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Radical Petrov–Galerkin Approach for the Time-Fractional KdV–Burgers’ Equation

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Abstract: This paper presents a novel numerical spectral scheme to handle the time-fractional KdV–Burgers’ equation, which is very important in both physics and engineering. The scheme basically uses the tau approach combined with Gegenbauer polynomials to provide accurate and stable numerical solutions. Instead of solving the differential problem together with the conditions, we solve a system of algebraic equations. The method can handle complex boundary conditions. Some numerical experiments are exhibited to demonstrate that this approach is highly efficient and produces results that are better than some existing numerical methods in the literature. This technique offers more advanced solutions for time-fractional problems in various fields.

Keywords: Gegenbauer polynomials; Petrov–Galerkin method; time-fractional KdV–Burgers’ equation

MSC: 35R11; 35Q53; 65M70; 65M12; 41A10



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1. Introduction

A mathematical model called the KdV–Burgers’ equation [1–3] is used to explain a variety of physical processes, including wave motion and fluid flow. Time-fractional derivatives, which depict processes with memory effects, make this equation very difficult to calculate. Recently, fractional derivatives have drawn interest because they can more closely simulate real-world phenomena than conventional techniques [4–7].

The time-fractional KdV–Burgers’ equation [8–12], which combines the well-known KdV equation [13–15] with Burgers’ equation, is the main topic of this study. Our objective is to suggest a straightforward and effective numerical approach. The Petrov–Galerkin approach, a potent method for solving differential equations, is the one we employ. Our solutions are more accurate and stable because we use shifted Gegenbauer polynomials, which have strong orthogonal characteristics.

The Caputo sense, which is frequently employed in fractional calculus due to its relevance to practical issues, is used to express the time-fractional derivative. Utilizing this method, we transform the problem and its boundary conditions into a set of algebraic equations that can be resolved through the application of widely recognized numerical methods, such as Gaussian elimination. We demonstrate the method’s advantages in terms of accuracy and computational economy by comparing it with existing techniques and demonstrating its high precision across a variety of examples.

Two important partial differential equations (PDEs) that are used to simulate a variety of physical phenomena, including shallow water waves and nonlinear acoustics, are the Burgers’ equation and the Korteweg–de Vries (KdV) equation. The KdV–Burgers equation, which combines these equations, is used to describe systems like fluid dynamics and traffic flow when nonlinear waves and dissipative processes occur [16–18].

In this work, we focus on solving the time-fractional KdV–Burgers’ equation [11,19,20], which can be written as follows:

$$D_t^\alpha \psi(x, t) + k_1 \psi(x, t) \psi_x(x, t) - k_2 \psi_{xx}(x, t) + k_3 \psi_{xxx}(x, t) = s(x, t), \quad 0 < \alpha < 1, \quad (1)$$

where D_t^α represents the Caputo fractional derivative of order α , k_1, k_2, k_3 are positive constants, and $s(x, t)$ is a source term. The combined effects of nonlinear advection, dissipation, and dispersion are simulated by the equation.

We use the Petrov–Galerkin method [21–24], a spectral approach that works well for solving boundary value problems involving fractional derivatives to solve Equation (1). Since the orthogonality qualities of the shifted Gegenbauer polynomials (SGPs) guarantee numerical stability and correctness, we specifically use them as trial and test functions. According to the SGPs, the trial functions $Z_i(x)$ and $Y_j(t)$ are as follows:

$$Z_i(x) = x(1-x)^2 G_i^\mu(x), \quad Y_j(t) = t^\alpha G_j^\mu(t), \quad (2)$$

where μ is a parameter that regulates the polynomial behavior, and $G_i^\mu(x)$ are the shifted Gegenbauer polynomials. Using the Petrov–Galerkin technique, we transform the time-fractional KdV–Burgers’ issue into a set of algebraic equations that can be efficiently solved by Gaussian elimination. This method’s primary advantage is its ability to precisely manage the equation’s fractional and nonlinear components.

This paper is organized as follows. The Gegenbauer polynomials and the characteristics of fractional derivatives are among the prerequisites that are introduced in Section 2. The Petrov–Galerkin method and its application to the time-fractional KdV–Burgers’ equation are explained in Section 3. We examine the performance of the method and show the numerical results in Section 4. In Section 5, we wrap up by providing a synopsis of the results and possible directions for further research.

2. Preliminaries and Essential Relations

2.1. The Fractional Derivative in the Caputo Sense

Definition 1 ([25]). The Caputo’s fractional derivative is defined as [25]:

$$D_z^\alpha Y(z) = \frac{1}{\Gamma(m-\alpha)} \int_0^z (z-t)^{m-\alpha-1} Y^{(m)}(t) dt, \quad \alpha > 0, \quad z > 0, \quad (3)$$

$$m-1 < \alpha < m, \quad m \in \mathbb{Z}^+.$$

We also have

$$D_z^\alpha C = 0, \quad (C \text{ is a constant}), \quad (4)$$

$$D_z^\alpha z^k = \begin{cases} 0, & \text{if } k \in \mathbb{Z}^{\geq 0} \text{ and } k < \lceil \alpha \rceil, \\ \frac{k!}{\Gamma(k+1-\alpha)} z^{k-\alpha}, & \text{if } k \in \mathbb{Z}^{\geq 0} \text{ and } k \geq \lceil \alpha \rceil, \end{cases} \quad (5)$$

where $\lceil \alpha \rceil$ is the ceiling function.

2.2. An Account on the SGPs

The SGPs on the interval $[0, 1]$ associated with the real parameters $\mu > -\frac{1}{2}$, are defined as

$$\mathcal{G}_m^\mu(x) = \frac{(2\mu)_m}{\left(\mu + \frac{1}{2}\right)_m} P_m^{(\mu-\frac{1}{2}, \mu-\frac{1}{2})}(x), \quad (6)$$

where $(y)_m = \frac{\Gamma(y+m)}{\Gamma(y)}$ is the Pochhammer symbol, and $P_m^{(\mu-\frac{1}{2}, \mu-\frac{1}{2})}(x)$ are the shifted classical Jacobi polynomials.

The orthogonality relation of $\mathcal{G}_m^\mu(x)$ is given by

$$\int_0^1 \mathcal{G}_n^\mu(x) \mathcal{G}_m^\mu(x) \omega(x) dx = h_n^\mu \delta_{n,m}, \quad (7)$$

where $\omega(x) = (x - x^2)^{\mu - \frac{1}{2}}$, $h_n^\mu = \frac{\pi \Gamma(2\mu + n)}{2^{4\mu - 1} (\mu + n) \Gamma(\mu)^2 \Gamma(n + 1)}$, and $\delta_{n,m}$ is the well known Kronecker delta.

The power form representation of the $\mathcal{G}_i^\mu(x)$ can be represented as

$$\mathcal{G}_i^\mu(x) = \sum_{k=0}^i \lambda_{k,i} x^k, \quad (8)$$

where

$$\lambda_{k,i} = \frac{(-1)^{i-k} (2\mu)_{i+k}}{\Gamma(k+1) \left(\mu + \frac{1}{2}\right)_k \Gamma(i-k+1)}. \quad (9)$$

Theorem 1 ([26]). *The following q -th derivative holds for $\mathcal{G}_n^\mu(x)$*

$$D^q \mathcal{G}_n^\mu(x) = (2)^q \sum_{i=0}^{n-q} \lambda_{i,n,q}^\mu \mathcal{G}_i^\mu(x), \quad (10)$$

where

$$\lambda_{i,n,q}^\mu = \frac{2^{1-q} (\mu + i) \Gamma\left(i + \mu + \frac{1}{2}\right) \Gamma(i + n + q + 2\mu)}{\Gamma(-i + n - q + 1)} {}_3F_2 \left(\begin{matrix} i - n + q, \mu + i + \frac{1}{2}, 2\mu + i + n + q \\ \mu + i + q + \frac{1}{2}, 2\mu + 2i + 1 \end{matrix} \middle| 1 \right), \quad (11)$$

where ${}_rF_s$ denotes the generalized hypergeometric function defined by

$${}_rF_s \left(\begin{matrix} p_1, p_2, \dots, p_r \\ q_1, q_2, \dots, q_s \end{matrix} \middle| x \right) = \sum_{m=0}^{\infty} \frac{(p_1)_m (p_2)_m \dots (p_r)_m}{(q_1)_m (q_2)_m \dots (q_s)_m} \frac{x^m}{m!}. \quad (12)$$

3. Petrov–Galerkin Approach for the Time-Fractional KdV–Burgers' Equation

In this section, we consider the following time-fractional KdV–Burgers' equation [27]

$$D_t^\alpha \psi(x, t) + k_1 \psi(x, t) \psi_x(x, t) - k_2 \psi_{xx}(x, t) + k_3 \psi_{xxx}(x, t) = s(x, t), \quad 0 < \alpha < 1, \quad (13)$$

subject to the following initial condition

$$\psi(x, 0) = a_0(x), \quad 0 < x \leq 1, \quad (14)$$

and boundary conditions

$$\psi(0, t) = a_1(t), \quad \psi(1, t) = a_2(t), \quad \psi_x(1, t) = a_3(t), \quad 0 < t \leq 1, \quad (15)$$

where $k_1 \neq 0$, k_2, k_3 are positive parameters, and $s(x, t)$ is the source term.

3.1. Trial Functions

Consider the following basis functions

$$\begin{aligned} \mathcal{Z}_i(x) &= x(1-x)^2 \mathcal{G}_i^\mu(x), \\ \mathcal{Y}_j(t) &= t^\alpha \mathcal{G}_j^\mu(t). \end{aligned} \quad (16)$$

Remark 1. The orthogonality relation (7) enables us to write the following relations

$$\int_0^1 \mathcal{Z}_n(x) \mathcal{Z}_m(x) \omega_1(x) dx = h_n^\mu \delta_{n,m}, \quad (17)$$

$$\int_0^1 \mathcal{Y}_n(t) \mathcal{Y}_m(t) \omega_2(t) dt = h_n^\mu \delta_{n,m}, \quad (18)$$

where $\omega_1(x) = \frac{\omega(x)}{x^2(1-x)^4}$ and $\omega_2(t) = \frac{\omega(t)}{t^{2\alpha}}$.

Remark 2. The basis functions defined in (16) satisfy the following conditions

$$\mathcal{Z}_i(0) = \mathcal{Z}_i(1) = \mathcal{Z}'_i(1) = \mathcal{Y}_j(0) = 0. \quad (19)$$

Lemma 1. The analytic and inversion form relations of $\mathcal{G}_i^\mu(x)$ are

$$\mathcal{G}_i^\mu(x) = \sum_{k=0}^i \frac{(-1)^{i-k} (2\mu)_{i+k}}{\Gamma(k+1) \left(\mu + \frac{1}{2}\right)_k \Gamma(i-k+1)} x^k, \quad (20)$$

$$x^r = \sum_{m=0}^r \frac{4^\mu (-1)^{2m} (\mu+m) \Gamma(\mu) \Gamma(r+1) \Gamma\left(\mu+r+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(r-m+1) \Gamma(2\mu+m+r+1)} \mathcal{G}_m^\mu(x). \quad (21)$$

Lemma 2. The following connection and linearization formulae are needed:

$$\mathcal{Z}_i(x) = \sum_{k=0}^{i+3} \frac{\gamma_{i,k}}{k_k} \mathcal{G}_r^\mu(x), \quad (22)$$

$$\mathcal{Z}_m(x) \mathcal{Z}_n(x) = \sum_{s=0}^{\min(m,n)} L_{m,n,s} \mathcal{G}_{m+n-2s}^\mu(x), \quad (23)$$

Theorem 2 ([26]). The q -th derivative for $\mathcal{G}_n^\mu(x)$ is

$$D^q \mathcal{G}_n^\mu(x) = (2)^q \sum_{i=0}^{n-q} \Gamma_{i,n,q}^\mu \mathcal{G}_i^\mu(x), \quad (24)$$

where

$$\Gamma_{i,n,q}^\mu = \frac{2^{1-q} (\mu+i) \Gamma\left(i+\mu+\frac{1}{2}\right) \Gamma(i+n+q+2\mu)}{\Gamma(-i+n-q+1)} {}_3F_2 \left(\begin{matrix} i-n+q, \mu+i+\frac{1}{2}, 2\mu+i+n+q \\ \mu+i+q+\frac{1}{2}, 2\mu+2i+1 \end{matrix} \middle| 1 \right), \quad (25)$$

Theorem 3. The elements a_{ir} , b_{kj} , c_{ior} , d_{ijr} , p_{ir} , q_{ij} , and g_{ir} are given by

$$\begin{aligned}
(1) \quad a_{ir} &= \int_0^1 \mathcal{Z}_i(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \gamma_{i,k} \delta_{k,r}, \\
(2) \quad c_{ior} &= \int_0^1 \mathcal{Z}_i(x) D_x \mathcal{Z}_o(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \sum_{s=0}^{\min(k,r)} \sum_{z=0}^{o+3} \frac{\gamma_{i,k} \gamma_{o,z} L_{k,r,s} v_{z,k+r-2s}^1}{h_k h_z}, \\
(3) \quad p_{ir} &= \int_0^1 D_x^2 \mathcal{Z}_i(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \frac{\gamma_{i,k} v_{k,r}^2}{h_k}, \\
(4) \quad g_{ir} &= \int_0^1 D_x^3 \mathcal{Z}_i(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \frac{\gamma_{i,k} v_{k,r}^3}{h_k}, \\
(5) \quad q_{ij} &= \int_0^1 \mathcal{Y}_i(t) \mathcal{G}_j^\mu(t) \omega(t) dt = e_{i,j,\alpha}^1, \\
(6) \quad d_{ijr} &= \int_0^1 \mathcal{Y}_i(t) \mathcal{Y}_j(t) \mathcal{G}_r^\mu(t) \omega(t) dt = \sum_{s=0}^{\min(i,j)} L(i,j,s) e(i+j-2s, r, 2\alpha), \\
(7) \quad b_{kj} &= \int_0^1 D_t^\alpha \mathcal{Y}_k(t) \mathcal{G}_j^\mu(t) \omega(t) dt = \sum_{r=0}^k \frac{\pi 4^{1-2\mu} (-1)^{k-r} \Gamma(j+2\mu) \Gamma(r+\alpha+1) \Gamma(k+r+2\mu)}{j! r! \Gamma(\mu)^2 \Gamma(-j+r+1) \Gamma(k-r+1) \Gamma(j+r+2\mu+1)},
\end{aligned} \tag{26}$$

where

$$L_{m,n,s} = \frac{\Gamma(s+\mu)(\mu+m+n-2s)\Gamma(m+n-2s+1)\Gamma(m-s+\mu)\Gamma(n-s+\mu)\Gamma(m+n-s+2\mu)}{\Gamma(\mu)^2\Gamma(s+1)\Gamma(m-s+1)\Gamma(n-s+1)\Gamma(m+n-s+\mu+1)\Gamma(m+n-2s+2\mu)}, \tag{27}$$

$$\gamma_{i,k}^1 = \begin{cases} \frac{\pi 2^{-4\mu} \Gamma(i+2\mu)}{(i+\mu)\Gamma(i+1)\Gamma(\mu)^2}, & \text{if } i = k, \\ \frac{\pi 2^{-4\mu-1} \Gamma(i+2\mu+1)}{(i+\mu)(i+\mu+1)\Gamma(i+1)\Gamma(\mu)^2}, & \text{if } k - i = 1, \\ \frac{\pi 2^{-4\mu-1} \Gamma(i+2\mu)}{(i+\mu-1)(i+\mu)\Gamma(i)\Gamma(\mu)^2}, & \text{if } k - i = -1, \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

$$\gamma_{i,k}^2 = \frac{\pi 2^{-4\mu} \Gamma(i+2\mu)}{i!(i+\mu+1)\Gamma(\mu)^2} \times \begin{cases} \frac{3i^2+6i\mu+2\mu^2+\mu-3}{4(i+\mu-1)(i+\mu)}, & \text{if } i = k, \\ \frac{i(i+\mu+1)}{2(i+\mu-1)(i+\mu)}, & \text{if } i - k = 1, \\ \frac{i+2\mu}{2(i+\mu)}, & \text{if } i - k = -1, \\ \frac{(i+2\mu+1)(i+2\mu)}{8(i+\mu+2)(i+\mu)}, & \text{if } i - k = -2, \\ \frac{(i-1)i(i+\mu+1)}{8(i+\mu-2)(i+\mu-1)(i+\mu)}, & \text{if } i - k = 2, \\ 0, & \text{otherwise,} \end{cases} \tag{29}$$

$$\gamma_{i,k}^3 = \frac{\pi 2^{-4\mu} \Gamma(i+2\mu+2)}{i! \Gamma(\mu)^2} \times \begin{cases} \frac{5i^2+10i\mu+2\mu^2+3\mu-5}{8(i+\mu)(i+\mu+1)(i^3+(5i^2-i-2)\mu+8i\mu^2-i+4\mu^3-2\mu^2)}, & \text{if } i=k, \\ \frac{3(5i^2+5i(2\mu+1)+4\mu^2+6\mu-10)}{32(i+\mu-1)(i+\mu)(i+\mu+1)(i+\mu+2)(i+2\mu+1)}, & \text{if } k-i=1, \\ \frac{3i(5i^2+10i\mu-5i+4\mu^2-4\mu-10)}{32(i+\mu-2)(i+\mu-1)(i+\mu)(i+\mu+1)(i+2\mu)(i+2\mu+1)}, & \text{if } k-i=-1, \\ \frac{3(i-1)i}{16(i+\mu-2)(i+\mu-1)(i+\mu)(i+2\mu)(i+2\mu+1)}, & \text{if } k-i=-2, \\ \frac{3}{16(i+\mu)_3}, & \text{if } k-i=2, \\ \frac{i+2\mu+2}{32(i+\mu)_4}, & \text{if } k-i=3, \\ \frac{(i-2)(i-1)i}{32(i+\mu-3)(i+\mu-2)(i+\mu-1)(i+\mu)(i+2\mu)(i+2\mu+1)}, & \text{if } k-i=-3, \\ 0, & \text{otherwise,} \end{cases} \quad (30)$$

$$\gamma_{i,k} = \gamma_{i,k}^1 - 2\gamma_{i,k}^2 + \gamma_{i,k}^3, \quad (31)$$

$$v_{k,r}^1 = \begin{cases} \frac{\pi 2^{3-4\mu} \Gamma(r+2\mu)}{r! \Gamma(\mu)^2}, & \text{if } k > r \text{ and } (k-r) \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \quad (32)$$

$$v_{k,r}^2 = \begin{cases} \frac{\pi 2^{3-4\mu} (k-r)(k+2\mu+r) \Gamma(r+2\mu)}{\Gamma(\mu)^2 \Gamma(r+1)}, & \text{if } k > r \text{ and } (k-r) \text{ even,} \\ 0, & \text{otherwise,} \end{cases} \quad (33)$$

$$v_{k,r}^3 = \begin{cases} \frac{\pi 4^{1-2\mu} (k-r-1)(k-r+1)(k+2\mu+r-1)(k+2\mu+r+1) \Gamma(r+2\mu)}{r! \Gamma(\mu)^2}, & \text{if } k > r+1 \text{ and } (k-r) \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \quad (34)$$

$$e_{i,j,\alpha} = (-1)^i \Gamma(\alpha+1) \Gamma\left(\mu + \frac{1}{2}\right)^2 \mathcal{G}_i^\mu(1) \mathcal{G}_j^\mu(1) \Gamma\left(\alpha + \mu + \frac{1}{2}\right) \\ \times {}_4\tilde{F}_3\left(\begin{matrix} -i, \alpha+1, \alpha+\mu+\frac{1}{2}, i+2\mu \\ -j+\alpha+1, \mu+\frac{1}{2}, j+\alpha+2\mu+1 \end{matrix} \middle| 1\right). \quad (35)$$

Proof.

(1) From (22), applying the orthogonality relation (7)

$$\int_0^1 \mathcal{Z}_i(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \frac{\gamma_{i,k}}{h_k} \int_0^1 \mathcal{G}_i^\mu(x) \mathcal{G}_r^\mu(x) \omega(x) dx = \sum_{k=0}^{i+3} \gamma_{i,k} \delta_{k,r}.$$

- (2) We begin with the derivative formula (24) for $q = 1$, then apply the linearization formula (23) and use the orthogonality relation (7) to derive the RHS.
- (3) Starting with the derivative formula (24) for $q = 2$ and applying the orthogonality relation (7), we derive the RHS.
- (4) Using the derivative formula (24) for $q = 3$, followed by the orthogonality relation (7), we obtain the RHS.
- (5) We use the linearization formula (23), which leads to a beta function integral. After simplification, we obtain the RHS.

- (6) We apply the linearization formula (23) and use the orthogonality relation (7) to derive the RHS.
- (7) Applying the Caputo operator to the analytic form (20) results in a beta function integral, and after simplification, we obtain the RHS.

□

3.2. Petrov–Galerkin Solution for the Time-Fractional KdV–Burgers’ Equation

To start our proposed collocation spectral approach, we will make use of the following transformation:

$$\eta(x, t) = \psi(x, t) + g(x, t), \quad (36)$$

where

$$g(x, t) = x(x-1)(\psi_x(1, 0) - \psi_x(1, t)) - (x-1)^2\psi(0, t) + (x-2)x\psi(1, t) + (x-1)^2\psi(0, 0) - (x-2)x\psi(1, 0) - \psi(x, 0), \quad (37)$$

to transform the time-fractional KdV–Burgers’ Equation (13) with the conditions (14)–(15) into the following modified equation:

$$D_t^\alpha \eta(x, t) + k_1 \eta(x, t) \eta_x(x, t) - k_2 \eta_{xx}(x, t) + k_3 \eta_{xxx}(x, t) - k_1 \eta(x, t) g_x(x, t) - k_1 \eta_x(x, t) g(x, t) = f(x, t), \quad 0 < \alpha < 1, \quad (38)$$

governed by the following homogeneous conditions

$$\eta(x, 0) = 0, \quad 0 < x \leq 1, \quad (39)$$

$$\eta(0, t) = \eta(1, t) = \eta_x(1, t) = 0, \quad 0 < t \leq 1, \quad (40)$$

where

$$f(x, t) = s(x, t) + D_t^\alpha g(x, t) - k_1 g(x, t) g_x(x, t) - k_2 g_{xx}(x, t) + k_3 g_{xxx}(x, t). \quad (41)$$

Thus, rather than solving (13) with the conditions (14)–(15), we can address the modified Equation (38) under the homogeneous conditions (39)–(40).

Now, one may set

$$\begin{aligned} \mathcal{L}^M &= \text{span}\{\mathcal{Z}_i(x) \mathcal{Y}_j(t) : i, j = 0, 1, \dots, M\}, \\ \Delta^M &= \{\eta \in \mathcal{L}^M : \eta(x, 0) = \eta(0, t) = \eta(1, t) = \eta_x(1, t) = 0\}, \end{aligned} \quad (42)$$

then, any function $\eta^M(x, t) \in \Delta^M$ may be written as

$$\eta^M(x, t) = \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} \mathcal{Z}_i(x) \mathcal{Y}_j(t). \quad (43)$$

The residual $\mathcal{R}(x, t)$ of Equation (38) can be written as

$$\begin{aligned} \mathcal{R}(x, t) &= D_t^\alpha \eta^M(x, t) + k_1 \eta^M(x, t) \eta_x^M(x, t) - k_2 \eta_{xx}^M(x, t) + k_3 \eta_{xxx}^M(x, t) \\ &\quad - k_1 \eta^M(x, t) g_x^M(x, t) - k_1 \eta_x^M(x, t) g^M(x, t) - f(x, t). \end{aligned} \quad (44)$$

Now, the application of the Petrov–Galerkin approach enables us to obtain

$$(\mathcal{R}(x, t), \mathcal{G}_r^\mu(x) \mathcal{G}_s^\mu(t))_{\omega(x)\omega(t)}, \quad 0 \leq r, s \leq M. \quad (45)$$

which can be written as

$$\begin{aligned}
& \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} (\mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (D_t^\alpha \mathcal{Y}_j(t), \mathcal{G}_s^\mu(t))_{\omega(t)} - k_2 \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} (D_{xx} \mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (\mathcal{Y}_j(t), \mathcal{G}_s^\mu(t))_{\omega(t)} \\
& + k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\eta}_{on} (\mathcal{Z}_i(x) D_x \mathcal{Z}_o(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (\mathcal{Y}_j(t) \mathcal{Y}_n(t), \mathcal{G}_s^\mu(t))_{\omega(t)} \\
& - k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\phi}_{on} (\mathcal{Z}_i(x) D_x \mathcal{Z}_o(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (\mathcal{Y}_j(t) \mathcal{Y}_n(t), \mathcal{G}_s^\mu(t))_{\omega(t)} \\
& - k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\phi}_{on} (D_x \mathcal{Z}_i(x) \mathcal{Z}_o(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (\mathcal{Y}_j(t) \mathcal{Y}_n(t), \mathcal{G}_s^\mu(t))_{\omega(t)} \\
& + k_3 \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} (D_{xxx} \mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)} (\mathcal{Y}_j(t), \mathcal{G}_s^\mu(t))_{\omega(t)} = (f(x, t), \mathcal{G}_r^\mu(x) \mathcal{G}_s^\mu(t))_{\omega(x) \omega(t)}, \quad 0 \leq r, s \leq M.
\end{aligned} \tag{46}$$

where $g(x, t)$ can be spectral as

$$g(x, t) = \sum_{o=0}^M \sum_{n=0}^M \hat{\phi}_{on} \mathcal{Z}_o(x) \mathcal{G}_n^\mu(t), \tag{47}$$

and $\hat{\phi}_{on}$ determined from the following relation

$$\hat{\phi}_{on} = \frac{1}{h_o^\mu h_n^\mu} \int_0^1 \int_0^1 \mathcal{Z}_o(x) \mathcal{Y}_n(t) \omega_1(x) \omega_2(t) dx dt. \tag{48}$$

Equation (46) can be rewritten in a simple form as

$$\begin{aligned}
& \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} a_{ir} b_{js} + k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\eta}_{on} c_{ior} d_{jns} - k_2 \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} p_{ir} q_{js} + k_3 \sum_{i=0}^M \sum_{j=0}^M \hat{\eta}_{ij} g_{ir} q_{js} \\
& - k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\phi}_{on} c_{ior} d_{jns} - k_1 \sum_{i=0}^M \sum_{j=0}^M \sum_{o=0}^M \sum_{n=0}^M \hat{\eta}_{ij} \hat{\phi}_{on} c_{oir} d_{jns} = (f(x, t), \mathcal{G}_r^\mu(x) \mathcal{G}_s^\mu(t))_{\omega(x) \omega(t)},
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
a_{ir} &= (\mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)}, \quad b_{js} = (D_t^\alpha \mathcal{Y}_j(t), \mathcal{G}_s^\mu(t))_{\omega(t)}, \\
c_{ior} &= (\mathcal{Z}_i(x) D_x \mathcal{Z}_o(x), \mathcal{G}_r^\mu(x))_{\omega(x)}, \quad d_{jns} = (\mathcal{Y}_j(t) \mathcal{Y}_n(t), \mathcal{G}_s^\mu(t))_{\omega(t)}, \\
p_{ir} &= (D_{xx} \mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)}, \quad q_{js} = (\mathcal{Y}_j(t), \mathcal{G}_s^\mu(t))_{\omega(t)}, \\
g_{ir} &= (D_{xxx} \mathcal{Z}_i(x), \mathcal{G}_r^\mu(x))_{\omega(x)}, \quad f_{r,s} = (f(x, t), \mathcal{G}_r^\mu(x) \mathcal{G}_s^\mu(t))_{\omega(x) \omega(t)}.
\end{aligned} \tag{50}$$

Hence, Equation (49) produces a nonlinear system of algebraic equations of dimension $(M+1)^2$ in the unknown expansion coefficients $\hat{\eta}_{ij}$ that may be solved using suitable procedure.

Remark 3. The elements a_{ir} , b_{js} , c_{ior} , d_{jns} , p_{ir} , q_{js} , and g_{ir} are given in Theorem 3.

4. Error Bound

Assume that the following Gegenbauer-weighted Sobolev space

$$\Delta_{\omega_2(t)}^{\alpha, m}(I_1) = \{u : u(0) = 0 \text{ and } D_t^{\alpha+k} u \in L_{\omega_2(t)}^2(I_1), 0 \leq k \leq m\}, \tag{51}$$

$$\Lambda_{\omega_1(x)}^m(I_2) = \{u : u(0) = u(1) = u'(1) = 0 \text{ and } D_x^k u \in L_{\omega_1(x)}^2(I_2), 0 \leq k \leq m\}, \tag{52}$$

where $I_1 = (0, 1)$ and $I_2 = (0, 1)$ equipped with the inner product, norm, and semi-norm

$$\begin{aligned} (u, v)_{\Delta_{\omega_2(t)}^{\alpha, m}} &= \sum_{k=0}^m (D_t^{\alpha+k} u, D_t^{\alpha+k} v)_{L^2_{\omega_2(t)}}, \quad \|u\|_{\Delta_{\omega_2(t)}^{\alpha, m}}^2 = (u, u)_{\Delta_{\omega_2(t)}^{\alpha, m}}, \quad |u|_{\Delta_{\omega_2(t)}^{\alpha, m}} = \|D_t^{\alpha+m} u\|_{L^2_{\omega_2(t)}}, \\ (u, v)_{\Lambda_{\omega_1(x)}^m} &= \sum_{k=0}^m (D_x^k u, D_x^k v)_{L^2_{\omega_1(x)}}, \quad \|u\|_{\Lambda_{\omega_1(x)}^m}^2 = (u, u)_{\Lambda_{\omega_1(x)}^m}, \quad |u|_{\Lambda_{\omega_1(x)}^m} = \|D_x^m u\|_{L^2_{\omega_1(x)}}, \end{aligned} \quad (53)$$

where $0 < \alpha < 1$ and $m \in \mathbb{N}$.

Also, assume that the two-dimensional Gegenbauer-weighted Sobolev space

$$\mathbf{GB}_{\bar{\omega}(x,t)}^{r,s}(\Omega) = \{u : u(x, 0) = u(0, t) = u(1, t) = u'(1, t) = 0 \text{ and } \frac{\partial^{\alpha+p+q} u}{\partial x^p \partial t^{\alpha+q}} \in L^2_{\bar{\omega}(x,t)}(\Omega), r \geq p \geq 0, s \geq q \geq 0\}, \quad (54)$$

equipped with the norm and semi-norm

$$\|u\|_{\mathbf{GB}_{\bar{\omega}(x,t)}^{r,s}} = \left(\sum_{p=0}^r \sum_{q=0}^s \left\| \frac{\partial^{\alpha+p+q} u}{\partial x^p \partial t^{\alpha+q}} \right\|_{L^2_{\bar{\omega}(x,t)}}^2 \right)^{\frac{1}{2}}, \quad |u|_{\mathbf{GB}_{\bar{\omega}(x,t)}^{r,s}} = \left\| \frac{\partial^{\alpha+r+s} u}{\partial x^r \partial t^{\alpha+s}} \right\|_{L^2_{\bar{\omega}(x,t)}}, \quad (55)$$

where $\bar{\omega}(x, t) = \omega_2(t) \omega_1(x)$, and $r, s \in \mathbb{N}$.

Theorem 4. Suppose $0 < \alpha < 1$, and $u^M(t) = \sum_{j=0}^M \hat{u}_j \mathcal{Y}_j(t)$ is the spectral solution of $u(t) \in \Delta_{\omega_2(t)}^{\alpha, m}(I_1)$. Then, for $0 \leq k \leq m \leq M+1$, we get

$$\|D_t^{\alpha+k} (u(t) - u^M(t))\|_{L^2_{\omega_2(t)}} \lesssim M^{\frac{-1}{2}(m-k)} |u(t)|_{\Delta_{\omega_2(t)}^{\alpha, m}}, \quad (56)$$

where $\mathcal{A} \lesssim \mathcal{B}$ indicates the existence of a constant n such that $\mathcal{A} \leq n \mathcal{B}$.

Proof. The definitions of $u(t)$ and $u^M(t)$ allow us to have

$$\begin{aligned} \|D_t^{\alpha+k} (u(t) - u^M(t))\|_{L^2_{\omega_2(t)}}^2 &= \sum_{n=M+1}^{\infty} |\hat{u}_n|^2 \|D_t^{\alpha+k} \mathcal{Y}_n(t)\|_{L^2_{\omega_2(t)}}^2 \\ &= \sum_{n=M+1}^{\infty} |\hat{u}_n|^2 \frac{\|D_t^{\alpha+k} \mathcal{Y}_n(t)\|_{L^2_{\omega_2(t)}}^2}{\|D_t^{\alpha+m} \mathcal{Y}_n(t)\|_{L^2_{\omega_2(t)}}^2} \|D_t^{\alpha+m} \mathcal{Y}_n(t)\|_{L^2_{\omega_2(t)}}^2 \\ &\leq \frac{\|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2}{\|D_t^{\alpha+m} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2} |u(t)|_{\Delta_{\omega_2(t)}^{\alpha, m}}^2. \end{aligned} \quad (57)$$

Now, our main aim is to estimate the factor $\frac{\|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2}{\|D_t^{\alpha+m} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2}$. To do this, we firstly used Equation (5) along with (8) to write $\|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2$, as follows

$$D_t^{\alpha+k} \mathcal{Y}_{M+1}(t) = \sum_{r=k}^{M+1} \lambda_{r,M+1} \frac{\Gamma(r+\alpha+1)}{(r-k)!} t^r, \quad (58)$$

and accordingly, we have

$$\begin{aligned} \|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2 &= \int_0^1 D_t^{\alpha+k} \mathcal{Y}_{M+1}(t) D_t^{\alpha+k} \mathcal{Y}_{M+1}(t) \omega_2(t) dt \\ &= \sum_{r=k}^{M+1} \lambda_{r,M+1}^2 \frac{\Gamma^2(r+\alpha+1)}{\Gamma^2(r-k+1)} \int_0^1 t^{2(r-\alpha)+\mu-\frac{1}{2}} (1-t)^{\mu-\frac{1}{2}} dt \\ &= \sum_{r=k}^{M+1} \lambda_{r,M+1}^2 \frac{\Gamma^2(r+\alpha+1) \Gamma\left(2(r-\alpha)+\mu+\frac{1}{2}\right) \Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma^2(r-k+1) \Gamma(2(r-\alpha+\mu)+1)}. \end{aligned} \quad (59)$$

Using the Stirling formula [28], one has

$$\frac{\Gamma^2(r+\alpha+1) \Gamma\left(2(r-\alpha)+\mu+\frac{1}{2}\right)}{\Gamma^2(r-k+1) \Gamma(2(r-\alpha+\mu)+1)} \lesssim r^{2(\alpha+k)-\frac{1}{2}(\mu+1)}. \quad (60)$$

Therefore, we can write

$$\begin{aligned} \|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2 &\lesssim \lambda^* (M+1)^{2(\alpha+k)-\frac{1}{2}(\mu+1)} \sum_{r=k}^{M+1} 1 \\ &= \lambda^* (M+1)^{2(\alpha+k)-\frac{1}{2}(\mu+1)} (M-k+2) \\ &= \lambda^* \left(\frac{\Gamma(M+2)}{\Gamma(M+1)} \right)^{2(\alpha+k)-\frac{1}{2}(\mu+1)} (M-k+2) \\ &\lesssim M^{2(\alpha+k)-\frac{1}{2}(\mu+1)} (M-k+2), \end{aligned} \quad (61)$$

where $\lambda^* = \max_{0 \leq r \leq M+1} \left\{ \lambda_{r,M+1}^2 \Gamma\left(\mu+\frac{1}{2}\right) \right\}$

Similarly, we have

$$\|D_t^{\alpha+m} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2 \lesssim M^{2(\alpha+m)-\frac{1}{2}(\mu+1)} (M-m+2). \quad (62)$$

Hence,

$$\begin{aligned} \frac{\|D_t^{\alpha+k} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2}{\|D_t^{\alpha+m} \mathcal{Y}_{M+1}(t)\|_{L^2_{\omega_2(t)}}^2} &\lesssim \frac{M^{2(\alpha+k)-\frac{1}{2}(\mu+1)} (M-k+2)}{M^{2(\alpha+m)-\frac{1}{2}(\mu+1)} (M-m+2)} \\ &= M^{-2(m-k)} \frac{\Gamma(M-k+3)}{\Gamma(M-m+3)} \\ &\lesssim M^{-(m-k)}. \end{aligned} \quad (63)$$

Inserting Equation (63) into Equation (57), one has

$$\|D_t^{\alpha+k} (u(t) - u^M(t))\|_{L^2_{\omega_2(t)}}^2 \lesssim M^{-(m-k)} |u(t)|_{\Lambda_{\omega_2(t)}^{\alpha,m}}^2. \quad (64)$$

Therefore, we obtain the desired result. $\square$

Theorem 5. Suppose $v^M(x) = \sum_{i=0}^M \hat{v}_i \mathcal{Z}_i(x)$, is the spectral solution of $v(x) \in \Lambda_{\omega_1(x)}^m(I_2)$. Then, for $0 \leq k \leq m \leq M+1$, we obtain

$$\|D_x^k (v(x) - v^M(x))\|_{L^2_{\omega_1(x)}} \lesssim M^{\frac{-1}{2}(m-k)(\mu-\frac{3}{2})} |v(x)|_{\mathbf{B}_{\omega_1(x)}^m}. \quad (65)$$

Proof. At first, based on the definitions of $v(x)$ and $\bar{v}(x)$, one has

$$\begin{aligned} \|D_x^k (v(x) - v^M(x))\|_{L^2_{\omega_1(x)}}^2 &= \sum_{n=M+1}^{\infty} |\hat{v}_n|^2 \|D_x^k \mathcal{Z}_n(x)\|_{L^2_{\omega_1(x)}}^2 \\ &= \sum_{n=M+1}^{\infty} |\hat{v}_n|^2 \frac{\|D_x^k \mathcal{Z}_n(x)\|_{L^2_{\omega_1(x)}}^2}{\|D_x^m \mathcal{Z}_n(x)\|_{L^2_{\omega_1(x)}}^2} \|D_x^m \mathcal{Z}_n(x)\|_{L^2_{\omega_1(x)}}^2 \\ &\leq \frac{\|D_x^k \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2}{\|D_x^m \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2} |v(x)|_{\Lambda_{\omega_1(x)}^m}^2. \end{aligned} \quad (66)$$

Now, we have

$$\begin{aligned} D_x^k \mathcal{Z}_{M+1}(x) &= \sum_{r=k}^{M+1} \lambda_{r,M+1} \frac{\Gamma(r+2)}{\Gamma(r-k+2)} x^{r-k+1} - 2 \sum_{r=k}^{M+1} \lambda_{r,M+1} \frac{\Gamma(r+3)}{\Gamma(r-k+3)} x^{r-k+2} \\ &\quad + \sum_{r=k}^{M+1} \lambda_{r,M+1} \frac{\Gamma(r+4)}{\Gamma(r-k+4)} x^{r-k+3}, \end{aligned} \quad (67)$$

Therefore, $\|D_x^k \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2$ is written as

$$\begin{aligned} \|D_x^k \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2 &= \sum_{r=k}^{M+1} \lambda_{r,M+1}^2 \frac{\Gamma^2(r+2) \Gamma(r-k+\mu-\frac{1}{2}) \Gamma(\mu-\frac{7}{2})}{\Gamma^2(r-k+2) \Gamma(r-k+2\mu-4)} \\ &\quad + 2 \sum_{r=k}^{M+1} \lambda_{r,M+1}^2 \frac{\Gamma^2(r+3) \Gamma(r-k+\mu+\frac{1}{2}) \Gamma(\mu-\frac{7}{2})}{\Gamma^2(r-k+3) \Gamma(r-k+2\mu-3)} \\ &\quad + \sum_{r=k}^{M+1} \lambda_{r,M+1}^2 \frac{\Gamma^2(r+4) \Gamma(r-k+\mu+\frac{3}{2}) \Gamma(\mu-\frac{7}{2})}{\Gamma^2(r-k+4) \Gamma(r-k+2\mu-2)}, \end{aligned} \quad (68)$$

The application of the Stirling formula leads to

$$\begin{aligned} \frac{\Gamma^2(r+2) \Gamma(r-k+\mu-\frac{1}{2})}{\Gamma^2(r-k+2) \Gamma(r-k+2\mu-4)} &\lesssim r^{2k} (r-k)^{-\mu+\frac{7}{2}}, \\ \frac{\Gamma^2(r+3) \Gamma(r-k+\mu+\frac{1}{2})}{\Gamma^2(r-k+3) \Gamma(r-k+2\mu-3)} &\lesssim r^{2k} (r-k)^{-\mu+\frac{7}{2}}, \\ \frac{\Gamma^2(r+4) \Gamma(r-k+\mu+\frac{3}{2})}{\Gamma^2(r-k+4) \Gamma(r-k+2\mu-2)} &\lesssim r^{2k} (r-k)^{-\mu+\frac{7}{2}}, \end{aligned} \quad (69)$$

Hence, we obtain

$$\|D_x^k \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2 \lesssim M^{2k} (M-k)^{-\mu+\frac{7}{2}}. \quad (70)$$

Similarly, we have

$$\|D_x^m \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2 \lesssim M^{2m} (M-m)^{-\mu+\frac{7}{2}}. \quad (71)$$

Finally, we obtain the following estimation

$$\frac{\|D_x^k \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2}{\|D_x^m \mathcal{Z}_{M+1}(x)\|_{L^2_{\omega_1(x)}}^2} \lesssim M^{-(m-k)(\mu-\frac{3}{2})}. \quad (72)$$

In the end, we obtain

$$\|D_x^k (v(x) - v^M(x))\|_{L^2_{\omega_1(x)}} \lesssim M^{\frac{-1}{2}(m-k)(\mu-\frac{3}{2})} |v(x)|_{\Lambda_{\omega_1(x)}^m}. \quad (73)$$

□

Theorem 6. Given the following assumptions, $0 \leq p \leq r \leq M+1$, and the approximation to $\eta(x, t) \in \mathbf{GB}_{\tilde{\omega}}^{r,s}(\Omega)$ is $\eta^M(x, t)$, as a result, the estimation that follows is applicable:

$$\left\| \frac{\partial^p}{\partial x^p} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}} \lesssim M^{\frac{-1}{4}(r-p)} |\eta(x, t)|_{\mathbf{GB}_{\tilde{\omega}(x,t)}^{r,0}}. \quad (74)$$

Proof. According to the definitions of $\eta(x, t)$ and $\eta^M(x, t)$, one has

$$\begin{aligned} \eta(x, t) - \eta^M(x, t) &= \sum_{i=0}^M \sum_{j=M+1}^{\infty} c_{ij} \mathcal{Z}_i(x) \mathcal{Y}_j(t) + \sum_{i=M+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \mathcal{Z}_i(x) \mathcal{Y}_j(t) \\ &\leq \sum_{i=0}^M \sum_{j=0}^{\infty} c_{ij} \mathcal{Z}_i(x) \mathcal{Y}_j(t) + \sum_{i=M+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \mathcal{Z}_i(x) \mathcal{Y}_j(t). \end{aligned} \quad (75)$$

Now, applying the same procedures as in Theorem 5, we obtain

$$\left\| \frac{\partial^p}{\partial x^p} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}} \lesssim M^{\frac{-1}{2}(r-p)(\mu-\frac{3}{2})} |\eta(x, t)|_{\mathbf{GB}_{\tilde{\omega}(x,t)}^{r,0}}. \quad (76)$$

□

Theorem 7. Let $\eta^M(x, t)$ be the spectral solution of $\eta(x, t) \in \mathbf{GB}_{\tilde{\omega}(x,t)}^{r,s}(\Omega)$, and assume that $0 < \alpha < 1$. Consequently, for $0 \leq q \leq s \leq M+1$, we obtain

$$\left\| \frac{\partial^{\alpha+q}}{\partial t^{\alpha+q}} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}} \lesssim M^{\frac{-1}{2}(s-q)} |\eta(x, t)|_{\mathbf{GB}_{\tilde{\omega}(x,t)}^{0,s}}. \quad (77)$$

Proof. The proof of Theorem 7 is similar to the proof of Theorem 6. □

Theorem 8. Let $\mathcal{R}(x, t)$ be the residual of Equation (38), then $\|\mathcal{R}(x, t)\|_{L^2_{\tilde{\omega}(x,t)}} \rightarrow 0$ as $M \rightarrow \infty$.

Proof. $\|\mathcal{R}(x, t)\|_{L^2_{\tilde{\omega}(x,t)}}$ of Equation (38) can be written as

$$\begin{aligned} \|\mathcal{R}(x, t)\|_{L^2_{\tilde{\omega}(x,t)}} &= \left\| D_t^\alpha \eta^M(x, t) + k_1 \eta^M(x, t) \eta_x^M(x, t) - k_2 \eta_{xx}^M(x, t) + k_3 \eta_{xxx}^M(x, t) - f(x, t) \right\|_{L^2_{\tilde{\omega}(x,t)}} \\ &\leq \left\| \frac{\partial^\alpha}{\partial t^\alpha} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}} + k_1 \left\| \eta^M(x, t) \frac{\partial \eta^M(x, t)}{\partial x} - \eta(x, t) \frac{\partial^2 \eta(x, t)}{\partial x^2} \right\|_{L^2_{\tilde{\omega}(x,t)}} \\ &\quad + k_2 \left\| \frac{\partial^2}{\partial x^2} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}} + k_3 \left\| \frac{\partial^3}{\partial x^3} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\tilde{\omega}(x,t)}}. \end{aligned} \quad (78)$$

that can be rewritten as

$$\begin{aligned} \|\mathcal{R}(x, t)\|_{L^2_{\hat{\omega}(x,t)}} &\leq \left\| \frac{\partial^\alpha}{\partial t^\alpha} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\hat{\omega}(x,t)}} + k_1 \|\eta^M(x, t)\|_{L^2_{\hat{\omega}(x,t)}} \left\| \frac{\partial}{\partial x} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\hat{\omega}(x,t)}} \\ &+ k_1 \left\| \frac{\partial \eta(x, t)}{\partial x} \right\|_{L^2_{\hat{\omega}(x,t)}} \|\eta(x, t) - \eta^M(x, t)\|_{L^2_{\hat{\omega}(x,t)}} + k_2 \left\| \frac{\partial^2}{\partial x^2} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\hat{\omega}(x,t)}} \\ &+ k_3 \left\| \frac{\partial^3}{\partial x^3} (\eta(x, t) - \eta^M(x, t)) \right\|_{L^2_{\hat{\omega}(x,t)}}. \end{aligned} \quad (79)$$

Now, the application of Theorems 6 and 7 leads to

$$\begin{aligned} \|\mathcal{R}(x, t)\|_{L^2_{\hat{\omega}(x,t)}} &\lesssim M^{-\frac{1}{2}s} |\eta(x, t)|_{\mathbf{GB}_{\hat{\omega}(x,t)}^{0,s}} + k_1 M^{-\frac{1}{2}(r-1)} |\eta(x, t)|_{\mathbf{GB}_{\hat{\omega}(x,t)}^{r,0}} + k_1 M^{-\frac{1}{2}r} |\eta(x, t)|_{\mathbf{GB}_{\hat{\omega}(x,t)}^{r,0}} \\ &+ k_2 M^{-\frac{1}{2}(r-2)} |\eta(x, t)|_{\mathbf{GB}_{\hat{\omega}(x,t)}^{r,0}} + k_3 M^{-\frac{1}{2}(r-3)} |\eta(x, t)|_{\mathbf{GB}_{\hat{\omega}(x,t)}^{r,0}}. \end{aligned} \quad (80)$$

Hence, $\|\mathcal{R}(x, t)\|_{L^2_{\hat{\omega}(x,t)}} \rightarrow 0$ as $M \rightarrow \infty$. $\square$

5. Illustrative Examples

Example 1 ([27]). Consider the time-fractional KdV–Burgers' equation of the form

$$D_t^\alpha \psi(x, t) + \psi(x, t) \psi_x(x, t) - \psi_{xx}(x, t) + \psi_{xxx}(x, t) = s(x, t), \quad 0 < \alpha < 1, \quad (81)$$

subject to the given initial and boundary conditions

$$\begin{aligned} \psi(x, 0) &= 0, \quad 0 < x \leq 1, \\ \psi(0, t) &= \psi(1, t) = \psi_x(1, t) = 0, \quad 0 < t \leq 1, \end{aligned} \quad (82)$$

where $\psi(x, t) = t^\alpha (x-1)^4 \sin(\pi x)$ is the exact known solution of this problem, and $s(x, t)$ is determined by Equation (81), which is consistent with the chosen solution. Table 1 shows a comparison of the L_2 error between our method and the method in [27] at $\alpha = 0.4$ and $\alpha = 0.8$. Figure 1 shows the absolute errors (left) and spectral solution (right) at $M = 12$, $\alpha = 0.7$, and $\mu = 1.8$. Figure 2 shows the absolute errors at different values of μ when $\alpha = 0.5$ and $M = 12$. Table 2 presents the absolute errors at different values of μ when $\alpha = 0.3$ and $M = 12$.

Table 1. Comparison of L_2 error of Example 1.

α	Method in [27] at $M = \lceil N^{\min\{r, 2\}} \rceil$, $r = 3$, $N = 32$	Our Method at $M = 12$		
		$\mu = 1$	$\mu = 1.5$	$\mu = 2$
0.4	9.8238×10^{-3}	4.16926×10^{-12}	1.93038×10^{-12}	9.64377×10^{-12}
0.8	6.4395×10^{-4}	3.39356×10^{-12}	1.57365×10^{-12}	8.3452×10^{-12}

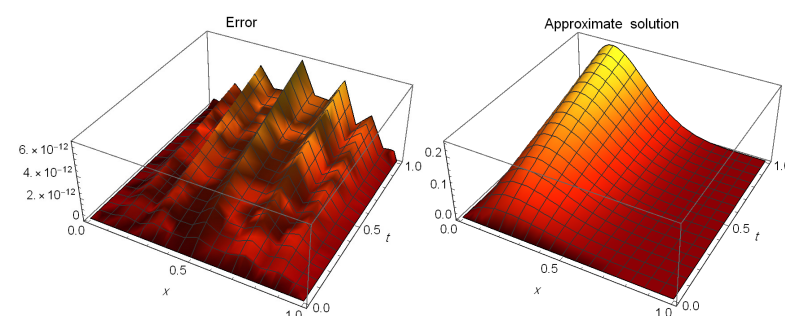


Figure 1. The absolute errors (left) and spectral solution (right) of Example 1 at $M = 12$, $\alpha = 0.7$, and $\mu = 1.8$.

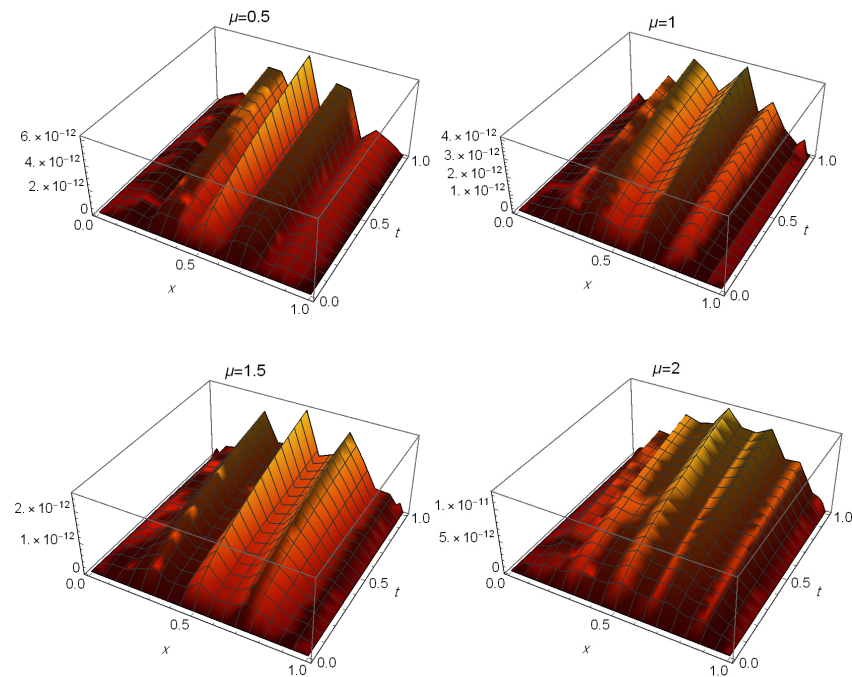


Figure 2. The absolute errors of Example 1 at different values of μ when $\alpha = 0.5$ and $M = 12$.

Table 2. The absolute errors of Example 1 at $\alpha = 0.3$, $M = 12$.

(x, t)	$\mu = 0.7$	$\mu = 1.3$	$\mu = 1.9$	$\mu = 2.5$
(0.1,0.1)	6.29025×10^{-13}	3.12347×10^{-13}	1.37257×10^{-12}	5.48636×10^{-12}
(0.2,0.2)	1.56244×10^{-12}	1.66714×10^{-12}	1.79409×10^{-12}	1.22696×10^{-11}
(0.3,0.3)	1.70239×10^{-12}	8.36803×10^{-13}	5.17145×10^{-12}	2.04907×10^{-11}
(0.4,0.4)	3.90749×10^{-12}	3.08369×10^{-12}	3.91269×10^{-12}	2.4729×10^{-11}
(0.5,0.5)	3.54023×10^{-12}	1.29931×10^{-12}	7.53037×10^{-12}	3.05061×10^{-11}
(0.6,0.6)	4.39656×10^{-12}	3.55601×10^{-12}	4.86968×10^{-12}	3.01709×10^{-11}
(0.7,0.7)	2.03082×10^{-12}	9.18957×10^{-13}	7.53591×10^{-12}	3.06087×10^{-11}
(0.8,0.8)	2.4376×10^{-12}	2.75191×10^{-12}	3.67946×10^{-12}	2.35486×10^{-11}
(0.9,0.9)	1.19161×10^{-12}	5.24651×10^{-13}	3.51299×10^{-12}	1.45983×10^{-11}

Example 2. Consider the time-fractional KdV–Burgers’ equation of the form

$$D_t^\alpha \psi(x, t) + \psi(x, t) \psi_x(x, t) - \psi_{xx}(x, t) + \psi_{xxx}(x, t) = s(x, t), \quad 0 < \alpha < 1, \quad (83)$$

subject to the given initial and boundary conditions

$$\begin{aligned} \psi(x, 0) &= 0, \quad 0 < x \leq 1, \\ \psi(0, t) &= t^\alpha, \quad \psi(1, t) = \psi_x(1, t) = e t^\alpha, \quad 0 < t \leq 1, \end{aligned} \quad (84)$$

where $\psi(x, t) = t^\alpha e^x$ is the exact known solution of this problem, and $s(x, t)$ is determined by Equation (83) consistent with the chosen solution. Figure 3 shows the absolute errors at different values of M when $\alpha = 0.9$ and $\mu = 1.2$. Table 3 presents the L_∞ errors at different values of μ when $\alpha = 0.5$ when $0 < x < 1$. Table 4 shows the maximum absolute errors at different values of M and α when $\mu = 1.5$.

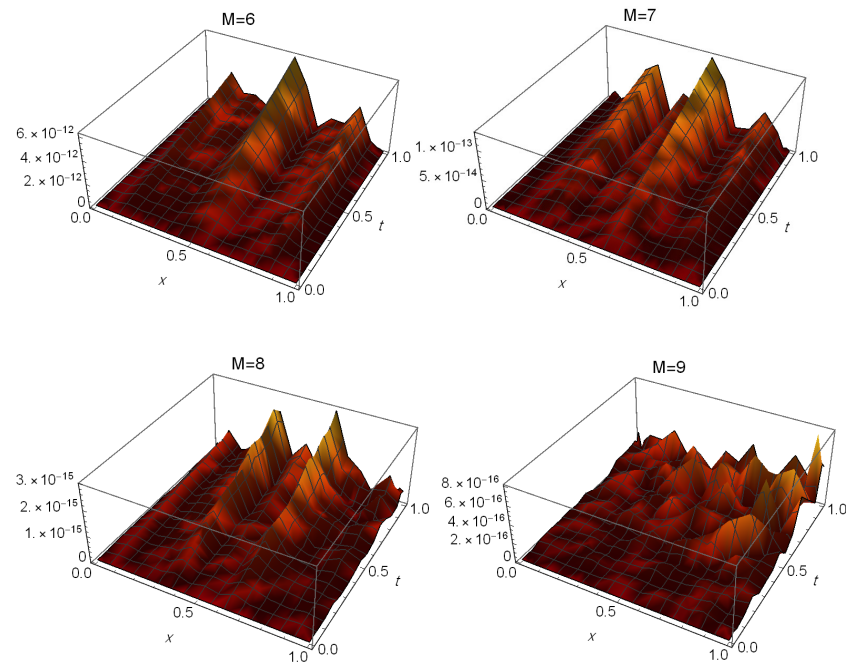


Figure 3. The absolute errors of Example 2 at different values of M when $\alpha = 0.9$ and $\mu = 1.2$.

Table 3. The L_∞ errors of Example 2 at $\alpha = 0.5$ when $0 < x < 1$.

t	$\mu = 0.5$	$\mu = 1.2$	$\mu = 1.9$	$\mu = 2.6$
0.1	3.42087×10^{-16}	2.72396×10^{-16}	3.0962×10^{-16}	3.82217×10^{-16}
0.2	4.8869×10^{-16}	6.82434×10^{-16}	5.27734×10^{-16}	5.17965×10^{-16}
0.3	5.3803×10^{-16}	5.70693×10^{-16}	7.6803×10^{-16}	5.70192×10^{-16}
0.4	6.84174×10^{-16}	5.44494×10^{-16}	6.19966×10^{-16}	7.64475×10^{-16}
0.5	5.45682×10^{-16}	6.07748×10^{-16}	6.81601×10^{-16}	6.81642×10^{-16}
0.6	9.97693×10^{-16}	8.4308×10^{-16}	1.02064×10^{-15}	9.20379×10^{-16}
0.7	8.91309×10^{-16}	1.0224×10^{-15}	1.13003×10^{-15}	1.19362×10^{-15}
0.8	9.77682×10^{-16}	1.36487×10^{-15}	1.0555×10^{-15}	1.03592×10^{-15}
0.9	6.03849×10^{-16}	6.72178×10^{-16}	1.04853×10^{-15}	1.05099×10^{-15}

Table 4. The maximum absolute errors of Example 2 at $\mu = 1.5$.

M	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.6$
2	4.69459×10^{-6}	4.08501×10^{-6}	3.31359×10^{-6}
4	4.52838×10^{-9}	3.94214×10^{-9}	3.20184×10^{-9}
6	2.90641×10^{-12}	2.53025×10^{-12}	2.15547×10^{-12}
8	1.69309×10^{-15}	1.40166×10^{-15}	9.26342×10^{-16}
9	3.81639×10^{-16}	3.27863×10^{-16}	3.68629×10^{-16}

Example 3. Consider the time-fractional KdV–Burgers’ equation of the form

$$D_t^\alpha \psi(x, t) + \psi(x, t) \psi_x(x, t) - \psi_{xx}(x, t) + \psi_{xxx}(x, t) = s(x, t), \quad 0 < \alpha < 1, \quad (85)$$

subject to the given initial and boundary conditions

$$\begin{aligned} \psi(x, 0) &= 0, \quad 0 < x \leq 1, \\ \psi(0, t) &= t^\alpha, \quad \psi(1, t) = \psi_x(1, t) = 0, \quad 0 < t \leq 1, \end{aligned} \quad (86)$$

where $\psi(x, t) = t^\alpha (1 - x)^2 \cos \pi x$ is the exact known solution of this problem, and $s(x, t)$ is determined by Equation (85), which is consistent with the chosen solution. Figure 4 shows the absolute errors at different values of M when $\alpha = 0.5$ and $\mu = 0.5$. Table 5 shows the absolute errors at different values of t when $\alpha = \mu = 0.8$ when $M = 12$. Table 6 shows the CPU time used and the maximum absolute errors at different values of M and α when $\mu = 1.5$. Figure 5 illustrates the absolute errors (left) and spectral solution (right) at $M = 12$, $\alpha = 0.2$, and $\mu = 2$.

Table 5. The absolute errors of Example 3 at $\alpha = \mu = 0.8$ when $M = 12$.

x	$t = 0.1$	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.1	4.41834×10^{-15}	1.0613×10^{-14}	1.84228×10^{-14}	2.5438×10^{-14}
0.2	7.04298×10^{-15}	1.70211×10^{-14}	2.96568×10^{-14}	4.10089×10^{-14}
0.3	7.59809×10^{-15}	1.81799×10^{-14}	3.16414×10^{-14}	4.37705×10^{-14}
0.4	1.04708×10^{-14}	2.53408×10^{-14}	4.41591×10^{-14}	6.10068×10^{-14}
0.5	8.38912×10^{-15}	2.01228×10^{-14}	3.50553×10^{-14}	4.85167×10^{-14}
0.6	7.02216×10^{-15}	1.70003×10^{-14}	2.96152×10^{-14}	4.09395×10^{-14}
0.7	2.69229×10^{-15}	6.46011×10^{-15}	1.11994×10^{-14}	1.54876×10^{-14}
0.8	2.53964×10^{-15}	6.13398×10^{-15}	1.06928×10^{-14}	1.48076×10^{-14}
0.9	1.03433×10^{-15}	2.48586×10^{-15}	4.32814×10^{-15}	5.9848×10^{-15}

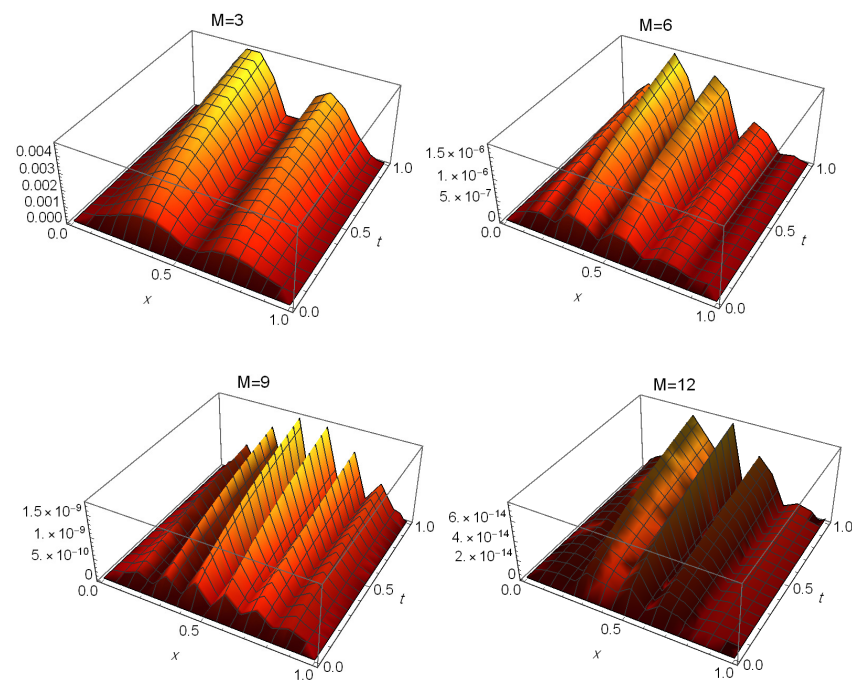


Figure 4. The absolute errors of Example 3 at different values of M when $\alpha = 0.5$ and $\mu = 0.5$.

Table 6. The maximum absolute errors of Example 3 at $\mu = 1.5$.

α	$M = 3$	CPU Time	$M = 6$	CPU Time	$M = 9$	CPU Time	$M = 12$	CPU Time
0.1	9.40161×10^{-4}	15.156	4.86794×10^{-7}	18.344	5.75219×10^{-10}	20.281	3.16691×10^{-14}	35.5
0.3	8.47965×10^{-4}	16.14	4.12545×10^{-7}	16.422	5.35616×10^{-10}	21.546	2.63123×10^{-14}	35.579
0.7	7.35108×10^{-4}	13.797	3.12559×10^{-7}	16.327	4.64397×10^{-10}	20.391	1.92624×10^{-14}	37.858

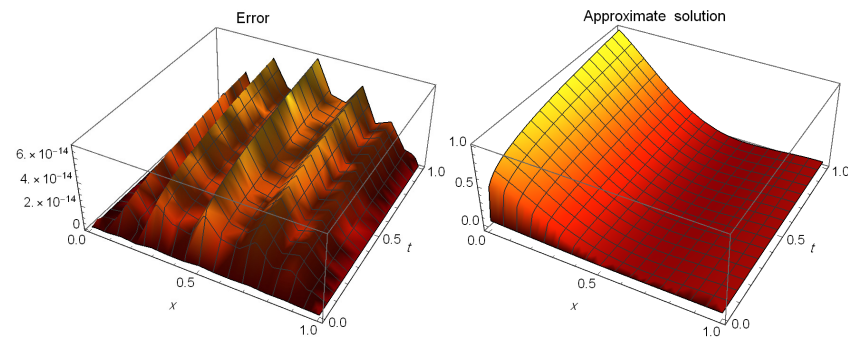


Figure 5. The absolute errors (left) and spectral solution (right) of Example 3 at $M = 12$, $\alpha = 0.2$, and $\mu = 2$.

6. Concluding Remarks

In conclusion, this paper presents a new numerical method for solving the time-fractional KdV–Burgers’ equation using the Petrov–Galerkin approach and Gegenbauer polynomials. The proposed method is both accurate and stable, offering significant improvements over some existing techniques. Through various examples, we have shown that this method can handle complex boundary conditions and produce results with minimal error. This study provides a valuable framework that can be extended to more complicated systems involving partial differential equations. The method’s efficiency and accuracy make it a promising tool for tackling fractional-order problems in fields like fluid dynamics, wave propagation, and other applications in mathematical physics and engineering. We believe that this approach could be further developed to solve a wider range of real-world problems.

Readers can implement the methodology proposed in this paper by adhering to the steps outlined in Algorithm 1.

Algorithm 1 Coding algorithm for the proposed technique

Input $\alpha, k_1, k_2, k_3, a_0(x), a_1(t), a_2(t), a_3(t)$ and $s(x, t)$.

Step 1. Using transformation (36) to convert (13) with the conditions (14)–(15), into modified equation (38) under the homogeneous conditions (39)–(40).

Step 2. Assume an approximate solution $\eta^M(x, t)$ as in (43).

Step 3. Apply Petrov–Galerkin method to obtain the system in (49).

Step 4. Use Theorem 3 to get the elements of $a_{ir}, b_{js}, c_{ior}, d_{jns}, p_{ir}, q_{js}$ and g_{ir}

Step 5. Use *FindRoot* command with initial guess $\{\hat{\eta}_{ij} = 10^{-i-j}, i, j : 0, 1, \dots, M\}$, to solve the system (49) to get $\hat{\eta}_{ij}$.

Output $\eta^M(x, t)$

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