

APPROXIMATE BOUNDARY LAYER SOLUTION OF A MOVING BEAM PROBLEM

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Abstract- The transverse vibrations of a simply supported beam moving with constant velocity is considered. The case of transition from string to beam effects is treated. In this model, the fourth order spatial derivative multiplies a small parameter and hence leads to a boundary layer problem. The problem is solved approximately using the method of multiple scales.

1. INTRODUCTION

Band saws, fiber textiles, paper sheets, aerial cable tramways, oil pipelines, magnetic tapes, power transmission belts are all classified as axially moving material problems. These problems are categorized into two groups: 1) axially moving strings 2) axially moving beams. Due to their technological importance, a vast amount of literature exists on the topic. The review papers by Ulsoy *et al.* [1] and Wickert and Mote [2] contains work on axially moving strings and beams upto 1988.

For more recent work on beam vibrations see Wickert and Mote [3], Wickert [4], Pellicano and Zirilli [5] and Öz *et al.* [6]. The transition behavior from string to beam has first been investigated by references [5] and [6]. In reference [5], a boundary layer solution was presented using a modified Lindstedt Poincare method. In [6], an outer solution was presented using the method of multiple scales. A stability analysis was also included.

Classical methods find the inner solution and outer solution separately and match the two solutions using physical constraints. The final solution is a composite expansion including the inner and outer solutions. On the other hand, using the method of multiple scales, the composite expansion can be retrieved at once using a single expansion [7]. In this work, the method of multiple scales is first applied to the small-stiffness axially-moving-beam problem. The approximate solution is constructed.

2. THE METHOD OF MULTIPLE SCALES

The linear dimensionless equation of motion for an axially moving beam with small flexural stiffness is [4]

$$\frac{\partial^2 y}{\partial t^2} + 2v \frac{\partial^2 y}{\partial x \partial t} + (v^2 - 1) \frac{\partial^2 y}{\partial x^2} + \varepsilon v_f^2 \frac{\partial^4 y}{\partial x^4} = 0 \quad (1)$$

where $y(x,t)$ is the transverse displacement, v is the constant velocity and v_f^2 is a dimensionless parameter [4]. A small parameter ε is introduced to represent small beam effects. x and t are the dimensionless spatial and time coordinates respectively. The simply supported boundary conditions for the problem are

$$y(0,t)=y(1,t)=0 \quad y''(0,t)=y''(1,t)=0 \quad (2)$$

Since the highest order derivative multiplies a small parameter, the problem is a boundary layer problem.

To apply the method of multiple scales, one defines the variables

$$x_0 = x, \quad x_1 = \frac{x}{\sqrt{\varepsilon}}, \quad T_0 = t, \quad T_1 = \varepsilon t \quad (3)$$

where x_0 is the outer variable and x_1 is the inner variable for the left boundary. T_0 is the usual fast time scale and T_1 is the slow time scale. The derivatives are defined as follows

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x_0} + \frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial x_1} \\ \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x_0^2} + \frac{2}{\sqrt{\varepsilon}} \frac{\partial^2}{\partial x_0 \partial x_1} + \frac{1}{\varepsilon} \frac{\partial^2}{\partial x_1^2} \\ \frac{\partial^4}{\partial x^4} &= \frac{\partial^4}{\partial x_0^4} + \frac{4}{\sqrt{\varepsilon}} \frac{\partial^4}{\partial x_0^3 \partial x_1} + \frac{6}{\varepsilon} \frac{\partial^4}{\partial x_0^2 \partial x_1^2} + \frac{4}{\varepsilon \sqrt{\varepsilon}} \frac{\partial^4}{\partial x_0 \partial x_1^3} + \frac{1}{\varepsilon^2} \frac{\partial^4}{\partial x_1^4} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} \\ \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} \end{aligned} \quad (4)$$

Substituting the derivatives into the original equation, one has

$$\begin{aligned}
& \frac{\partial^2 y}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 y}{\partial T_0 \partial T_1} + 2v \left(\frac{\partial^2 y}{\partial x_0 \partial T_0} + \varepsilon \frac{\partial^2 y}{\partial x_0 \partial T_1} + \frac{1}{\sqrt{\varepsilon}} \frac{\partial^2 y}{\partial x_1 \partial T_0} + \sqrt{\varepsilon} \frac{\partial^2 y}{\partial x_1 \partial T_1} \right) \\
& + (v^2 - 1) \left(\frac{\partial^2 y}{\partial x_0^2} + \frac{2}{\sqrt{\varepsilon}} \frac{\partial^2 y}{\partial x_0 \partial x_1} + \frac{1}{\varepsilon} \frac{\partial^2 y}{\partial x_1^2} \right) \\
& + \varepsilon v_f^2 \left(\frac{\partial^4 y}{\partial x_0^4} + \frac{4}{\sqrt{\varepsilon}} \frac{\partial^4 y}{\partial x_0^3 \partial x_1} + \frac{6}{\varepsilon} \frac{\partial^4 y}{\partial x_0^2 \partial x_1^2} + \frac{4}{\varepsilon \sqrt{\varepsilon}} \frac{\partial^4 y}{\partial x_0 \partial x_1^3} + \frac{1}{\varepsilon^2} \frac{\partial^4 y}{\partial x_1^4} \right) = 0
\end{aligned} \quad (5)$$

A suitable expansion for $y(x, t)$ would be

$$y(x, t; \varepsilon) = y_0 + \sqrt{\varepsilon} y_1 + \varepsilon y_2 + \varepsilon \sqrt{\varepsilon} y_3 + \varepsilon^2 y_4 + \dots \quad (6)$$

where y_i all depend on x_0, x_1, T_0 and T_1

Substituting (6) into (5) and separating each order of ε , one obtains the set of equations

$$O\left(\frac{1}{\varepsilon}\right): (v^2 - 1) \frac{\partial^2 y_0}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_0}{\partial x_1^4} = 0 \quad (7)$$

$$O\left(\frac{1}{\sqrt{\varepsilon}}\right): (v^2 - 1) \frac{\partial^2 y_1}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_1}{\partial x_1^4} = -2v \frac{\partial^2 y_0}{\partial x_1 \partial T_0} - 2(v^2 - 1) \frac{\partial^2 y_0}{\partial x_0 \partial x_1} - 4v_f^2 \frac{\partial^4 y_0}{\partial x_0 \partial x_1^3} \quad (8),$$

$$\begin{aligned}
O(1): (v^2 - 1) \frac{\partial^2 y_2}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_2}{\partial x_1^4} = & -\frac{\partial^2 y_0}{\partial T_0^2} - 2v \frac{\partial^2 y_0}{\partial x_0 \partial T_0} - 2v \frac{\partial^2 y_1}{\partial x_1 \partial T_0} - (v^2 - 1) \left(\frac{\partial^2 y_0}{\partial x_0^2} \right. \\
& \left. + 2 \frac{\partial^2 y_1}{\partial x_0 \partial x_1} \right) - v_f^2 \left(6 \frac{\partial^4 y_0}{\partial x_0^2 \partial x_1^2} + 4 \frac{\partial^4 y_1}{\partial x_0 \partial x_1^3} \right) \quad (9)
\end{aligned}$$

$$\begin{aligned}
O(\sqrt{\varepsilon}): (v^2 - 1) \frac{\partial^2 y_3}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_3}{\partial x_1^4} = & -\frac{\partial^2 y_1}{\partial T_0^2} - 2v \left(\frac{\partial^2 y_1}{\partial x_0 \partial T_0} + \frac{\partial^2 y_2}{\partial x_1 \partial T_0} + \frac{\partial^2 y_0}{\partial x_1 \partial T_1} \right) \\
& - (v^2 - 1) \left(\frac{\partial^2 y_1}{\partial x_0^2} + 2 \frac{\partial^2 y_2}{\partial x_0 \partial x_1} \right) - v_f^2 \left(4 \frac{\partial^4 y_0}{\partial x_0^3 \partial x_1} \right. \\
& \left. + 6 \frac{\partial^4 y_1}{\partial x_0^2 \partial x_1^2} + 4 \frac{\partial^4 y_2}{\partial x_0 \partial x_1^3} \right) \quad (10)
\end{aligned}$$

$$\begin{aligned}
O(\varepsilon): \quad (v^2-1) \frac{\partial^2 y_4}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_4}{\partial x_1^4} = & -\frac{\partial^2 y_2}{\partial T_0^2} - 2 \frac{\partial^2 y_0}{\partial T_0 \partial T_1} - 2v \left(\frac{\partial^2 y_2}{\partial x_0 \partial T_0} + \frac{\partial^2 y_3}{\partial x_1 \partial T_0} \right. \\
& + \frac{\partial^2 y_0}{\partial x_0 \partial T_1} + \frac{\partial^2 y_1}{\partial x_1 \partial T_1} \Big) - (v^2-1) \left(\frac{\partial^2 y_2}{\partial x_0^2} + 2 \frac{\partial^2 y_3}{\partial x_0 \partial x_1} \right) \\
& - v_f^2 \left(\frac{\partial^4 y_0}{\partial x_0^4} + 4 \frac{\partial^4 y_1}{\partial x_0^3 \partial x_1} + 6 \frac{\partial^4 y_2}{\partial x_0^2 \partial x_1^2} + 4 \frac{\partial^4 y_3}{\partial x_0 \partial x_1^3} \right) \quad (11)
\end{aligned}$$

3. SOLUTIONS

In this section, one solves the equations consecutively. At order $(1/\varepsilon)$, the general solution is

$$y_0 = A(x_0, T_0, T_1) e^{\frac{\sqrt{1-v^2}}{v_f} x_1} + B(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_1} + C(x_0, T_0, T_1) x_1 + D(x_0, T_0, T_1) \quad (12)$$

For finite solutions, one chooses $A=C=0$. The term with B is a part of the inner solution. One may require $B=0$ for not allowing the inner solution to appear at the first order. Hence

$$y_0 = D(x_0, T_0, T_1) \quad (13)$$

is the solution at this order.

For order $(1/\sqrt{\varepsilon})$, one substitutes (13) into (8)

$$(v^2-1) \frac{\partial^2 y_1}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_1}{\partial x_1^4} = 0 \quad (14)$$

One may now choose

$$y_1 = 0 \quad (15)$$

At order (1), inserting (13) and (15) into (9) one obtains

$$(v^2-1) \frac{\partial^2 y_2}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_2}{\partial x_1^4} = -\frac{\partial^2 D}{\partial T_0^2} - 2v \frac{\partial^2 D}{\partial x_0 \partial T_0} - (v^2-1) \frac{\partial^2 D}{\partial x_0^2} \quad (16)$$

In order not to introduce secular terms, D should be selected such that

$$\frac{\partial^2 D}{\partial T_0^2} + 2v \frac{\partial^2 D}{\partial x_0 \partial T_0} + (v^2 - 1) \frac{\partial^2 D}{\partial x_0^2} = 0 \quad (17)$$

Note that this equation is the equation for a constantly moving strip. A solution for y_2 would then be

$$y_2 = E_1(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_1} + F_1(x_0, T_0, T_1) \quad (18)$$

In above, only the decaying solutions are included.

At order ($\sqrt{\varepsilon}$), the solvability condition is

$$v \frac{\partial E_1}{\partial T_0} + (1 - v^2) \frac{\partial E_1}{\partial x_0} = 0 \quad (19)$$

and y_3 could be selected as

$$y_3 = G_1(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_1} \quad (20)$$

Finally at order (ε), one only determines the solvability conditions

$$\frac{\partial^2 F_1}{\partial T_0^2} + 2v \frac{\partial^2 F_1}{\partial x_0 \partial T_0} + (v^2 - 1) \frac{\partial^2 F_1}{\partial x_0^2} = -v_f^2 \frac{\partial^4 D}{\partial x_0^4} - 2 \frac{\partial^2 D}{\partial T_0 \partial T_1} - 2v \frac{\partial^2 D}{\partial x_0 \partial T_1} \quad (21)$$

$$\frac{\partial^2 E_1}{\partial T_0^2} + 2v \frac{\partial^2 E_1}{\partial x_0 \partial T_0} + 5(1 - v^2) \frac{\partial^2 E_1}{\partial x_0^2} = 2v \frac{\sqrt{1 - v^2}}{v_f} \frac{\partial G_1}{\partial T_0} + 2 \frac{(1 - v^2)^{3/2}}{v_f} \frac{\partial G_1}{\partial x_0} \quad (22)$$

Substituting the solutions obtained to the expansion, one has the approximate boundary layer solution

$$y = D(x_0, T_0, T_1) + \varepsilon (E_1(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_1} + F_1(x_0, T_0, T_1)) + \dots \quad (23)$$

The above solution contains the inner expansion at the left hand side and the outer expansion. One may now calculate the outer expansion and the right hand side boundary layer solution by defining

$$x_2 = \frac{1 - x}{\sqrt{\varepsilon}} \quad (24)$$

A similar calculation with only inserting x_2 instead of x_1 makes some sign changes in the equations. Finally the solution for this case is

$$y = D(x_0, T_0, T_1) + \varepsilon (E_2(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_2} + F_2(x_0, T_0, T_1)) + \dots \quad (25)$$

To obtain the composite expansion valid for all parts of the domain, one has to add solutions (23) and (25) and subtract the outer solution which is common. Hence the final solution is

$$y = D(x_0, T_0, T_1) + \varepsilon (E_1(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_1} + E_2(x_0, T_0, T_1) e^{-\frac{\sqrt{1-v^2}}{v_f} x_2} + F_1(x_0, T_0, T_1)) + \dots \quad (26)$$

4. BOUNDARY CONDITIONS

In this section, the boundary conditions will be applied to the composite expansion (26). One has to solve equation (17) first subject to the boundary conditions

$$D(0, T_0, T_1) = D(1, T_0, T_1) = 0 \quad (27)$$

Since equation (17) is of second order, these two conditions are the only applicable conditions from the set of boundary conditions given in (2). The solution is

$$D(x_0, T_0, T_1) = A_n(T_1) e^{i\omega_n T_0} \psi_n(x_0) + \bar{A}_n(T_1) e^{-i\omega_n T_0} \bar{\psi}_n(x_0) \quad (28)$$

where

$$\psi_n = C_n e^{i\alpha_n x_0} \sin n\pi x_0, \quad \omega_n = n\pi(1-v^2), \quad \alpha_n = n\pi v \quad (29)$$

Substituting solution (28) into (21), the solvability condition requires

$$\frac{\partial A_n}{\partial T_1} + k_1 A_n = 0 \quad (30)$$

where

$$k_1 = -\frac{i}{2} v_f^2 n^3 \pi^3 (v^4 + 6v^2 + 1) \quad (31)$$

The solution of (30) is

$$A_n = A_0 e^{-k_1 T_1} \quad (32)$$

Inserting (32) into (28) and defining $A_0 = \frac{1}{2} a e^{i\theta}$, one finally obtains

$$D(x_0, T_0, T_1) = C_n \cos\{n\pi(1-v^2)T_0 + \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1)T_1 + n\pi v x_0 + \theta\} \sin n\pi x_0 \quad (33)$$

The right hand side of (21) is zero and the left hand side may now be chosen as

$$F_1(x_0, T_0, T_1) = 0 \quad (34)$$

Since the composite expansion is constructed upto order ε , there is no need to determine G_1 from (22)

One has to determine E_1 now. A solution satisfying equation (19) is

$$E_1 = C_n \alpha_1 \sin[n\pi(1-v^2)T_0 + \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1)T_1 - n\pi v x_0 + \theta_1] \quad (35)$$

Similarly

$$E_2 = C_n \alpha_2 \sin[n\pi(1-v^2)T_0 + \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1)T_1 - n\pi v x_0 + \theta_2] \quad (36)$$

Substituting (33)-(36) into (26) and imposing the boundary conditions, one sees that the solution has an order (ε) error for conditions $y(0, t) = y(1, t) = 0$.

For the remaining boundary conditions, to eliminate the error at the first order of approximation, α_1 , θ_1 , α_2 and θ_2 in (35) and (36) are selected as

$$\begin{aligned} \alpha_1 &= \frac{2v_f^2 n^2 \pi^2 v}{1-v^2} & \theta_1 &= \theta \\ \alpha_2 &= \frac{2v_f^2 n^2 \pi^2 v}{1-v^2} \cos n\pi & \theta_2 &= \theta + 2n\pi v \end{aligned} \quad (37)$$

Hence the solution in terms of original variables is

$$\begin{aligned} y(x, t) &= C_n \left\{ \cos\left[(n\pi(1-v^2) + \varepsilon \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1))t + n\pi v x + \theta\right] \sin n\pi x \right. \\ &\quad + \varepsilon \left(\frac{2v_f^2 n^2 \pi^2 v}{1-v^2} \sin\left[(n\pi(1-v^2) + \varepsilon \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1))t - n\pi v x + \theta\right] e^{-\frac{\sqrt{1-v^2}}{v_f} \frac{x}{\sqrt{\varepsilon}}} \right. \\ &\quad + \frac{2v_f^2 n^2 \pi^2 v}{1-v^2} \cos n\pi \sin\left[(n\pi(1-v^2) + \varepsilon \frac{1}{2} n^3 \pi^3 (v^4 + 6v^2 + 1))t \right. \\ &\quad \left. \left. - n\pi v x + 2n\pi v + \theta\right] e^{-\frac{\sqrt{1-v^2}}{v_f} \frac{1-x}{\sqrt{\varepsilon}}} \right) + \dots \left. \right\} \quad (38) \end{aligned}$$

The above approximate solution has an $O(\varepsilon)$ error for boundary conditions $y(0,t)=y(1,t)=0$ and an $O(\sqrt{\varepsilon})$ error for boundary conditions $y''(0,t)=y''(1,t)=0$.

5. CONCLUDING REMARKS

An approximate boundary layer solution is presented for an axially moving beam with small flexural stiffness. The method of multiple scales is applied to the problem and the composite expansion including two inner solutions and one outer solution is found.

Comparing the solutions with those of Pellicano and Zirilli [5], one may note that both solutions yield the same error for boundary conditions $y(0,t)=y(1,t)=0$, namely $O(\varepsilon)$ error. While multiple scales yield an $O(\sqrt{\varepsilon})$ error for boundary conditions $y''(0,t)=y''(1,t)=0$, their solution satisfies the boundary conditions exactly. However contrary to their modified Lindstedt Poincare method, using the method of multiple scales, a single expansion is sufficient and requires no matching between the expansions. This advantage brings another disadvantage: constructing the solutions at each order of ε is not as straightforward in the method of multiple scales as in the method of matched expansions and requires some experience.

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