtained from the stress functions as follows:

$$\dot{t}_i^{PI}(P,Q) = \sum_{m=1}^{M} T_{iml}^{PI}(Q,P_m)\ \dot{\phi}_{lm}(P_m) \tag{18}$$

where n_j is the unit outward normal.

4.2. Numerical Implementation

By considering a particular load increments, the complementary displacement and traction rates at the boundary can be written as follows:

$$\left[\dot{u}^{CF}\right] = \left[\dot{u}\right] - \left[\dot{u}^{PF}\right]$$

$$\left[\dot{t}^{CF}\right] = \left[\dot{t}\right] - \left[\dot{t}^{PF}\right] \tag{19}$$

It should be noted that the complementary solution is simply the elastic solution. Therefore, by introducing the particular integrals into the system equations, the elasto-plastic system equations can be formed as follows:

$$[A]\left[\dot{u}\right] - [A]\left[\dot{u}^{PF}\right] = [B]\left[\dot{t}\right] - [B]\left[\dot{t}^{PI}\right] \tag{20}$$

which can be rearranged as follows:

$$[A]\left[\dot{u}\right] = [B]\left[\dot{t}\right] - [B]\left[\dot{t}^{PI}\right] + [A]\left[\dot{u}^{PI}\right] \tag{21}$$

As discussed previously, the displacement and traction rates are arranged such that all unknown variables are placed in the left hand side, and all prescribed values in the right hand side. The system equations can be formed as followed.

$$\left[A^*\right]\left[\dot{x}\right] = \left[B^*\right]\left[\dot{y}\right] - [B]\left[\dot{t}^{PI}\right] + [A]\left[\dot{u}^{PI}\right] \tag{22}$$

where the matrices $[A^*]$ and$[B^*]$ are the modified forms of $[A]$ and $[B]$, while the vector $[x]$ and $[y]$ contain the unknown and known values respectively, of either traction or displacement rates. By using equations (17) and (18), the particular solutions for displacement and traction rates are given in matrix form as follows:

$$\left[\dot{u}^{PI}\right]=\left[D^{PI}\right]\left[\dot{\phi}\right]$$

$$\left[\dot{t}^{PI}\right]=\left[T^{PI}\right]\left[\dot{\phi}\right] \tag{23}$$

In this expression the rectangular matrices $[D^{PI}]$ and $[T^{PI}]$ are the size $(2N \times 3M)$ where N is the total number of boundary nodes and M is the total number of the boundary and interior (fictitious) nodes in the solution domain. Details of the calculation of the calculation of the initial stress and plastic strain increments can be found in reference[31].

4.3. Comparison of the Initial Strain and the Particular Integral Approaches

As mentioned earlier, in order to include the effect of plasticity in the initial strain BE formulation, interior discretization is required, which results in strongly singular integrals. A significant increase in analytical and numerical work is required to maintain the accuracy of the BE formulation. However, by using the displacement-gradient approach, the strongly singular integrals can be avoided. The particular integral approach uses fictitious nodes (internal nodes) in the solution domain to evaluate the internal variables. Comparing the numerical implementation of the particular integral approach to the initial strain approach several advantages and disadvantages of the particular integral approach can be identified.

The main advantage of the particular integral approach is the elimination of the interior discretization into domain cells. However, the penalty for this is a greater degree of the mathematical complexity, and the fact that the distribution of the fictitious nodes needs to be consistent with the boundary discretization of the solution domain. Furthermore, a great deal of effort is needed to produce a robust general-purpose BE computer program which may be used by inexperienced users. Therefore, it is concluded that, despite its disadvantage of full or partial interior discretization, the initial strain method with numerical differentiation of the Displacement is a more attractive option for general robust BE software.

5. INCREMENTAL/ITERATIVE SOLUTION PROCEDURES

In the elasto-plastic FE analysis, depending on the formulation of the stiffness matrix, either the tangential stiffness technique or initial stiffness technique can be employed. However, because of the nature of the elasto-plastic BE formulation, neither of these formulation can be used. Instead, the concepts of the initial strain and initial stress techniques, used in the FE approach, can be modified in order to be applicable to the elasto-plastic BE analysis.

In the elasto-plastic BE analysis, the plastic strain increments can be calculated by using the flow rule expressions in equations(4) and (5). In equation(5) the actual stress increments are required in order to obtain the initial strain increments. The initial stresses can be obtained by using equation(4) in which the total strain increments are assumed to be known. In the initial strain approach it is possible to obtain the plastic strain increments by using equation (4), which can handle the perfectly plastic material behaviour. In BE formulations, there is no significant difference between the initial stress approach and initial strain approach, because the integral equations in both approaches include the effect of the plasticity. The initial strain formulation is considered more suitable for traction-control problems, because the first approximations for the stress increments are usually reasonably accurate.

A robust incremental iterative procedure and the details of the evaluation of the plastic strain rates and the initial plastic stress rates for both approaches are presented by Gun[29].

6. EXAMPLES OF ELASTOPLASTIC APPLICATIONS

The present initial strain formulation with the displacement-gradient approach is implemented in a computer program and applied to some classical test problems. The results are compared with either analytical solutions or the corresponding FE solutions.

6.1. Uniaxial Tensile Problem (plane stress)

This problem concerns a square plate subjected to uniform tension in the x-direction. The BE discretization is shown in Figure 3 where 4 boundary elements and 1 cell are used. The material is loaded in tension up to 596 N/mm^2 using 26 increments. The material properties are:

$$\sigma = \varepsilon E \qquad \text{for} \quad \sigma \prec \sigma_y$$

$$\sigma = \sigma_y \left(\frac{E\ \varepsilon}{\sigma_y} \right)^n \qquad \text{for} \quad \sigma \succ \sigma_y$$

where the yield stress, $\sigma_y = 483$ N/mm^2, E=207000 N/mm^2, n=0.1, and $\nu = 0.3$.

The BE solutions are compared with the corresponding analytical and FE solutions obtained using ABAQUS[33] in Figure 4 where the agreement is excellent.

6.2. Thick Cylinder under Internal Pressure

This problem concerns a thick cylinder under an internal pressure in which the diameter ratio R_2/R_1 is taken to be 2. The analytical solution of this problem was presented by Hodge and White[34]. The geometry and loading conditions are shown in Figure 5. The material is assumed to be elastic-perfectly plastic with the following material properties: $\sigma_y = 200$ N/mm^2, E=200000 N/mm^2 and $\nu = 0.33$.

The boundary element discretization is shown in Figure 6 where a 15^0 sector is used to represent the cylinder with 4 cells.

The variation of the load factor P/σ_y with the non-dimensionalised displacement ($4\mu\, u_b/\sigma_y a$) at the outer radius for BE mesh and the corresponding FE solutions are depicted in Figure 7. In the BE analysis 5 load increments were used to reach the load factor, P/σ_y, value of 0.79. The BE solutions show very good agreement with the analytical and FE results.

Figure 8 shows the hoop stress distribution along the radius for the load factor, P/σ_y, value of 0.76. The BE results, computed using 5 load increments, are again in good agreement with the analytical and FE results.

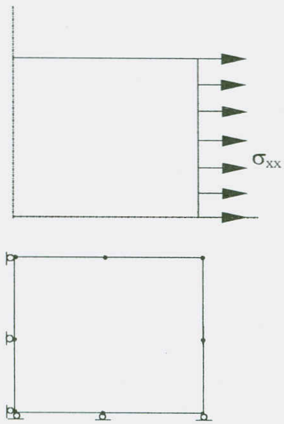

Figure 3: Uniaxial tension problem and BE mesh design

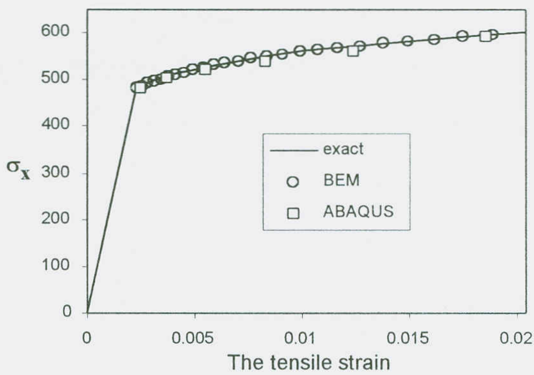

Figure 4: Solutions for the uniaxial problem

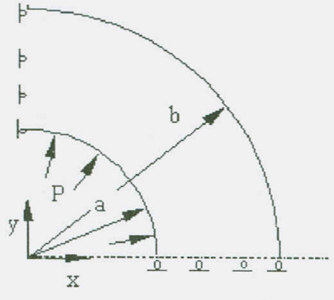

Figure 5: The pressured cylinder problem

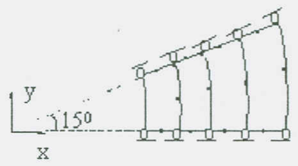

Figure 6: BE discretization for the Pressurized thick cylinder

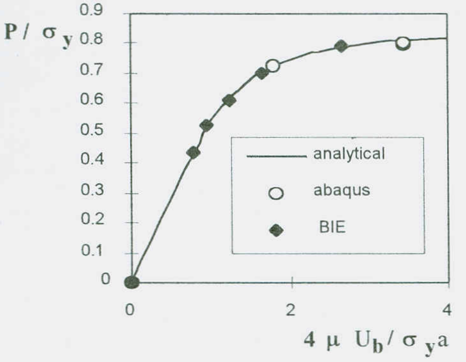

Figure 7: Non-dimensionalized radial displacement at the outer radius of thick cylinder

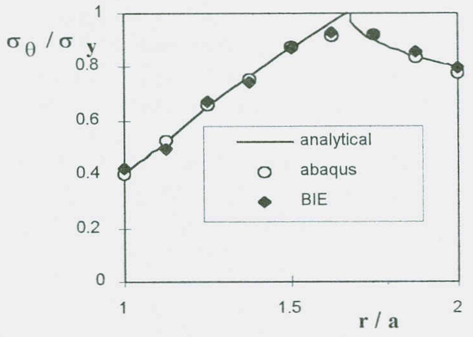

Figure 8: hoop stresses with radius at the load factor P / σ_y

7. CONCLUSION

A brief review of elastoplastic BE formulations was presented. In the initial strain approach, the plastic strain rates are treated as initial strain rates. The initial stress approach is very similar to the initial strain except that the initial plastic stress rates is used as a primary domain unknown in the integral equations. The choice between the use of the initial strain and initial stress formulation is not critical because the effect of plasticity is catered for in the integral equations. The initial strain approach, however, is considered more suitable for traction-control problems because the first approximations for the stress increments are usually accurate. In order to circumvent the strongly singular integrals arising in the domain kernels, the stress and strain rates inside the domain are calculated by numerical differentiation of the displacement rates obtained from the boundary integral equations in an element-wise manner. This approach is referred to as the displacement-gradient approach.

As an alternative formulation to the initial strain, the particular integral formulation is also presented. The main advantage of this method is that it is possible to eliminate the domain integrals. However, the main disadvantage of the particular integral approach is that it requires a fictitious interior node distribution in order to avoid the discretization of the entire solution domain, which makes its implementation in a general-purpose computer program very complex.

The initial strain approach is implemented in a computer program and applied to elasto-plasticity problems. The BE solutions obtained from the computer program are shown to be accurate and in good agreement with the FE and other solutions.

The present elasto-plastic BE formulation can be extended to cover contact mechanics problems. In contact applications, introducing frictional stick-slip behaviour would require a carefully designed robust numerical algorithm to incorporate nested iterations and load increments that are capable of monitoring contact development and marching the solution along the elasto-plastic material path. This is the ongoing research work.

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