

Article

Structural Stability of Pseudo-Parabolic Equations for Basic Data

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Abstract: This article investigates the spatial decay properties and continuous dependence on the basic geometric structure. Assuming that the total potential energy is bounded and the homogeneous Dirichlet condition is satisfied on the side of the solution within the cylindrical domain, we establish an auxiliary function related to the solution. By extending the data at the finite end forward, we can establish the continuous dependence on the perturbation of base data.

Keywords: pseudo parabolic equation; disturbance data; continuous dependence

MSC: 35B40; 35Q30; 76D05

1. Introduction

This article studies the pseudo parabolic equation that has already appeared in the theory of dual temperature heat conduction equation

$$\dot{u} = \Delta u + \eta \Delta \dot{u}, \quad (1)$$

where $\dot{u} = \frac{\partial u}{\partial t}$, Δ is the Laplace operator and $\eta > 0$. In 1968, Chen and Gurtin [1] developed a new theory of heat conduction. This pseudo parabolic equation appears in the theory of double thermal conduction. Assuming that (1) satisfies nonstandard condition, Horgan and Quintanilla [2] obtained the spatial decay estimate for solution to (1) in a three-dimensional semi-infinite cylinder. The existence and blow-up phenomenon of solutions to pseudo-parabolic equations can be seen in [3,4]. The semi-infinite cylinder was defined as (see Figure 1)

$$R = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq 0 \right\},$$

where D was a bounded domain on $x_1 O x_2$.

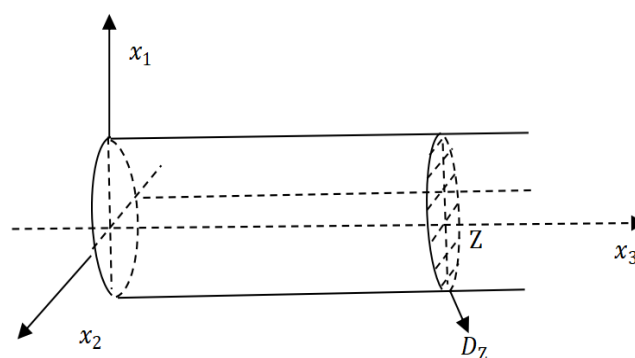


Figure 1. Cylindrical pipe R .

For more results about the spatial decay estimate, one can see [5–10].



Citation: Wang, Y.; Li, Y. Structural Stability of Pseudo-Parabolic Equations for Basic Data. *Math. Comput. Appl.* **2024**, *29*, 105. <https://doi.org/10.3390/mca29060105>

Academic Editor: Gianluigi Rozza

Received: 2 September 2024

Revised: 11 November 2024

Accepted: 13 November 2024

Published: 15 November 2024



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The first purpose of this article is to study the spatial properties of solutions to Equation (1). Previous articles have always focused on the decay of time variable in the solution of equations. Baranovskii [11] considered an initial-boundary value problem for the Navier-Stokes-Voigt equations and proved that the solution decayed exponentially as $t \rightarrow \infty$. Our innovation lies in abandoning non-standard conditions and obtaining solutions that decay exponentially with spatial variable. To do this, we suppose that the solution to (1) satisfy the following initial-boundary conditions

$$u = 0, x \in \partial D \times \{x_3 > 0\}, t > 0, \quad (2)$$

$$u(x_1, x_2, 0, t) = g(x_1, x_2, t), (x_1, x_2) \in D, t > 0, \quad (3)$$

$$u(x, 0) = 0, x \in R, \quad (4)$$

where ∂D is the boundary of D and g is a positive given function. If the initial-boundary conditions satisfy certain constraints, we prove that the solution of Equations (1)–(4) decays exponentially as the axial variable tends infinite. Obviously, this research is a continuation of the spatial properties of solutions.

The second purpose of this article is to investigate the continuous dependence of the solution of Equations (1)–(4) on the base perturbation. Let D_f represent the disturbed base, i.e.,

$$D_f = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = f(x_1, x_2) \geq 0 \right\},$$

where the given sufficiently smooth function f satisfies

$$|f(x_1, x_2)| < \epsilon, \epsilon > 0. \quad (5)$$

ϵ is called as perturbation parameter which is a measure of the maximum perturbation of the actual base from its assumed planar shape $D(0)$. When the perturbation parameters are subjected to small perturbations, we investigate whether the solutions of Equations (1)–(4) will also be subject to small perturbations.

The cylinder with a disturbed base is defined as (see Figure 2)

$$R_f = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq f(x_1, x_2) \geq 0 \right\}.$$

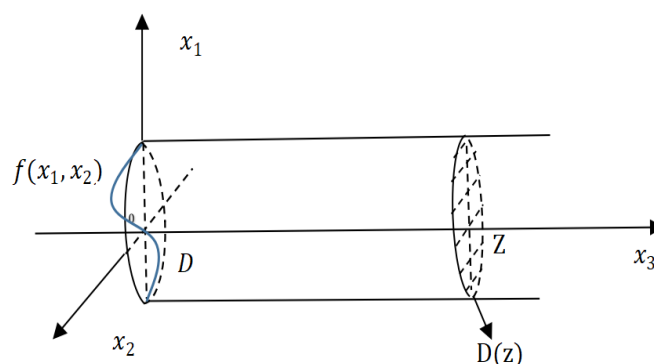


Figure 2. Cylindrical pipe R_f .

We assume that v is the perturbation solution of Equations (1)–(4) when R is replaced by R_f . But (3) is changed to

$$v(x_1, x_2, f(x_1, x_2), t) = h(x_1, x_2, t), (x_1, x_2) \in D_f, t > 0, \quad (6)$$

where h is a given function. If the total potential energy on the cylinder is bounded, many authors have studied the continuous dependence of various types of partial differential equations on planar substrates and have achieved significant results (see [12–17]). This

article considers the error caused by the solution of Equation (1) on actual non-planar substrates and assumed planar substrates. There are relatively few studies of this type. Knops and Payne [12] established the continuous dependence of the displacement on the base geometry and load of the decay behaviour in a prismatic semi-infinite cylinder occupied by an anisotropic homogeneous linear elastic material. Li [18] investigated the continuous dependence of solutions to layered composite materials in binary mixtures on perturbation parameters defined in a semi-infinite cylinder. Different from [12,18], the model in this article has two Laplacian operators, which seems a bit cumbersome to handle. Our proof method is to introduce an auxiliary function and derive a differential inequality. Then we prove that the solution is continuously dependent on perturbation parameter and base data.

In this paper, we shall also introduce the notations

$$R(z) = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq z \geq 0 \right\},$$

$$D(z) = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = z \geq 0 \right\}.$$

2. Spatial Decay Bound

First, we give two useful Lemmas.

Lemma 1. (see [19,20]) If $\phi = 0$, on ∂D , then

$$\lambda \int_D \psi^2 dA \leq \int_D \sum_{\alpha=1}^2 \left(\frac{\partial \psi}{\partial x_\alpha} \right)^2 dA,$$

where λ is the smallest positive eigenvalue of

$$\Delta_2 \vartheta + \lambda \vartheta = 0, \text{ in } D, \vartheta = 0, \text{ on } \partial D.$$

Here, Δ_2 is a two-dimensional Laplace operator.

Lemma 2. (see p182 in [21]) If $\phi(0) = 0$ and k is a positive integer, then

$$\int_0^1 \phi^{2k} dx \leq C \int_0^1 (\phi')^{2k} dx,$$

where $C = \frac{1}{2k-1} \left(\frac{2k}{\pi} \sin \frac{\pi}{2k} \right)^{2k}$.

Specifically, if we take $k = 1$ in Lemma 2 and use the integration technique, we can obtain the following result. If $\phi(-m) = 0$, then

$$\int_{-m}^m \phi^2 dx \leq \frac{16m^2}{\pi^2} \int_{-m}^m (\phi')^2 dx, m > 0. \quad (7)$$

To derive the spatial decay estimate, we establish an “energy” function, i.e.,

$$F^u(z, t) = - \int_0^t \int_{D(z)} e^{-b_1 \tau} \frac{\partial u}{\partial x_3} \dot{u} dA d\tau - \eta \int_0^t \int_{D(z)} e^{-b_1 \tau} \frac{\partial \dot{u}}{\partial x_3} \dot{u} dA d\tau, \quad (8)$$

where b_1 is a positive constant. The purpose of setting this parameter is to obtain an appropriate energy function, which can easily control the gradient terms of the solution.

We suppose that the total potential energy $F^u(0, t)$ is bounded.

Using Equations (1), (2) and (4) and the divergence theorem, we have

$$\begin{aligned}
 -\frac{\partial}{\partial z} F^u(z, t) &= \int_0^t \int_{D(z)} e^{-b_1 \tau} \left[\frac{\partial^2 u}{\partial x_3^2} \dot{u} + \frac{\partial u}{\partial x_3} \frac{\partial \dot{u}}{\partial x_3} + \eta \left(\frac{\partial \dot{u}}{\partial x_3} \right)^2 + \eta \frac{\partial^2 \dot{u}}{\partial x_3^2} \dot{u} \right] dA d\tau \\
 &= \frac{1}{2} e^{-b_1 t} \int_{D(z)} \left(\frac{\partial u}{\partial x_3} \right)^2 dA \Big|_{\tau=t} + \int_0^t \int_{D(z)} e^{-b_1 \tau} \left[\frac{1}{2} b_1 \left(\frac{\partial u}{\partial x_3} \right)^2 + \eta \left(\frac{\partial \dot{u}}{\partial x_3} \right)^2 \right] dA d\tau \\
 &\quad + \int_0^t \int_{D(z)} e^{-b_1 \tau} \dot{u} \left[\dot{u} - \Delta_2 u - \eta \Delta_2 \dot{u} \right] dA d\tau \\
 &= \frac{1}{2} e^{-b_1 t} \int_{D(z)} \sum_{i=1}^3 \left(\frac{\partial u}{\partial x_i} \right)^2 dA \Big|_{\tau=t} + \int_0^t \int_{D(z)} e^{-b_1 \tau} \dot{u}^2 dA d\tau \\
 &\quad + \int_0^t \int_{D(z)} e^{-b_1 \tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial u}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{u}}{\partial x_i} \right)^2 \right] dA d\tau.
 \end{aligned} \tag{9}$$

An application of the Hölder inequality and the A-G mean inequality in (8) leads to

$$\begin{aligned}
 F^u(z, t) &\leq \sqrt{\frac{2}{b_1}} \left[\frac{1}{2} b_1 \int_0^t \int_{D(z)} e^{-b_1 \tau} \left(\frac{\partial u}{\partial x_3} \right)^2 dA d\tau \int_0^t \int_{D(z)} e^{-b_1 \tau} \dot{u}^2 dA d\tau \right]^{\frac{1}{2}} \\
 &\quad + \sqrt{\eta} \left[\eta \int_0^t \int_{D(z)} e^{-b_1 \tau} \left(\frac{\partial \dot{u}}{\partial x_3} \right)^2 dA d\tau \int_0^t \int_{D(z)} e^{-b_1 \tau} \dot{u}^2 dA d\tau \right]^{\frac{1}{2}} \\
 &\leq \left[\sqrt{\frac{2}{b_1}} + \sqrt{\eta} \right] \left[-\frac{\partial}{\partial z} F^u(z, t) \right].
 \end{aligned} \tag{10}$$

Integrating (10) from 0 to ∞ , we obtain

$$F^u(z, t) \leq F^u(0, t) e^{-b_2 z}, \quad z > 0, \tag{11}$$

where $b_2 = \frac{1}{\sqrt{\frac{2}{b_1}} + \sqrt{\eta}}$. Equation (10) shows that $\lim_{z \rightarrow \infty} F^u(z, t) = 0$.

Now, integrating (9) from z to ∞ , we have

$$\begin{aligned}
 F^u(z, t) &= \frac{1}{2} e^{-b_1 t} \int_{R(z)} \sum_{i=1}^3 \left(\frac{\partial u}{\partial x_i} \right)^2 dx \Big|_{\tau=t} + \int_0^t \int_{R(z)} e^{-b_1 \tau} \dot{u}^2 dx d\tau \\
 &\quad + \int_0^t \int_{R(z)} e^{-b_1 \tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial u}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{u}}{\partial x_i} \right)^2 \right] dx d\tau.
 \end{aligned} \tag{12}$$

Combining (11) and (12), the following result can be obtained.

Theorem 1. Assume that u is a solution to (1)–(3) which is defined in a semi-infinite cylinder R and the total potential energy is bounded, then

$$\begin{aligned}
 &\frac{1}{2} e^{-b_1 t} \int_{R(z)} \sum_{i=1}^3 \left(\frac{\partial u}{\partial x_i} \right)^2 dx \Big|_{\tau=t} + \int_0^t \int_{R(z)} e^{-b_1 \tau} \dot{u}^2 dx d\tau \\
 &\quad + \int_0^t \int_{R(z)} e^{-b_1 \tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial u}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{u}}{\partial x_i} \right)^2 \right] dx d\tau \\
 &\leq F^u(0, t) e^{-b_2 z}.
 \end{aligned}$$

Remark 1. Assume that v is an unperturbed solution of (1)–(4) which is defined in R_f . If we define

$$F^v(z, t) = - \int_0^t \int_{D(z)} e^{-b_1 \tau} \left[\frac{\partial v}{\partial x_3} \dot{v} + \eta \frac{\partial \dot{v}}{\partial x_3} \dot{v} \right] dA d\tau, \tag{13}$$

using a similar method to Theorem 1 we can also obtain

$$\begin{aligned} F^v(z, t) &= \frac{1}{2} e^{-b_1 t} \int_{R_f(z)} \sum_{i=1}^3 \left(\frac{\partial v}{\partial x_i} \right)^2 dx \Big|_{\tau=t} + \int_0^t \int_{R_f(z)} e^{-b_1 \tau} \phi^2 dx d\tau \\ &\quad + \int_0^t \int_{R_f(z)} e^{-b_1 \tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial v}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{v}}{\partial x_i} \right)^2 \right] dx d\tau \\ &\leq F^v(\epsilon, t) e^{-b_2(z-\epsilon)}, \quad z > \epsilon. \end{aligned} \quad (14)$$

Remark 2. Theorem 1 and (14) show that the perturbed and unperturbed solutions decay exponentially to zero as $z \rightarrow \infty$.

Remark 3. Using Lemma 1, we can obtain the bounds of L^2 -norm of unperturbed solution u and perturbed solution v . That is

$$\begin{aligned} \int_0^t \int_{D(z)} u^2 dA d\tau &= -2 \int_0^t \int_{R(z)} u \frac{\partial u}{\partial x_3} dx d\tau \\ &\leq 2 \left[\int_0^t \int_{R(z)} u^2 dx d\tau \int_0^t \int_{R(z)} \left(\frac{\partial u}{\partial x_3} \right)^2 dx d\tau \right] \\ &\leq \frac{2}{\sqrt{\lambda}} \left[\int_0^t \int_{R(z)} \sum_{\alpha=1}^2 \left(\frac{\partial u}{\partial x_\alpha} \right)^2 dx d\tau \int_0^t \int_{R(z)} \left(\frac{\partial u}{\partial x_3} \right)^2 dx d\tau \right] \\ &\leq \frac{2}{\sqrt{\lambda}} F^u(0, t) e^{-b_2 z} e^{b_1 t}, \quad z > 0, \end{aligned} \quad (15)$$

and

$$\int_0^t \int_{D(z)} v^2 dA d\tau \leq \frac{2}{\sqrt{\lambda}} F^v(\epsilon, t) e^{-b_2(z-\epsilon)} e^{b_1 t}, \quad z > \epsilon. \quad (16)$$

3. Important Lemmas

Let w represent the difference between the perturbed solution and the unperturbed solution, i.e.,

$$w = u - v,$$

then w satisfies

$$\dot{w} = \Delta w + \eta \Delta \dot{w}, \quad x \in R(\epsilon), \quad t > 0, \quad (17)$$

$$w = 0, \quad x \in \partial D \times \{x_3 > \epsilon\}, \quad t > 0, \quad (18)$$

$$w(x, 0) = 0, \quad x \in R(\epsilon). \quad (19)$$

According to the triangle inequality, it can be concluded that

$$\int_0^t \int_{D(x_3)} w^2 dA d\tau \leq \int_0^t \int_{D(x_3)} u^2 dA d\tau + \int_0^t \int_{D(x_3)} v^2 dA d\tau, \quad x_3 \geq \epsilon.$$

Combining (15) and (16), we can conclude that w decays to zero as $z \rightarrow \infty$.

To obtain the continuous dependence of the solution on the perturbation parameter, we establish a new energy function

$$V(x_3, t) = \int_0^t \int_{R(x_3)} w^2 dx d\tau, \quad x_3 \geq \epsilon, \quad (20)$$

We introduce an auxiliary function φ such that

$$\dot{\varphi} + \Delta\varphi - \eta\Delta\dot{\varphi} = -w, x \in R(x_3 > 0), 0 < \tau < t, \quad (21)$$

$$\varphi(x, \tau) = 0, x \in \partial D \times \{x_3 > 0\}, 0 < \tau < t, \quad (22)$$

$$\varphi(x, \tau) = 0, (x_1, x_2) \in D, 0 < \tau < t, \quad (23)$$

$$\varphi(x, t) = 0, x \in R(x_3), \quad (24)$$

$$\varphi, \nabla\varphi \rightarrow 0 (\text{uniformly in } x_1, x_2, \tau) \text{ as } x_3 \rightarrow \infty, \quad (25)$$

where $x_3 > \epsilon$.

We derive several lemmas related to auxiliary function φ .

Lemma 3. *The auxiliary function φ satisfies the following identity*

$$\int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \leq V(x_3, t), x_3 \geq \epsilon.$$

Proof. We begin with

$$\int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau = \int_0^t \int_{R(x_3)} \dot{\varphi} [-\Delta\varphi + \eta\Delta\dot{\varphi} - w] dx d\tau. \quad (26)$$

Using the divergence theorem, (18) and (19), we have

$$\begin{aligned} \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau &= -\frac{1}{2} \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx \Big|_{\tau=0} - \eta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \dot{\varphi}}{\partial x_i} \right)^2 dx d\tau \\ &\quad - \int_0^t \int_{R(x_3)} w \dot{\varphi} dx d\tau \\ &\leq \left[\int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \int_0^t \int_{R(x_3)} w^2 dx d\tau \right]^{\frac{1}{2}}. \end{aligned} \quad (27)$$

Using the Schwarz inequality and dropping the non-positive terms in (27), Lemma 3 can be obtained. $\square$

Lemma 4. *The auxiliary function φ satisfies the following identity*

$$\int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx d\tau \leq \frac{2t}{\pi} V(x_3, t), x_3 \geq \epsilon.$$

Proof. We begin with

$$\int_0^t \int_{R(x_3)} \varphi [\dot{\varphi} + \Delta\varphi - \eta\Delta\dot{\varphi} + w] dx d\tau = 0. \quad (28)$$

Using the divergence theorem, (18), (19) and Lemma 2, we have

$$\begin{aligned} \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx d\tau &= -\frac{1}{2} \int_{R(x_3)} \varphi^2 dx \Big|_{\tau=0} - \frac{1}{2} \eta \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx \Big|_{\tau=0} \\ &\quad + \int_0^t \int_{R(x_3)} w \varphi dx d\tau \\ &\leq \left[\int_0^t \int_{R(x_3)} \varphi^2 dx d\tau \int_0^t \int_{R(x_3)} w^2 dx d\tau \right]^{\frac{1}{2}} \\ &\leq \frac{2t}{\pi} \left[\int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \int_0^t \int_{R(x_3)} w^2 dx d\tau \right]^{\frac{1}{2}}. \end{aligned} \quad (29)$$

Using Lemma 3 in (29), Lemma 4 can be obtained. $\square$

Lemma 5. The auxiliary function φ satisfies the following identity

$$\int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \dot{\varphi}}{\partial x_i \partial x_3} \frac{\partial \varphi}{\partial x_i} dx d\tau + \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i \partial x_3} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau = \frac{1}{2} \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dx \Big|_{\tau=0}, x_3 \geq \epsilon.$$

Proof. By using the divergence theorem and (22)–(25), we can obtain Lemma 5. $\square$

Lemma 6. The auxiliary function φ satisfies the following identity

$$\begin{aligned} \frac{1}{2} \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA \Big|_{\tau=0} + \frac{1}{2} \eta \int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dA d\tau &\leq \frac{1}{2\epsilon_1} V(x_3, t) \\ &+ \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \dot{\varphi}}{\partial x_i \partial x_3} \frac{\partial \varphi}{\partial x_i} dx d\tau + \frac{1}{2} \epsilon_1 \int_0^t \int_{R(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dx d\tau, \quad x_3 \geq \epsilon, \end{aligned}$$

where ϵ_1 is a positive constant.

Proof. We begin with the following identity

$$\int_0^t \int_{R(x_3)} \frac{\partial \dot{\varphi}}{\partial x_3} [\dot{\varphi} + \Delta \varphi - \eta \Delta \dot{\varphi} + w] dx d\tau = 0. \quad (30)$$

Integrating (30) by parts and using (18), (19), we have

$$\begin{aligned} \frac{1}{2} \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA \Big|_{\tau=0} + \frac{1}{2} \eta \int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dA d\tau \\ = \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \dot{\varphi}}{\partial x_i \partial x_3} \frac{\partial \varphi}{\partial x_i} dx d\tau + \int_0^t \int_{R(x_3)} w \frac{\partial \dot{\varphi}}{\partial x_3} dx d\tau \\ \leq \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \dot{\varphi}}{\partial x_i \partial x_3} \frac{\partial \varphi}{\partial x_i} dx d\tau + \frac{1}{2\epsilon_1} V(x_3, t) \\ + \frac{1}{2} \epsilon_1 \int_0^t \int_{R(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dx d\tau. \end{aligned}$$

Therefore Lemma 6 can be obtained. $\square$

Lemma 7. If then the auxiliary function φ satisfies the following inequality

$$\begin{aligned} \frac{1}{2} \int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA d\tau + \frac{1}{2} \eta \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA \Big|_{\tau=0} + \delta \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \\ + \frac{1}{2} \delta \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx \Big|_{\tau=0} + \eta \delta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \dot{\varphi}}{\partial x_i} \right)^2 dx d\tau \\ \leq \eta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i \partial x_3} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau + a_1(t) V(x_3, t), \end{aligned}$$

where $a_1(t) = 2\sqrt{\frac{2t}{\pi}} + \delta$.

Proof. Letting δ be a positive constant, we compute

$$\int_0^t \int_{R(x_3)} \left[\frac{\partial \varphi}{\partial x_3} - \delta \dot{\varphi} \right] [\dot{\varphi} + \Delta \varphi - \eta \Delta \dot{\varphi} + w] dx d\tau = 0. \quad (31)$$

Using the divergence theorem, (18) and (19) in (31), we obtain

$$\begin{aligned} & \int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} \dot{\varphi} dx d\tau - \int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA d\tau - \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i \partial x_3} \frac{\partial \varphi}{\partial x_i} dx d\tau \\ & + \eta \int_0^t \int_{D(x_3)} \frac{\partial \varphi}{\partial x_3} \frac{\partial \dot{\varphi}}{\partial x_3} dA d\tau + \eta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i \partial x_3} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau \\ & - \delta \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau + \delta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial \varphi}{\partial x_i} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau \\ & - \eta \delta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial \dot{\varphi}}{\partial x_i} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau \\ & + \int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} w dx d\tau - \delta \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA d\tau + \frac{1}{2} \eta \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA \Big|_{\tau=0} + \delta \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \\ & + \frac{1}{2} \delta \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx \Big|_{\tau=0} + \eta \delta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \dot{\varphi}}{\partial x_i} \right)^2 dx d\tau \\ & = \int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} \dot{\varphi} dx d\tau + \eta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i \partial x_3} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau \\ & + \int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} w dx d\tau - \delta \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau. \end{aligned} \quad (32)$$

Using the Schwarz inequality and Lemmas 3 and 4, we obtain

$$\int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} \dot{\varphi} dx d\tau \leq \left[\int_0^t \int_{R(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dx d\tau \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \right]^{\frac{1}{2}} \leq \sqrt{\frac{2t}{\pi}} V(x_3, t), \quad (33)$$

$$\int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} w dx d\tau \leq \left[\int_0^t \int_{R(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dx d\tau \int_0^t \int_{R(x_3)} w^2 dx d\tau \right]^{\frac{1}{2}} \leq \sqrt{\frac{2t}{\pi}} V(x_3, t), \quad (34)$$

$$-\delta \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau \leq \delta \left[\int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \int_0^t \int_{R(x_3)} w^2 dx d\tau \right]^{\frac{1}{2}} \leq \delta V(x_3, t). \quad (35)$$

Inserting (33)–(35) into (32), we can complete the proof of Lemma 7.

Combining Lemma 7 and Lemma 6, using Lemma 5 and choosing $\delta > \varepsilon_1$, we have the following lemma. $\square$

Lemma 8. The auxiliary function φ satisfies the following inequality

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA d\tau + \frac{1}{2} \eta \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dA \Big|_{\tau=0} + \frac{1}{2} \eta^2 \int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dA d\tau \\ & + \delta \int_0^t \int_{R(x_3)} \dot{\varphi}^2 dx d\tau \\ & + \frac{1}{2} \delta \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx \Big|_{\tau=0} + \eta \delta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \left(\frac{\partial \dot{\varphi}}{\partial x_i} \right)^2 dx d\tau \\ & \leq a_2(t) V(x_3, t), \quad x_3 \geq \epsilon, \end{aligned}$$

where $a_2(t) = a_1(t) + \frac{1}{2\varepsilon_1} \eta$.

Remark 4. Lemma 8 indicates the following important result

$$\int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dAd\tau + \eta^2 \int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dAd\tau \leq 2a_2(t)V(x_3, t), \quad x_3 \geq \epsilon. \quad (36)$$

4. Main Results

The third section has already derived the required properties for the auxiliary function. In this section, we will use these properties to derive a first-order differential inequality about auxiliary function. By solving this inequality, the continuous dependence of the solution can be obtained.

Theorem 2. Let w be solution of the Equations (17)–(20) in $R(\epsilon)$. If (17) and (23) hold, then the function $V(x_3, t)$ satisfies

$$V(x_3, t) \leq V(\epsilon, t)e^{-\frac{1}{2a_2(t)}(x_3 - \epsilon)}, \quad x_3 \geq \epsilon.$$

Proof. Let $x_3 \geq \epsilon$ is a fixed point on the coordinate axis x_3 . Using (17)–(20) and the divergence theorem, we can have

$$\begin{aligned} V(x_3, t) &= - \int_0^t \int_{R(x_3)} [\dot{\varphi} + \Delta \varphi - \eta \Delta \dot{\varphi}] w dx d\tau \\ &= - \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau + \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial w}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx d\tau \\ &\quad - \eta \int_0^t \int_{R(x_3)} \sum_{i=1}^3 \frac{\partial w}{\partial x_i} \frac{\partial \dot{\varphi}}{\partial x_i} dx d\tau \\ &\quad + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dAd\tau - \eta \int_0^t \int_{D(x_3)} w \frac{\partial \dot{\varphi}}{\partial x_3} dAd\tau \\ &= - \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau - \int_0^t \int_{R(x_3)} \Delta w \varphi dx d\tau \\ &\quad - \eta \int_0^t \int_{R(x_3)} \Delta \dot{w} \varphi dx d\tau \\ &\quad + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dAd\tau - \eta \int_0^t \int_{D(x_3)} w \frac{\partial \dot{\varphi}}{\partial x_3} dAd\tau \\ &= - \int_0^t \int_{R(x_3)} \dot{\varphi} w dx d\tau - \int_0^t \int_{R(x_3)} \varphi \dot{w} dx d\tau \\ &\quad + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dAd\tau - \eta \int_0^t \int_{D(x_3)} w \frac{\partial \dot{\varphi}}{\partial x_3} dAd\tau \\ &= \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dAd\tau - \eta \int_0^t \int_{D(x_3)} w \frac{\partial \dot{\varphi}}{\partial x_3} dAd\tau. \end{aligned} \quad (37)$$

A combination of (36), (37) and the Hölder inequality leads to

$$\begin{aligned} V(x_3, t) &\leq \left[\int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dAd\tau \int_0^t \int_{D(x_3)} w^2 dAd\tau \right]^{\frac{1}{2}} \\ &\quad + \eta \left[\int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dAd\tau \int_0^t \int_{D(x_3)} w^2 dAd\tau \right]^{\frac{1}{2}} \\ &\leq \left[\int_0^t \int_{D(x_3)} \left(\frac{\partial \varphi}{\partial x_3} \right)^2 dAd\tau + \eta^2 \int_0^t \int_{D(x_3)} \left(\frac{\partial \dot{\varphi}}{\partial x_3} \right)^2 dAd\tau \right]^{\frac{1}{2}} \left[\int_0^t \int_{D(x_3)} w^2 dAd\tau \right]^{\frac{1}{2}} \\ &\leq \sqrt{2a_2(t)} \sqrt{V(x_3, t)} \left[- \frac{\partial}{\partial x_3} V(x_3, t) \right]^{\frac{1}{2}} \end{aligned}$$

or

$$V(x_3, t) \leq -2a_2(t) \frac{\partial}{\partial x_3} V(x_3, t), x_3 > \epsilon. \quad (38)$$

An integration of (38) yields Theorem 2. $\square$

Remark 5. Theorem 2 only indicates that w decays exponentially as x_3 increases. Including Theorem 2, these decay results are not rigorous because we do not yet know whether $V(\epsilon, t)$ depends on the perturbation parameter ϵ . Therefore, we derive the explicit bound of $V(\epsilon, t)$ in term of ϵ and g, h .

Now, we extend the perturbed data (see (6)) and unperturbed data (see (3)) to $(-\epsilon, f)$ and $(-\epsilon, 0)$, respectively. That is

$$u(x_1, x_2, x_3, t) = g(x_1, x_2, t), (x_1, x_2) \in D(0), -\epsilon \leq x_3 \leq 0, t > 0, \quad (39)$$

$$v(x_1, x_2, f(x_1, x_2), t) = h(x_1, x_2, t), (x_1, x_2) \in D(0), -\epsilon \leq x_3 \leq f(x_1, x_2), t > 0. \quad (40)$$

We also define

$$w(x_1, x_2, x_3, t) = u(x_1, x_2, x_3, t) - v(x_1, x_2, x_3, t), (x_1, x_2) \in D(0), -\epsilon \leq x_3 \leq \epsilon, t > 0.$$

Combining (39) and (40), it is easy to have

$$w(x_1, x_2, -\epsilon, t) = g(x_1, x_2, t) - h(x_1, x_2, f(x_1, x_2), t), (x_1, x_2) \in D(0), t > 0. \quad (41)$$

We now derive some lemmas.

Lemma 9. If $h \in H^1(R_f \times [0, \infty))$, then

$$F^v(f, t) \leq c_1(f, t),$$

where $c_1(f, t)$ is a positive function which depends on f and t .

Proof. We introduce a new known function

$$H(x_1, x_2, x_3, t) = h(x_1, x_2, t)e^{-\sigma(x_3-f)}. \quad (42)$$

In (13), we choose $x_3 = f$, and then use the divergence theorem and (42) to have

$$\begin{aligned} F^v(f, t) &= - \int_0^t \int_{D(f)} e^{-b_1\tau} \left[\frac{\partial v}{\partial x_3} \dot{H} + \eta \frac{\partial \dot{v}}{\partial x_3} \dot{H} \right] dA d\tau \\ &= \int_0^t \int_{R_f} e^{-b_1\tau} \sum_{i=1}^3 \left[\frac{\partial}{\partial x_i} \left(\frac{\partial v}{\partial x_i} \dot{H} \right) + \eta \frac{\partial}{\partial x_i} \left(\frac{\partial \dot{v}}{\partial x_i} \dot{H} \right) \right] dx d\tau \\ &= \int_0^t \int_{R_f} e^{-b_1\tau} \sum_{i=1}^3 \left(\frac{\partial v}{\partial x_i} \frac{\partial \dot{H}}{\partial x_i} + \eta \frac{\partial \dot{v}}{\partial x_i} \frac{\partial \dot{H}}{\partial x_i} \right) dx d\tau + \int_0^t \int_{R_f} e^{-b_1\tau} \dot{v} \dot{H} dx d\tau \end{aligned} \quad (43)$$

Using the Schwarz inequality, we obtain from (43)

$$\begin{aligned} F^v(f, t) &\leq \frac{1}{2} \int_0^t \int_{R_f} e^{-b_1\tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial v}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{v}}{\partial x_i} \right)^2 \right] dx d\tau \\ &\quad + \left(\frac{1}{2} \eta + \frac{1}{b_1} \right) \int_0^t \int_{R_f} e^{-b_1\tau} \sum_{i=1}^3 \left(\frac{\partial \dot{H}}{\partial x_i} \right)^2 dx d\tau \\ &\quad + \frac{1}{2} \int_0^t \int_{R_f} e^{-b_1\tau} \dot{v}^2 dx d\tau + \frac{1}{2} \int_0^t \int_{R_f} e^{-b_1\tau} \dot{H}^2 dx d\tau. \end{aligned} \quad (44)$$

On the other hand, we choose $x_3 = f$ in (14) to obtain

$$\begin{aligned} F^v(f, t) &= \frac{1}{2} e^{-b_1 t} \int_{R_f} \sum_{i=1}^3 \left(\frac{\partial v}{\partial x_i} \right)^2 dx \Big|_{\tau=t} + \int_0^t \int_{R_f} e^{-b_1 \tau} \dot{v}^2 dx d\tau \\ &\quad + \int_0^t \int_{R_f} e^{-b_1 \tau} \sum_{i=1}^3 \left[\frac{1}{2} b_1 \left(\frac{\partial v}{\partial x_i} \right)^2 + \eta \left(\frac{\partial \dot{v}}{\partial x_i} \right)^2 \right] dx d\tau. \end{aligned} \quad (45)$$

Combining (44) and (45), we can conclude that

$$\begin{aligned} F^v(f, t) &\leq \frac{1}{2} F^v(f, t) + \frac{1}{2} \int_0^t \int_{R_f} e^{-b_1 \tau} \dot{H}^2 dx d\tau \\ &\quad + \left(\frac{1}{2} \eta + \frac{1}{b_1} \right) \int_0^t \int_{R_f} e^{-b_1 \tau} \sum_{i=1}^3 \left(\frac{\partial \dot{H}}{\partial x_i} \right)^2 dx d\tau \end{aligned}$$

or

$$F^v(f, t) \leq \int_0^t \int_{R_f} e^{-b_1 \tau} \dot{H}^2 dx d\tau + \left(1 + \frac{2}{b_1} \right) \int_0^t \int_{R_f} e^{-b_1 \tau} \sum_{i=1}^3 \left(\frac{\partial \dot{H}}{\partial x_i} \right)^2 dx d\tau. \quad (46)$$

Choosing $c_1(f, t) = \int_0^t \int_{R_f} e^{-b_1 \tau} \dot{H}^2 dx d\tau + \left(1 + \frac{2}{b_1} \right) \int_0^t \int_{R_f} e^{-b_1 \tau} \sum_{i=1}^3 \left(\frac{\partial \dot{H}}{\partial x_i} \right)^2 dx d\tau$ in (46), we can complete the proof of Lemma 9.

By adopting a similar method to Lemma 9, we can obtain the following lemma. $\square$

Lemma 10. If $g \in H^1(R \times [0, \infty))$, then

$$F^u(0, t) \leq c_2(0, t),$$

where

$$c_2(0, t) = \int_0^t \int_R e^{-b_1 \tau} \dot{G}^2 dx d\tau + \left(1 + \frac{2}{b_1} \right) \int_0^t \int_R e^{-b_1 \tau} \sum_{i=1}^3 \left(\frac{\partial \dot{G}}{\partial x_i} \right)^2 dx d\tau$$

and

$$G(x_1, x_2, x_3, t) = g(x_1, x_2, t) e^{-\sigma x_3}.$$

By using the triangle inequality, Lemmas 9 and 10, we have

$$\begin{aligned} \int_0^t \int_{R(-\epsilon)} \left(\frac{\partial w}{\partial x_3} \right)^2 dx d\tau &\leq \int_0^t \int_{R(-\epsilon)} \left(\frac{\partial u}{\partial x_3} \right)^2 dx d\tau + \int_0^t \int_{R(-\epsilon)} \left(\frac{\partial v}{\partial x_3} \right)^2 dx d\tau \\ &\leq \int_0^t \int_R \left(\frac{\partial u}{\partial x_3} \right)^2 dx d\tau + \int_0^t \int_{R_f} \left(\frac{\partial v}{\partial x_3} \right)^2 dx d\tau \\ &\leq \frac{2}{b_1} e^{b_1 t} [c_1(f, t) + c_2(0, t)] \doteq c_3(f, t). \end{aligned} \quad (47)$$

Now, we derive the upper bound of $V(\epsilon, t)$ in term of ϵ . If we let $x_3 = \epsilon$ in (38), we have

$$\begin{aligned} V(\epsilon, t) &\leq 2a_2(t) \int_0^t \int_{D(\epsilon)} w^2 dA d\tau \\ &= 2a_2(t) \int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} w \frac{\partial w}{\partial x_3} dx d\tau + 2a_2(t) \int_0^t \int_D (g - h)^2 dA d\tau \\ &\leq 2a_2(t) \left[\int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} w^2 dx d\tau \int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} \left(\frac{\partial w}{\partial x_3} \right)^2 dx d\tau \right]^{\frac{1}{2}} \\ &\quad + 2a_2(t) \int_0^t \int_D (g - h)^2 dA d\tau. \end{aligned} \quad (48)$$

Using (7), we have

$$\int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} w^2 dx d\tau \leq \frac{16\epsilon^2}{\pi^2} \int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} \left(\frac{\partial w}{\partial x_3} \right)^2 dx d\tau + 2\epsilon \int_0^t \int_D (g-h)^2 dAd\tau. \quad (49)$$

Inserting (49) into (48) and using the Schwarz inequality, we obtain

$$V(\epsilon, t) \leq \frac{32}{\pi} a_2(t) \epsilon \int_0^t \int_{-\epsilon}^{\epsilon} \int_{D(x_3)} \left(\frac{\partial w}{\partial x_3} \right)^2 dx d\tau + \left[\frac{a_2(t)\pi}{2} + 2a_2(t) \right] \int_0^t \int_D (g-h)^2 dAd\tau. \quad (50)$$

In view of (47) and (50), we can have the following theorem.

Theorem 3. If $h \in H^1(R_f \times [0, \infty))$ and $g \in H^1(R \times [0, \infty))$, then function $V(\epsilon, t)$ defined in (21) satisfies

$$V(\epsilon, t) \leq \frac{32}{\pi} a_2(t) \epsilon \int_0^t \int_{-\epsilon}^{\epsilon} c_3(f, t) + \left[\frac{a_2(t)\pi}{2} + 2a_2(t) \right] \int_0^t \int_D (g-h)^2 dAd\tau. \quad (51)$$

Remark 6. Theorems 2 and 3 demonstrate that $V(x_3, t)$ continuously depends on ϵ and the base data. That is, when ϵ approaches 0 and g approaches h , then $v(x_3, t)$ approaches 0. If $\epsilon = 0$, Theorem 2 is the Saint-Venant's principle type decay result. When the disturbance parameter and baseline data are subjected to small disturbances, the system can recover to its original state after being disturbed, or at least remain within an acceptable range of fluctuations, which is very important for many practical applications, such as control systems and signal processing.

Remark 7. In any cross-section of R , the continuous dependence result can still be obtained. We compute

$$\begin{aligned} \int_0^t \int_{D(x_3)} w^2 dAd\eta &= -2 \int_0^t \int_{R(x_3)} w \frac{\partial w}{\partial x_3} dx d\tau \\ &\leq 2 \left[V(x_3) \int_0^t \int_{R(-\epsilon)} \left(\frac{\partial w}{\partial x_3} \right)^2 dx d\eta \right]^{\frac{1}{2}}. \end{aligned} \quad (52)$$

Using (47) and Theorems 2 and 3, we can obtain the continuous dependence result.

Author Contributions: Software, Y.W.; writing—original draft, Y.W.; validation, Y.L.; data curation, Y.L. The authors have equal contributions to each part of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Tutor System Project of Guangzhou Huashang College (2023HSDS29) and the Research team project of Guangzhou Huashang College (2021HSKT01).

Data Availability Statement: This paper focuses on theoretical analysis, not involving experiments and data.

Acknowledgments: The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions.

Conflicts of Interest: The authors declare that they have no competing interests.

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