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Loss of the Sturm–Liouville Property of Time-Varying Second-Order Differential Equations in the Presence of Delayed Dynamics

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Abstract: This paper considers a nominal undelayed and time-varying second-order Sturm–Liouville differential equation on a finite time interval which is a nominal version of another perturbed differential equation subject to a delay in its dynamics. The nominal delay-free differential equation is a Sturm–Liouville system in the sense that it is subject to prescribed two-point boundary conditions. However, the perturbed differential system is not a Sturm–Liouville system, in general, due to the presence of delayed dynamics. The main objective of the paper is to investigate the loss of the boundary values of the Sturm–Liouville nominal undelayed system in the presence of the delayed dynamics. Such a delayed dynamics is considered a perturbation of the nominal dynamics related to the Sturm–Liouville system with given two-point boundary values. In particular, this loss of the Sturm–Liouville exact tracking of the prescribed two-point boundary values might happen because the nominal boundary values may become lost by the state trajectory solution in the presence of delays, related to the undelayed case, due to the presence of the delayed dynamics. The worst-case error description of the deviation of the two-point boundary values of the current perturbed differential with respect to those of the nominal Sturm–Liouville system is characterized in terms of error norms related to the nominal system. Under sufficiently small deviations of the parameterization of the perturbed system with respect to the nominal one, such a worst-error characterization makes the current perturbed system an approximate Sturm–Liouville system of the nominal undelayed one.

Keywords: ordinary differential equations; Sturm–Liouville system; undelayed system; delayed system; approximate Sturm–Liouville system



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1. Introduction

Some of the well-known problems within the so-called Sturm–Liouville class are, for instance, the Bessel’s and Legendre’s equations and some typical second-order ordinary differential equations with periodic solutions [1–11]. One of the main properties of Sturm–Liouville systems is that their solution trajectory tracks the prescribed two-boundary given points under an infinite countable set of values of a constant real parameter λ which parameterizes the differential equation. Such a set of values is said to be the eigenvalue set of the Sturm–Liouville system [1–3]. In [4], a Sturm–Liouville-type description is developed for the wave equation, while in [5–7], some Sturm–Liouville type systems of interest are described under the formalism of fractional calculus. In particular, in [7], the solvability and the stability of nonlinear impulsive Langevin and Sturm–Liouville equations involving Caputo–Hadamard fractional derivatives and multipoint boundary value conditions are focused on. To unify the two types of equations, we investigate a general nonlinear impulsive coupled implicit system. On the other hand, Sturm–Liouville problems have been studied in the presence of delays. In particular, the uniqueness of the nonlinear inverse problem associated with Sturm–Liouville operators with multiple delays has been investigated in [8], while the multiplicity and the non-existence of positive solutions for a class of impulsive Sturm–Liouville boundary value problems has been

discussed in [9]. On the other hand, the optimal state estimation based on deterministic Kalman filtering in both time and the frequency domains is considered in [10]. In particular, the spectral factorization is considered for Kalman filtering, and a class of Riesz-spectral systems is considered in this research. In the study performed in [11], the inverse Sturm–Liouville problem is considered with polynomials of the spectral parameter in the first boundary condition and with entire analytic functions in the second one. More recently, in [12], one proposes a novel approach to inverse Sturm–Liouville eigenvalue problems under Dirichlet boundary conditions. In that way, when a Sturm–Liouville eigenvalue problem with unknown integrable potential interacts with function potentials, a family of perturbation problems is obtained. Also, it is proved in [13] that the Sturm–Liouville problem with mixed boundary conditions can be transformed to a generalized Sturm–Liouville problem endowed with Dirichlet or Neumann boundary conditions. In [14], the eigenvalues of a fractional-type Sturm–Liouville problem are investigated, while a generalization of Picard’s Theorem on Sturm–Liouville Equation with Lebesgue integrable function on a finite interval is addressed in [15].

On the other hand, time-delay systems appear in a natural way in the description of many problems of real life like, for instance, in many biological problems (such as epidemic models, ecology and medical models, sunflower equation, etc.), in diffusion problems, in war and peace problems, in missile/anti-missile dynamics, etc. The background literature on this subject has been very rich since several decades ago. See, for instance, [16–25] and the references therein. Delays can typically be either constant or time-varying point delays or distributed delays over intervals of finite or infinite durations. Within the classes of distributed delays, one includes some typical Volterra-type integro-differential equations. Delays can also appear in the internal dynamics, that is, in the states or in the forcing or control functions, that is, in the inputs or outputs. The delays can also be single or multiple and, in the second case, they can be measurable; that is, integer multiples of a basic minimum delay or not then referred to as not measurable. Sets of the above types of delays can appear also in a combined manner in a particular problem. A main characteristic of internal time-delay systems is that they are infinite-dimensional so that they possess infinitely many characteristic zeros. In this context, stability and stabilization criteria and methodologies for time-delay systems are investigated in [16,17]. See also references therein. In particular, in [17], the system’s states are assumed to be not measurable. In [18], the effect of time delay on the dynamics of an eco-epidemiological system is investigated. In [19], the existence and the global exponential stability of the periodic solution of a class of Cohen–Grossberg neural networks with time-varying delays is investigated. In [20], the h -manifold stability of neural networks with variable impulsive perturbations and time-varying delays of Cohen–Grossberg-type bidirectional associative memory is investigated. On the other hand, the h -manifold stability for impulsive delayed SIR epidemic models (that is, with Susceptible–Infectious–Recovered subpopulations) have been investigated in [21]. In [22], the dynamics of a Lotka–Volterra competitor–competitor–mutualist system with time-varying delays is investigated and, in [23], the μ analysis and synthesis type for uncertain time-delay systems via Padé approximations is investigated.

It can also be pointed out that the influence of delays is very relevant also in control theory and applications. For instance, the controlled system itself can possess delays through its own nature, which leads to increased stabilization difficulties compared with the cases of absence of delays. Also, the stabilizing controller of an uncontrolled system can be synthesized with designed delays, in order to facilitate the closed-loop stabilization process. In typical problems, there is no precise knowledge of the dynamics or the values of the delays, but only some nominal approximated values, which makes the appropriate resulting stabilization technique still more involved.

The main objective of this paper is to investigate the loss of the boundary values of properties of the Sturm–Liouville nominal undelayed system in the presence of delays in the dynamics, which is considered a perturbation of the nominal dynamics. In particular, this loss of the Sturm–Liouville exact tracking of the prescribed two-point boundary values

might happen because the nominal boundary values may become lost in the presence of delays related to the undelayed case, due to the presence of the delayed dynamics, and, respectively, the eventual presence of time-varying parameterizations. See, for instance, refs. [16–25] and, respectively, refs. [26–28], and some of the references therein.

The paper is organized as follows. Section 2 discusses two linear time-varying second-order differential equations associated with the two-boundary values of Sturm–Liouville-type problems. The first one is a Sturm–Liouville system which plays the role of a nominal differential system, while the second one is a current version of the above one which is considered a perturbation caused by an internal constant delay in its dynamics. Both differential equations are equivalently reformulated, for convenience of subsequent analysis, as second-order differential systems of differential equations of first-order. The respective solutions of the undelayed system and the delayed one are calculated analytically in the appropriate closed forms for given finite initial conditions, through the definition and use of “ad hoc” evolution operators. The comparison between such solutions is explicitly addressed. The solution of the nominal undelayed system is calculated under finite-point initial conditions, while that of the delayed one has, in general, a piecewise bounded function of initial conditions on a time interval of length being equal to the delay size value. The dynamics of such a nominal undelayed system is decomposed as a non-unique representation in a time-invariant part plus an incremental time-varying one, whose representative matrix has bounded piecewise continuous entries. Such a decomposition is then useful in the next section of this article, to characterize the boundedness and the asymptotic properties of the evolution operator and the state-trajectory solution, based on the stability, critical stability or instability properties of the time-invariant part of the matrix of dynamics. Sections 3 and 4 are devoted to the characterization of the two-point boundary conditions when the undelayed differential system is a Sturm–Liouville one of uniform boundedness and the asymptotic properties of the evolution operators when the time-invariant part of the undelayed system is stable, critically stable or unstable. The necessary and sufficient condition to keep the prescribed two-point boundary values of the nominal Sturm–Liouville system by the current delayed system is characterized explicitly in Section 3. Such a condition is very difficult to accomplish in general and, therefore, the errors of the two boundary points between those of the nominal Sturm–Liouville system and the current delayed one are characterized in the subsequent section. Section 5 is devoted to characterizing the worst-case errors in terms of error norms of the two-point boundary values between those of the nominal differential system, assumed to be a Sturm–Liouville system, and those of the perturbed current one. The sources of error between the respective two-point boundary values are the norms of the matrix of delayed dynamics and the delay size. The main idea behind the given error approach is to estimate the error norms of the two-point point boundary values related to those of the nominal undelayed system, which is a Sturm–Liouville one. It is seen that, under sufficiently small deviations of the parameterization of the perturbed system with respect to the nominal one, such a worst-error characterization makes the current perturbed system an approximate Sturm–Liouville system of the nominal one. Finally, some concluding remarks end the paper.

Notation

I_n is the n -th identity matrix.

$$\bar{n} = \{1, 2, \dots, n\}$$

The superscript T denotes transposition.

$$e_1 = (1, 0)^T; e_2 = (0, 1)^T$$

If $f : \mathbf{R} \rightarrow \mathbf{R}^n$ then $\hat{f}_{[\beta, \alpha]}$ is the strip of $f(t)$ on $[\alpha, \beta]$, namely, $\hat{f}_{[\beta, \alpha]} = f(\tau)$ for $\tau \in [\alpha, \beta]$ and $\hat{f}_{[\beta, \alpha]} = 0$ for $\tau \notin [\alpha, \beta]$. In a similar way, one defines $\hat{f}_{(a,b)}$, $\hat{f}_{[a, b)}$ and $\hat{f}_{(a, b]}$.

M^+ denotes the Moore–Penrose pseudoinverse of $M \in \mathbb{R}^{n \times m}$. In particular, $M^+ = M^{-1}$ if $n = m$ and M is non-singular.

$R(M)$ denotes the Range (or Image) subspace of the matrix M .

2. Second-Order Nominal and Delayed Time-Varying Differential Equations

Through this section, we discuss two linear differential equations associated with the two-boundary value Sturm–Liouville-type problems. One of them is considered a nominal version of the Sturm–Liouville problem while the other one is the current version, which is considered a perturbation of the former one. A problem that arises is that the characterization of the conditions under the current differential equation is a Sturm–Liouville system provided that the nominal one is a Sturm–Liouville system. Another problem to be addressed is the characterization of the worst-case errors of the solution deviation of the current differential equation from an achievable nominal Sturm–Liouville system.

Assume that the following linear time-varying differential equation of second-order

$$(p_0(t)y'_0(t))' + (q_0(t) + \lambda_0\omega_0(t))y_0(t) = 0 \quad (1)$$

is a nominal version of the perturbed current differential equation with a finite-point time delay $h > 0$:

$$(p_0(t)y'(t))' + (q_0(t) + \lambda_0\omega_0(t))y(t) + (q_d(t) + \tilde{\lambda}\omega_d(t))y(t-h) = 0 \quad (2)$$

where $p_0(t) > 0$, $\omega_0(t) > 0$, $q_0(t)$, λ_0 and $\tilde{\lambda}$ are real numbers, $q_d(t)$ and $\omega_d(t) > 0$ are bounded real functions defined in the real interval $[0, b]$ and piecewise-continuous in $(0, b)$, and $p_0(t)$ is differentiable in $(0, b)$.

Note that Equation (1) can be rewritten, equivalently, as a second-order linear time-varying differential system of two coupled first-order differential equations by defining the second-order vector function $x_0(t) = (y_0(t), y'_0(t))^T$ as follows:

$$x'_0(t) = A_0(t)x_0(t); \quad t \in [a, b] \quad (3)$$

with $x_0(0) = x_{00}$, where

$$A_0(t) = \begin{bmatrix} 0 & 1 \\ -p_0^{-1}(t)(q_0(t) + \lambda_0\omega_0(t)) & 0 \end{bmatrix} \quad (4)$$

is a piecewise-continuous matrix from the given assumptions on (1), and then (3) and (4) is a second-order piecewise-continuous differential system of first-order differential equations [29], while Equation (2) becomes, in the same way, to be equivalent to the second-order piecewise-continuous delayed differential system of first-order differential equations:

$$x'(t) = A_0(t)x(t) + A_d(t)x(t-h); \quad t \in [a, b] \quad (5)$$

with $x(t) = (y(t), y'(t))^T$ subject to a bounded piecewise-continuous function of initial conditions $\varphi: [-h, 0] \rightarrow \mathbb{R}^2$ with $\varphi(0^+) = x(0^+) = x_0(0^+) = x_{00}$ (note that the piecewise-continuous $\phi(t)$ can have a finite jump at $t = 0$), where

$$A_d(t) = \begin{bmatrix} 0 & 0 \\ -p_0^{-1}(t)(q_d(t) + \tilde{\lambda}\omega_d(t)) & 0 \end{bmatrix} \quad (6)$$

The unique solutions of (3), subject to (4), and (5), subject to (6), on $[0, \infty)$ for each given initial condition are, respectively:

$$x_0(t) = T_0(t, 0)x_{00} \quad (7)$$

$$\begin{aligned}
x(t) &= (\hat{T}(t, 0))(\varphi) = (\hat{T}(t, 0))(\hat{x}_{[0, -h]}) \\
&= T_0(t, 0)x_{00} + \int_0^t T_0(t, \tau)A_d(\tau)x(\tau - h)d\tau \\
&= T_0(t, 0)x_{00} + \int_{-h}^0 T_0(t, \tau)A_d(h + \tau)\varphi(\tau)d\tau + \left(\int_0^{\max(t-h, 0)} T_0(t, \tau)A_d(\tau)T_0(\tau - h, 0)d\tau \right)x_{00} \\
&\quad + \int_0^{\max(t-h, 0)} \int_{-h}^0 T_0(t, \tau)A_d(\tau)T_0(\tau - h, \theta)A_d(h + \theta)\varphi(\theta)d\theta d\tau; \quad t \in \mathbf{R}_0
\end{aligned} \tag{8}$$

where $x(t) = \varphi(t)$ for $t \in [-h, 0]$ and $T_0 : [t, \tau] \rightarrow \mathbf{R}^{2 \times 2}$ and $\hat{T} : [t, \tau] \rightarrow \mathbf{R}^{2 \times 2}$ are the respective evolution operators of (3) and (5) which satisfy $T_0(t, t) = T(t, t) = I_2$ for $t \geq 0$ and $T_0(t, \tau) = T(t, \tau) = 0$ for $t < \tau$, as well as the respective differential systems (3) and (5), which is

$$T_0'(t) = A_0(t)T_0(t, 0) \tag{9}$$

$$T'(t) = A_0(t)T(t, 0) + A_d(t)T(t - h, 0) \tag{10}$$

for $t \geq 0$. Note that (7) is the obvious unique solution of the unforced differential system (3) built with the evolution operator $T_0 : [t, \tau] \rightarrow \mathbf{R}^{2 \times 2}$ which satisfies (3), in fact, the same differential system (3). In the same way, (8) is the unique solution of (5) subject to the evolution operator which satisfies (10), and (5) might be considered indistinctly as an unforced delayed differential system subject to interval initial conditions on $[-h, 0]$ or as a forced one whose forcing term is the delayed dynamics. Therefore, the solution can be expressed equivalently as the first expression of (8), via the evolution operator $\hat{T} : [t, \tau] \rightarrow \mathbf{R}^{2 \times 2}$, or as the subsequent equivalent ones, which are more useful for their evaluation, again via the evolution operator $T_0 : [t, \tau] \rightarrow \mathbf{R}^{2 \times 2}$ which builds separately the unforced and forced parts of the solution.

Then, the solutions of (3) and (5) for $t \geq 0$ are, respectively,

$$T_0(t, 0) = I_2 + \int_0^t A_0(\tau)T_0(\tau, 0)d\tau \tag{11}$$

$$\begin{aligned}
T(t, 0) &= I_2 + \int_0^t (A_0(\tau)T(\tau, 0) + A_d(\tau)T(\tau - h, 0))d\tau \\
&= T_0(t, 0) + \int_0^t (A_0(\tau)\tilde{T}(\tau, 0) + A_d(\tau)T(\tau - h, 0))d\tau \\
&= T_0(t, 0) + \int_0^t T_0(t, \tau)A_d(\tau)T(\tau - h, 0)d\tau
\end{aligned} \tag{12}$$

where the error evolution operator between the current and nominal differential systems satisfies the following:

$$\tilde{T}(t, 0) = T(t, 0) - T_0(t, 0) = \int_0^t (A_0(\tau)\tilde{T}(\tau, 0) + A_d(\tau)T(\tau - h, 0))d\tau \tag{13}$$

for all $t \geq 0$, which is the unique solution of the differential system:

$$\begin{aligned}
\tilde{T}'(t) &= A_0(t)\tilde{T}(t, 0) + A_d(t)T(t - h, 0) \\
&= A_0(t)\tilde{T}(t, 0) + A_d(t)\tilde{T}(t - h, 0) + A_dT_0(t - h, 0)
\end{aligned} \tag{14}$$

Note that $T(t, h) = T_0(t, 0)$ for $t \in (-\infty, h]$.

Remark 1. Since $T(\sigma, \sigma) = T_0(\sigma, \sigma) = I_2$ for any real $\sigma \geq 0$ then (11) and (12) obey the more general expressions:

$$T_0(t, \sigma) = I_2 + \int_\sigma^t A_0(\tau)T_0(\tau, \sigma)d\tau \tag{15}$$

$$T(t, \sigma) = I_2 + \int_\sigma^t (A_0(\tau)T(\tau, \sigma) + A_d(\tau)T(\tau - h, \sigma))d\tau \tag{16}$$

for any real $\sigma \in [0, t]$.

Remark 2. Since $A_0(t)$ is not constant, then the evolution operator $T_0(t, \tau)$ is not a strongly continuous one-parameter semigroup (commonly referred to also as a C_0 -semigroup) from $\mathbf{R}_{0+}^2$ to $L(C_0^2(\mathbf{R}_{0+}))$ (with $L(C_0^2(\mathbf{R}_{0+}))$ being the Banach space of bounded continuous real vector functions of dimension two with domain in $C_0^2(\mathbf{R}_{0+})$ endowed with the sup norm) of infinitesimal generator for any $t_0 \in \mathbf{R}_{0+}$ so that it does not hold the property $A_0(t_0)x(t_0) = \lim_{t \rightarrow t_0^+} \frac{1}{t} (T(t, t_0) - I_2)$. It can be seen easily that the operator is not a one-parameter semigroup since it does not fulfil the associative relation [27], $T_0(t + \tau, 0) = T_0(t, 0)T_0(\tau, 0) = T_0(\tau, 0)T_0(t, 0)$ for all $t, \tau \in \mathbf{R}_{0+}$ and, in particular, the operator composition is not commutative. In the same way, $T(t, \tau) : \mathbf{R}_{0+}^2 \rightarrow C_0^2(\mathbf{R}_{0+})$ is not a strongly continuous semigroup either.

It is now seen that, although the evolution operator $T_0(t, \tau) : \mathbf{R}_{0+}^2 \rightarrow C_0^2(\mathbf{R}_{0+})$ is not a one-parameter semigroup, it is still a strongly continuous semigroup, since it satisfies the associative property as it is now discussed.

Theorem 1. The undelayed system has the following properties:

(i) $T_0(t, \tau) : \mathbf{R}_{0+}^2 \rightarrow C_0^2(\mathbf{R}_{0+})$ satisfies the associative property $T_0(t, 0) = T_0(t, \sigma)T_0(\sigma, 0)$ for any $\sigma, t(\geq \sigma) \in \mathbf{R}_{0+}$.

(ii) Decompose $A_0(t)$, in a non-unique way, as $A_0(t) = A_{00} + \tilde{A}_0(t)$ for all $t \in \mathbf{R}_{0+}$, where $A_{00} \in \mathbf{R}^{2 \times 2}$ is constant. Then, one has for any $\sigma, t(\geq \sigma) \in \mathbf{R}_{0+}$, which the following equivalent expressions hold:

$$\begin{aligned} x_0(t) &= T_0(t, \sigma)x_0(\sigma) = T_0(t, 0)x_{00} \\ &= e^{A_{00}(t-\sigma)} \left(x_0(\sigma) + \int_0^{t-\sigma} e^{-A_{00}\tau} \tilde{A}_0(\tau + \sigma)x(\tau + \sigma)d\tau \right) \\ &= e^{A_{00}(t-\sigma)} \left(I_2 + \int_0^{t-\sigma} e^{-A_{00}\tau} \tilde{A}_0(\tau + \sigma)T_0(\tau + \sigma, \sigma)d\tau \right) T_0(\sigma, 0)x_{00} \\ &= e^{A_{00}(t-\sigma)} \left(I_2 + \int_\sigma^t e^{-A_{00}(\tau-\sigma)} \tilde{A}_0(\tau)T_0(\tau, \sigma)d\tau \right) T_0(\sigma, 0)x_{00} \\ &= e^{A_{00}t} \left(I_2 + \int_0^t e^{-A_{00}\tau} \tilde{A}_0(\tau)T_0(\tau, 0)d\tau \right) x_{00} \end{aligned} \quad (17)$$

(iii) The evolution operator of the solution of (3) satisfies for any $\sigma, t(\geq \sigma) \in \mathbf{R}_{0+}$ and any decomposition $A_0(t) = A_{00} + \tilde{A}_0(t)$ for all $t \in \mathbf{R}_{0+}$:

$$T_0(t, \sigma) = e^{A_{00}(t-\sigma)} \left(T_0(\sigma, 0) + \int_0^{t-\sigma} e^{-A_{00}\tau} \tilde{A}_0(\tau + \sigma)T_0(\tau + \sigma, 0)d\tau \right) \quad (18)$$

Proof. Note that, since the solution of (3) is unique for each given initial condition, the relations

$$x_0(t) = T_0(t, 0)x_{00} = T_0(t, \sigma)x_0(\sigma) = T_0(t, \sigma)T_0(\sigma, 0)x_{00} \quad (19)$$

Hold, irrespective of x_{00} . This proves Property (i). On the other hand, if $A_0(t) = A_{00} + \tilde{A}_0(t)$, then the expressions (17) follow from the solution of (3) on $[0, +\infty)$ and Property (i). This proves Property (ii). Property (ii) follows from the fact that (17) hold for any initial condition. $\square$

Now, it is seen that the evolution operator of the delayed system is not a semigroup in the sense that the associative property for composing the solution on a time interval by using intermediate time instants to fix the initial conditions does not work, in general. See also, Remark 2. First, one expresses the solution trajectory of the delayed system $x(t)$ from the previous truncated two-dimensional real vector function $\hat{x}_{[\sigma, \sigma-h]}$ through the evolution operator $\hat{T}(t, \tau) : [-h, \infty) \times [-h, \infty) \rightarrow L(C_0^2(\mathbf{R}))$. For any $\sigma, t(\geq \sigma) \in \mathbf{R}_{0+}$, the following result follows directly from (8) and (17) and (18):

Proposition 1. The solution trajectory of the delayed system for any given bounded piecewise-continuous function of initial conditions $\varphi : [-h, 0] \rightarrow \mathbf{R}^2$, with $\varphi(0^+) = x(0^+) = x_0(0^+) = x_{00}$, is given by the following expressions:

$$\begin{aligned}
x(t) &= (\hat{T}(t, \sigma - h)) \hat{x}_{[\sigma, \sigma - h]} \\
&= T_0(t, \sigma) x(\sigma) + \int_0^{h-\sigma} T_0(t, \tau + \sigma) A_d(\tau + \sigma) \varphi(\tau + \sigma - h) d\tau + \int_{h-\sigma}^t T_0(t, \tau + \sigma) A_d(\tau + \sigma) x(\tau + \sigma - h) U(\tau + \sigma - h) d\tau \\
&= T_0(t, \sigma) (\hat{T}(\sigma, 0)) \hat{x}_{[0, -h]} + \int_0^{h-\sigma} T_0(t, \tau + \sigma) A_d(\tau + \sigma) \varphi(\tau + \sigma - h) d\tau + \left(\int_{h-\sigma}^t T_0(t, \tau + \sigma) A_d(\tau + \sigma) \hat{T}(\tau + \sigma - h, 0) \right) \hat{x}_{[0, -h]} d\tau \\
&= T_0(t, 0) x_{00} + \int_{-h}^0 T_0(t, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(t-h, 0)} T_0(t, \tau) A_d(\tau) T_0(\tau - h, 0) d\tau \right) x_{00} \\
&\quad + \int_0^{\max(t-h, 0)} \int_{-h}^0 T_0(t, \tau) A_d(\tau) T_0(\tau - h, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \\
&= e^{A_{00}t} \left(I_2 + \int_0^t e^{-A_{00}\tau} \tilde{A}_0(\tau) T_0(\tau, 0) d\tau \right) x_{00} \\
&\quad + \left(\int_0^{\max(t-h, 0)} T_0(t, \tau) A_d(\tau) \left[e^{A_{00}(\tau-h)} \left(T_0(\tau - h, 0) + \int_0^{\tau-h} e^{-A_{00}\sigma} \tilde{A}_0(\tau - h + \sigma) T_0(\tau - h + \sigma, 0) d\sigma \right) \right] d\tau \right) x_{00} \\
&\quad + \int_{-h}^0 \left[e^{A_{00}(t-\tau)} \left(T_0(\tau, 0) + \int_0^{t-\tau} e^{-A_{00}\sigma} \tilde{A}_0(\tau + \sigma) T_0(\tau + \sigma, 0) d\sigma \right) \right] A_d(h + \tau) \varphi(\tau) d\tau \\
&\quad + \int_0^{\max(t-h, 0)} \int_{-h}^0 \left[e^{A_{00}(t-\tau)} \left(T_0(\tau, 0) + \int_0^{t-\tau} e^{-A_{00}\sigma} \tilde{A}_0(\tau + \sigma) T_0(\tau + \sigma, 0) d\sigma \right) \right] \\
&\quad \times A_d(\tau) \left[e^{A_{00}(\tau-h-\theta)} \left(T_0(\tau - h, 0) + \int_0^{\tau-h} e^{-A_{00}\sigma} \tilde{A}_0(\tau - h + \sigma) T_0(\tau - h + \sigma, 0) d\sigma \right) \right] A_d(h + \sigma) \varphi(\sigma) d\sigma d\tau
\end{aligned} \tag{20}$$

3. Solutions at the Boundary Points of $[a, b]$ for $+\infty > b > a \geq 0$ and Related Results

This section discusses the loss of the Sturm–Liouville properties associated with a two-point boundary problem in a nominal undelayed system if delayed dynamics is incorporated.

For the undelayed system, the point solutions at the boundary points of the interval of interest are taken from (7) by picking up initial conditions at $t = 0$:

$$x_0(a) = T_0(a, 0) x_{00}$$

$$x_0(b) = T_0(b, 0) x_{00} = T_0(b, 0) T_0^{-1}(a, 0) x_0(a) = T_0(b, a) x_0(a)$$

then, by using the associative semigroup property of $T_0(\cdot, \cdot)$ proved in Property 1

$$\begin{aligned}
x(a) &= T_0(a, 0) x_{00} + \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(a-h, 0)} T_0(a, \tau) A_d(\tau) T_0(\tau - h, 0) d\tau \right) x_{00} \\
&\quad + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(\tau) T_0(\tau - h, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau
\end{aligned} \tag{21}$$

$$\begin{aligned}
&= x_0(a) + \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(a-h, 0)} T_0(a, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau \right) x_{00} \\
&\quad + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau
\end{aligned} \tag{22}$$

$$\begin{aligned}
x(b) &= T_0(b, 0) x_{00} + \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(b-h, 0)} T_0(b, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau \right) x_{00} \\
&\quad + \int_0^{\max(b-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau
\end{aligned} \tag{23}$$

which leads to the equivalent expression:

$$\begin{aligned}
x(b) &= x_0(b) + T_0(b, a) \left[\int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(b-h, 0)} T_0(a, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau \right) x_{00} \right. \\
&\quad \left. + \int_0^{\max(b-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \right] \\
&= x_0(b) + T_0(b, a) \left[\int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau \right. \\
&\quad \left. + \left(\int_0^{\max(a-h, 0)} T_0(a, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau + \left(\int_{\max(a-h, 0)}^b T_0(a, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau \right) \right) x_{00} \right. \\
&\quad \left. + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \right. \\
&\quad \left. + \int_{\max(a-h, 0)}^b \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \right]
\end{aligned} \tag{24}$$

Since $T_0(b, a)$ is non-singular, then $x(a) = x_0(a)$ if, and only if,

$$\begin{aligned}
&\int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \left(\int_0^{\max(a-h, 0)} T_0(a, \tau) A_d(h + \tau) T_0(\tau, 0) d\tau \right) x_{00} \\
&\quad + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau = 0
\end{aligned} \tag{25}$$

and $x(b) = x_0(b)$ from (24), if, and only if,

$$\begin{aligned} & \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) \varphi(\tau) d\tau \\ & + \left(\int_0^{\max(a-h, 0)} T_0(b, \tau) A_d(\tau) T(\tau - h, 0) d\tau + \int_{\max(a-h, 0)}^b T_0(b, \tau) A_d(h + \tau) T(\tau, 0) d\tau \right) x_{00} \\ & + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \\ & + \int_{\max(a-h, 0)}^{\max(b-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau = 0 \end{aligned} \quad (26)$$

In view of (25) and (26), it follows that $x(a) = x_0(a)$ and $x(b) = x_0(b)$ if, and only if, the constraint (25) holds together with the subsequent constraint which makes $x(b) = x_0(b)$ if $x(a) = x_0(a)$:

$$\begin{aligned} & \left(\int_{\max(a-h, 0)}^b T_0(a, \tau) A_d(\tau) T(\tau - h, 0) d\tau \right) x_{00} \\ & + \int_{\max(a-h, 0)}^{\max(b-h, 0)} \int_{-h}^0 T_0(a, \tau) A_d(\tau) T_0(\tau - h, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau = 0 \end{aligned} \quad (27)$$

Since $\varphi : [-h, 0]$ is piecewise continuous, then $x(0^+) = x_{00} = \varphi(0^+)$ if $\varphi(t)$ is discontinuous at $t = 0$ and $x(0) = x_{00} = \varphi(0^+)$ if $\varphi(t)$ is continuous at $t = 0$.

Define the two-dimensional real matrices of respective orders (4×2) and (4×1) :

$$\Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} = \begin{bmatrix} \int_0^{\max(a-h, 0)} T_0(a, \tau) A_d(h + \tau) T(\tau, 0) d\tau + \int_{\max(a-h, 0)}^b T_0(a, \tau) A_d(h + \tau) T(\tau, 0) d\tau \\ \int_{\max(a-h, 0)}^b T_0(a, \tau) A_d(h + \tau) T(\tau, 0) d\tau \end{bmatrix} \quad (28)$$

$$\begin{aligned} & \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\ & = - \begin{bmatrix} \int_{-h}^0 T_0(a, \tau) A_d(h + \tau) \varphi(\tau) d\tau + \int_{\max(a-h, 0)}^{\max(b-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau + \int_0^{\max(a-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \\ \int_{\max(a-h, 0)}^{\max(b-h, 0)} \int_{-h}^0 T_0(b, \tau) A_d(h + \tau) T_0(\tau, \theta) A_d(h + \theta) \varphi(\theta) d\theta d\tau \end{bmatrix} \end{aligned} \quad (29)$$

Note that Ω and Γ depend on the piecewise-continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbb{R}^2$. Then, the constraints (25) and (27) can be jointly rewritten in a compact form as $\Omega x_{00} = \Gamma$, which is a compatible algebraic system if, and only if, $\text{rank}[\Omega] = \text{rank}[\Omega, \Gamma]$ from the Rouché–Capelli theorem from Linear Algebra and, in this case, the whole set of solutions is:

$$x_{00} = \Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi \quad (30)$$

where $\xi \in \mathbb{R}^2$ is arbitrary. Thus, the given x_{00} has to be one of the solutions of (30) parameterized by ξ . In other words (30) has to be solvable in ξ for the given x_{00} which holds if, and only if,

$$\text{rank}[I_2 - \Omega^\dagger \Omega] = \text{rank}[I_2 - \Omega^\dagger \Omega, x_{00} - \Omega^\dagger \Gamma] \quad (31)$$

Since $x_0(a) = T_0(a, 0)x_{00}$ then the Sturm–Liouville system of the undelayed system has to fulfil the boundary condition:

$$x_0(a) = T_0(a, 0) \left(\Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi \right) \quad (32)$$

The following results have been proved as a direct conclusion of the above discussion:

Theorem 2. Assume that the differential system is a Sturm–Liouville one on $[a, b] \subset (0, \infty)$ subject to given boundary conditions $x(a)$ and $x(b)$ if $A_d = 0$ and that $x(0^+) = x_0(0^+) = \varphi(0^+) = x_{00}$. Assume also that (28) and (29) satisfy the rank condition $\text{rank}[\Omega] = \text{rank}[\Omega, \Gamma]$ for the given piecewise continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbb{R}^2$.

Then, the delayed system is also a Sturm–Liouville system with the same two-point boundary values as the undelayed one if, and only if, $x_0(a) = T_0(a, 0) (\Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi)$ for some $\xi \in \mathbf{R}^2$.

Theorem 3. Assume that the undelayed system is a Sturm–Liouville system on $[a, b] \subset (0, \infty)$ subject to given boundary conditions $x(a)$ and $x(b)$. Then, the following properties hold:

(i) If $\text{rank}[\Omega] < \text{rank}[\Omega, \Gamma]$ for the given piecewise continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbf{R}^2$ then the delayed system cannot be a Sturm–Liouville system with the same two-point boundary values as the undelayed one; that is, $x_0(a) = x(a)$ and $x_0(b) = x(b)$ are not fulfilled.

(ii) If $\text{rank}[\Omega] = \text{rank}[\Omega, \Gamma]$ for the given piecewise continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbf{R}^2$ but there is no $\xi \in \mathbf{R}^2$ such that $x_0(a) = T_0(a, 0) (\Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi)$ then the delayed system is not a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one; that is, $x(a) = x_0(a)$ and $x(b) = x_0(b)$ are not fulfilled.

(iii) If $\text{rank}[\Omega] = \text{rank}[\Omega, \Gamma]$ for the given piecewise continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbf{R}^2$ and there is some $\xi \in \mathbf{R}^2$ such that $x_0(a) = T_0(a, 0) (\Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi)$ then the delayed system is a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one; that is, $x(a) = x_0(a)$ and $x(b) = x_0(b)$ hold.

(iv) The delayed system is a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one, that is, $x(a) = x_0(a)$ and $x(b) = x_0(b)$, for the given piecewise continuous function of initial conditions $\varphi : [-h, 0) \rightarrow \mathbf{R}^2$, if, and only if, the following constraints hold together:

$$\text{rank}[\Omega] = \text{rank}[\Omega, \Gamma] \quad (33)$$

$$\text{rank} \left[T_0(a, 0) (I_2 - \Omega^\dagger \Omega) \right] = \text{rank} \left[T_0(a, 0) (I_2 - \Omega^\dagger \Omega), x_0(a) - T_0(a, 0) \Omega^\dagger \Gamma \right] \quad (34)$$

Outline of Proof. Properties (i)–(iii) follow directly from the previous discussion. Property (iv) is identical to Property (iii), since (34) guarantees the existence of some $\xi \in \mathbf{R}^2$ such that (32) holds for the given $x_0(a)$. $\square$

In the event that $a = 0$, one fixes $x_{00} = x_0(a^+) = x(a^+)$ and (25) holds trivially but, in order that $x(b) = x_0(b)$, (27) has to hold as an individual constraint for $x_{00} = x_0(a)$. Equation (27) can be written compactly as $\Omega_2 x_0(a) = \Gamma_2$, where Ω_2 and Γ_2 are, respectively, 2×2 and 2×1 real matrices defined in the second block matrices of (28) and (29). Then, provided that $\text{rank}[\Omega_2] = \text{rank}[\Omega_2, \Gamma_2]$, there should exist some $\varsigma \in \mathbf{R}^2$ such that $x_0(a) = \Omega_2^\dagger \Gamma_2 + (I_2 - \Omega_2^\dagger \Omega_2) \varsigma$, which would also imply that $x(b) = x_0(b)$. As a result, we conclude the following relaxed counterpart of Theorem 3:

Theorem 4. Assume that the undelayed system is a Sturm–Liouville system on $[a, b] \subset [0, \infty)$ with $a = 0$ subject to given boundary conditions $x(0)$ and $x(b)$. Then, the following properties hold:

(i) If $\text{rank}[\Omega_2] < \text{rank}[\Omega_2, \Gamma_2]$ then the delayed system cannot be a Sturm–Liouville system with the same two-point boundary values as the undelayed one; that is, $x_0(a) = x(a)$ and $x_0(b) = x(b)$ are not fulfilled.

(ii) If $\text{rank}[\Omega_2] = \text{rank}[\Omega_2, \Gamma_2]$ but there is no $\varsigma \in \mathbf{R}^2$ such that $x_0(a) = T_0(a, 0) (\Omega^\dagger \Gamma + (I_2 - \Omega^\dagger \Omega) \xi)$ then the delayed system is not a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one; that is, $x(a) = x_0(a)$ and $x(b) = x_0(b)$ are not fulfilled.

(iii) If $\text{rank}[\Omega_2] = \text{rank}[\Omega_2, \Gamma_2]$ and there is some $\varsigma \in \mathbf{R}^2$ such that $x_0(a) = \Omega_2^\dagger \Gamma_2 + (I_2 - \Omega_2^\dagger \Omega_2) \varsigma$ then the delayed system is a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one; that is, $x(a) = x_0(a) = x_{00}$ and $x(b) = x_0(b)$ hold.

(iv) The delayed system is a Sturm–Liouville one with the same two-point boundary value problems as the undelayed one; that is, $x(a) = x_0(a) = x_{00}$ and $x(b) = x_0(b)$ if, and only if, the following constraints hold together:

$$\text{rank}[\Omega_2] = \text{rank}[\Omega_2, \Gamma_2] \quad (35)$$

$$\text{rank}\left[I_2 - \Omega_2^\dagger \Omega_2\right] = \text{rank}\left[I_2 - \Omega_2^\dagger \Omega_2, x_0(a) - \Omega_2^\dagger \Gamma_2\right] \quad (36)$$

4. Boundedness and Asymptotic Properties of the Evolution Operators

One understands from (18) that the evolution operator of the free of delayed dynamics system, that is, the undelayed system with $A_d(t) \equiv 0$, is

$$T_0(t, 0) = e^{A_{00}t} \left(I_2 + \int_0^t e^{-A_{00}\tau} \tilde{A}_0(\tau) T_0(\tau, 0) d\tau \right); t \in \mathbf{R}_{0+} \quad (37)$$

$$T_0(t, \tau) = e^{A_{00}(t-\tau)} \left(I_2 + \int_0^{t-\tau} e^{-A_{00}\sigma} \tilde{A}_0(\tau + \sigma) T_0(\tau + \sigma, 0) d\sigma \right); \tau, t(\geq \tau) \in \mathbf{R}_{0+} \quad (38)$$

Note that the system referred to as the undelayed system is the one free of delayed dynamics; that is, $A_d = 0$ in (5). Also, one understands from (5) that if $h = 0$, the so-called delay-free system of matrix of dynamics being $A_0(t) = A_0(t) + A_d(t)$ which has an evolution operator similar to (18) by replacing $\tilde{A}_0(t) \rightarrow A_0(t) + A_d(t)$, resulting as

$$T_{00}(t, 0) = e^{A_{00}t} \left(I_2 + \int_0^t e^{-A_{00}\tau} (\tilde{A}_0(\tau) + A_d(\tau)) T_{00}(\tau, 0) d\tau \right); t \in \mathbf{R}_{0+} \quad (39)$$

$$T_{00}(t, \tau) = e^{A_{00}(t-\tau)} \left(I_2 + \int_0^{t-\tau} e^{-A_{00}\sigma} (\tilde{A}_0(\tau + \sigma) + A_d(\tau + \sigma)) T_{00}(\tau + \sigma, 0) d\sigma \right); \tau, t(\geq \tau) \in \mathbf{R}_{0+} \quad (40)$$

And (12), combined with (37) and (38), yields that the evolution operator of the delayed system can be equivalently expressed as

$$\begin{aligned} T(t, 0) &= e^{A_{00}t} \left(I_2 + \int_0^t e^{-A_{00}\tau} \tilde{A}_0(\tau) T_0(\tau, 0) d\tau \right) + \\ &+ \int_0^t \left(e^{A_{00}(t-\tau)} \left(I_2 + \int_0^{t-\tau} e^{-A_{00}\sigma} \tilde{A}_0(\tau + \sigma) T_0(\tau + \sigma, 0) d\sigma \right) \right) A_d(\tau) T(\tau - h, 0) d\tau; t \in \mathbf{R}_{0+} \end{aligned} \quad (41)$$

Now, some upper-bounding growing laws of the evolution operators with time are calculated in order to obtain approximate Sturm–Liouville boundary conditions of the delayed system, provided that the undelayed one is a Sturm–Liouville system.

It turns out that there exist non-unique real constants $K \geq 1$ (which are norm-dependent) and ρ associated with the dominant eigenvalue of A_{00} such that $\|e^{A_{00}t}\| \leq Ke^{\rho t}$. In particular, if ρ_{00} is the stability abscissa of A_{00} , that is, the largest of the real parts of its eigenvalues, then $\rho \geq \rho_{00}$ if the dominant eigenvalues are single and $\rho > \rho_{00}$ otherwise [24,25]. It is well-known that if A_{00} is a stability matrix (i.e., all its eigenvalues have a negative real part) then $\rho < 0$. If it is unstable, i.e., at least one of its eigenvalues has a positive real part then $\rho > 0$. If it is critically stable, i.e., at least one of its eigenvalues has null real part and no eigenvalue is unstable, then $\rho = 0$. See, for instance, [26–32]. Then, from (17), by assuming $\sup_{0 \leq \tau \leq t} \|\tilde{A}_0(\tau)\| \leq \tilde{a}_0$, one has the following:

(a) if $\rho < 0$, then

$$\|T_0(t, 0)\| \leq K \left(1 + |\rho|^{-1} \tilde{a}_0 \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \right); t \in \mathbf{R}_{0+} \quad (42)$$

thus,

$$\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K \left(1 + |\rho|^{-1} \tilde{a}_0 \sup_{0 \leq \sigma \leq \tau} \|T_0(\sigma, 0)\| \right) \leq K \left(1 + |\rho|^{-1} \tilde{a}_0 \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \right); t, \tau(\leq t) \in \mathbf{R}_{0+} \quad (43)$$

and, if $\tilde{a}_0 < |\rho|/K$, then

$$\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq \frac{K|\rho|}{|\rho| - K\tilde{a}_0} \quad (44)$$

(b) For any admissible real ρ , one has from (17) that

$$\|T_0(t, 0)\| \leq Ke^{\rho t} \left(1 + \tilde{a}_0 \int_0^t e^{-\rho \tau} \|T_0(\tau, 0)\| d\tau \right); t \in \mathbf{R}_{0+} \quad (45)$$

Then,

$$v(t) = e^{-\rho t} \|T_0(t, 0)\| \leq K + K\tilde{a}_0 \int_0^t v(\tau) d\tau; t \in \mathbf{R}_{0+} \quad (46)$$

Now, Gronwall's inequality, [2], applied to (46) yields

$$v(t) \leq K + K^2\tilde{a}_0 \int_0^t e^{\int_\tau^t K\tilde{a}_0 dr} d\tau = K + K^2\tilde{a}_0 \int_0^t e^{K\tilde{a}_0(t-\tau)} d\tau = Ke^{K\tilde{a}_0 t}; t \in \mathbf{R}_{0+} \quad (47)$$

Thus, one gets from (46) and (47),

$$\|T_0(t, 0)\| \leq Ke^{(\rho + K\tilde{a}_0)t}; t \in \mathbf{R}_{0+} \quad (48)$$

so that:

(b1) If $\rho < 0$ and $\tilde{a}_0 < |\rho|/K$ then $\lim_{t \rightarrow +\infty} \|T_0(t, 0)\| = 0$ with exponential convergence and also the bounding condition $\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K|\rho|/(|\rho| - K\tilde{a}_0) < +\infty$ for all $t \in \mathbf{R}_{0+}$ according to (44) can be refined to $\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K$ for $t \in \mathbf{R}_{0+}$, according to (48).

(b2) If $\rho = \tilde{a}_0 = 0$ then $\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K$ for all $t \in \mathbf{R}_{0+}$.

Example 1. Assume that the time-varying component $\tilde{A}_0(t) = A_0(t) - A_{00}$ of $A_0(t)$ is zero, except on a subset S of $[0, \infty)$ of nonzero finite Lebesgue measure L_S and that $\rho = 0$, that is, the constant part A_{00} of $A_0(t)$ is critically stable. Then, (47) becomes

$$\|T_0(\tau, 0)\| = v(t) \leq K + K^2\tilde{a}_0 \int_0^t e^{\int_\tau^t K\tilde{a}_0 dr} d\tau \leq K \left(1 + K\tilde{a}_0 e^{K\tilde{a}_0 L_S} t \right); t \in \mathbf{R}_{0+} \quad (49)$$

since

$$e^{\int_\tau^t K\|\tilde{A}_0(r)\| dr} \leq e^{K \int_S \|\tilde{A}_0(r)\| dr} \leq e^{K\tilde{a}_0 L_S} < +\infty \quad (50)$$

for all $0 \leq \tau \leq t$, $t \leq +\infty$.

Therefore, Gronwall's inequality does not guarantee the boundedness of the undelayed evolution operator if $A_0(t)$ is constant, except in a set of zero measure, while the constant additive component is critically stable.

Then, one understands from (12) and (48), if $\tilde{a}_0 < |\rho|/K$ and $\rho < 0$, that

$$\|T(t, 0)\| \leq \|T_0(t, 0)\| + \int_0^t \|T_0(t, \tau)\| \|A_d(\tau)\| \|T(\tau - h, 0)\| d\tau; t \in \mathbf{R}_{0+} \quad (51)$$

If $a_d = \sup_{0 \leq \tau \leq t} \|A_d(\tau)\|$ then

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \|T(\tau, 0)\| &\leq \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| + \sup_{0 \leq \tau \leq t-h} \|T(\tau, 0)\| \int_0^t \|T_0(t, \tau)\| \|A_d(\tau)\| d\tau \\ &\leq \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| + a_d \sup_{0 \leq \tau \leq t} \|T(\tau, 0)\| \int_0^t \|T_0(t, \tau)\| d\tau; t \in \mathbf{R}_{0+} \end{aligned} \quad (52)$$

Now, if $\tilde{a}_0 < |\rho|/K$, $\rho < 0$, $a_d < 1/(\int_0^\infty \|T_0(t, \tau)\| d\tau)$, so that $\|T_0(t, 0)\| \leq Ke^{-(|\rho| - K\tilde{a}_0)t}$ and $\int_0^t \|T_0(t, \tau)\| d\tau \leq K/(|\rho| - K\tilde{a}_0)$ from (48) for all $t \in \mathbf{R}_{0+}$, so that the above constraint $a_d < 1/(\int_0^\infty \|T_0(t, \tau)\| d\tau)$ is guaranteed if $a_d < (|\rho| - K\tilde{a}_0)/K$, equivalently if $(\tilde{a}_0 + a_d) < |\rho|/K$, and one understands from (52) that

$$\left(1 - \frac{Ka_d}{|\rho| - K\tilde{a}_0}\right) \sup_{0 \leq \tau \leq t} \|T(\tau, 0)\| \leq \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\|; t \in \mathbf{R}_{0+} \quad (53)$$

Then, it follows from (52) that

$$\sup_{0 \leq \tau \leq t} \|T(\tau, 0)\| \leq \frac{|\rho| - K\tilde{a}_0}{|\rho| - K(\tilde{a}_0 + a_d)} \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| = \frac{K(|\rho| - K\tilde{a}_0)}{|\rho| - K(\tilde{a}_0 + a_d)}; t \in \mathbf{R}_{0+} \quad (54)$$

Then, the subsequent result holds from the above discussion and partial results (51)–(53):

Theorem 5. Assume that $\rho < 0$. Then, the following properties hold:

(i) if $\tilde{a}_0 < |\rho|/K$ then

$$\sup_{\sigma \leq \tau \leq t} \|T_0(t, \sigma)\| \leq \frac{K|\rho|}{|\rho| - K\tilde{a}_0}; \sigma, t(\geq \sigma) \in \mathbf{R}_{0+} \quad (55)$$

(ii) if $(\tilde{a}_0 + a_d) < |\rho|/K$ then $\sup_{\sigma \leq \tau \leq t} \|T_0(\tau, \sigma)\| \leq K$; $\sigma, t(\geq \sigma) \in \mathbf{R}_{0+}$, and

$$\sup_{\sigma \leq \tau \leq t} \|T(\tau, \sigma)\| \leq \frac{|\rho| - K\tilde{a}_0}{|\rho| - K(\tilde{a}_0 + a_d)} \sup_{\sigma \leq \tau \leq t} \|T_0(\tau, \sigma)\| = \frac{K(|\rho| - K\tilde{a}_0)}{|\rho| - K(\tilde{a}_0 + a_d)}; \sigma, t(\geq \sigma) \in \mathbf{R}_{0+} \quad (56)$$

$$\lim_{\sigma, t \rightarrow \infty} \sup_{t \leq \tau \leq t+\sigma} \|T(t+\sigma, t)\| \leq \frac{|\rho| - K\tilde{a}_0}{|\rho| - K(\tilde{a}_0 + a_d)} \lim_{\sigma, t \rightarrow \infty} \left(\sup_{\sigma \leq \tau \leq t} \|T_0(t+\sigma, t)\| \right) = 0 \quad (57)$$

It is of interest in the context of Sturm–Liouville systems to investigate the uniform boundedness of non-asymptotically vanishing evolution operators of the undelayed system; that is, when A_{00} is critically stable so that $\rho = 0$. This covers a wide class of situations when defining two-boundary value problems on finite intervals, including periodicity of such boundary values. A typical and elementary example is concerned with second-order linear time-invariant differential equations whose matrix of dynamics is driven by a couple of imaginary complex-conjugate eigenvalues. Sufficiency-type conditions of boundedness of the evolution operators when $\rho = 0$ are now given:

Proposition 2. Assume that $\rho = 0$ and $\left\| \begin{pmatrix} \left(\int_0^\infty e^{A_{00}(t-\tau)} \right)_{12} A_{021}(\tau) d\tau \\ \left(\int_0^\infty e^{A_{00}(t-\tau)} \right)_{22} A_{021}(\tau) d\tau \end{pmatrix} \right\| < 1$. Then, $\|T_0(t, 0)\| < +\infty$; $t \in \mathbf{R}_{0+}$.

A stronger sufficient condition which guarantees that $\|T_0(t, 0)\| < +\infty$; $t \in \mathbf{R}_{0+}$ is $\tilde{a}_{021}^I = \int_0^\infty |A_{021}(\tau)| d\tau < K^{-1}$.

Proof. Note from (37) that, if $\rho = 0$, then

$$\begin{aligned} \|T_0(t, 0)\| &\leq K + \left\| \int_0^t e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) T_0(\tau, 0) d\tau \right\| \\ &\leq K + \left(\left\| \int_0^t e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right) \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\|; t \in \mathbf{R}_{0+} \end{aligned} \quad (58)$$

which implies that

$$\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K + \left(\left\| \int_0^t e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right) \sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\|; \quad t \in \mathbf{R}_{0+} \quad (59)$$

so that, if $1 > \left\| \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\|$, which is equivalent to $1 > \left\| \left(\int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \right)_{12} \left(\int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \right)_{22} \right\|$ which is guaranteed under the sufficiency-type condition $\tilde{a}_{021}^I = \int_0^\infty \tilde{A}_{021}(\tau) d\tau < K^{-1}$ (since $\|e^{A_{00}(t-\tau)}\| \leq K$; $\tau, t(\geq \tau) \in \mathbf{R}_{0+}$) because of the structure of (4), which makes the entry $A_{021}(\cdot)$ of $A_0(t)$ being the only one which is eventually non-null for any established decomposition $A_0(t) = A_{00} + \tilde{A}_0(t)$. Then, one has

$$\sup_{0 \leq \tau \leq t} \|T_0(\tau, 0)\| \leq K \left(1 - \left\| \int_0^t e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} < +\infty; \quad t \in \mathbf{R}_{0+} \quad (60)$$

$$\sup_{0 \leq t < \infty} \|T_0(t, 0)\| \leq K \left(1 - \left\| \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} < +\infty \quad (61)$$

If, furthermore, $\tilde{a}_{021}^I < K$ then $\sup_{0 \leq t < \infty} \|T_0(t, 0)\| \leq K / (1 - K \tilde{a}_{021}^I)^{-1} < +\infty$.

Which proves the result. $\square$

Proposition 3. Assume that $\rho = 0$ and that the following conditions hold:

$$(1) \left\| \left(\int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \right)_{12} \left(\int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \right)_{22} \right\| < 1 \quad (62)$$

$$(2) A_d : [-h, +\infty) \rightarrow \mathbf{R}^{2 \times 2} \text{ is in } L^1([0, +\infty); \mathbf{R}^{2 \times 2}) \text{ so that } a_d^I = \int_0^\infty \|A_d(\tau)\| d\tau < +\infty \text{ fulfils} \\ a_d^I < 1 / \left(\sup_{0 \leq t \leq \infty} \|T_0(t, 0)\| \right) < +\infty \quad (63)$$

Then, the following properties hold:

(i) The constraint (62) is guaranteed if

$$a_d^I < K^{-1} \left(1 - \left\| \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right) < +\infty \quad (64)$$

$$(ii) \sup_{0 \leq t \leq \infty} \|T(t, 0)\| \leq \frac{\sup_{0 \leq t \leq \infty} \|T_0(t, 0)\|}{1 - a_d^I \sup_{0 \leq t \leq \infty} \|T_0(t, 0)\|} < +\infty \quad (65)$$

(iii) If (64) holds, then

$$\sup_{0 \leq t \leq \infty} \|T(t, 0)\| \leq \frac{K \left(1 - \left\| \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)}{1 - a_d^I K \left(1 - \left\| \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)} < +\infty \quad (66)$$

Proof. From Proposition 2, the condition 1, Equation (62), implies that $\|T_0(t, 0)\| < +\infty$. This proves Property (i). From (12) and the condition 2, it follows that (65) holds, which proves Property (ii). If the condition 2 holds under the strongest one (64) then (65) implies (66). Also, (64) and (65) imply (66) and then Property (iii). $\square$

5. Worst-Case Deviation of the Delayed System from a Sturm–Liouville Behavior Compared to the Undelayed Sturm–Liouville Nominal One

Theorem 2 and Theorem 3 establish conditions for the Sturm Liouville two-boundary value conditions to be kept in the delayed system, from the undelayed one. The respective deviations from zero of the left-hand sides of (25) and (26), in terms of error norms, give estimations of the errors in the two-point boundary conditions of the delayed system at $t = a$ and $t = b$, with respect to the undelayed Sturm–Liouville one. Note that if the delayed system loses the Sturm–Liouville property, then the zero right-hand-side values in (25) and (26) are replaced by $(x(a) - x_0(a))$ and $(x(b) - x_0(b))$, respectively, since the error between the current and nominal systems at the boundary points is no longer zero. That concern is now addressed by using the results about the boundedness and the asymptotic behaviors of the evolution operators which have been characterized in the above section, together with the results of Theorem 2, Theorem 3, and Equations (25) and (26) of Section 2.

Now, suppose that

$$\bar{\varphi} = \sup_{-h \leq \tau \leq 0} \|\varphi(\tau)\|; a_{da} = \sup_{0 \leq \tau \leq a} \|A_d(\tau)\|; a_{db} = \sup_{0 \leq \tau \leq b} \|A_d(\tau)\| = \max \left(a_{da}, \sup_{a \leq \tau \leq b} \|A_d(\tau)\| \right) \quad (67)$$

Taking norms in the right-hand sides of (25) and (26), one obtains:

$$\begin{cases} x(a) = x_0(a) \text{ if } a = 0 \\ \|x(a) - x_0(a)\| \leq a a_{da} \bar{\varphi} \sup_{0 \leq \tau \leq a} \|T_0(a, \tau)\| \left(1 + h a_{da} \sup_{0 \leq \tau \leq a} \|T_0(a, \tau)\| \right) \text{ if } a > 0 \end{cases} \quad (68)$$

and

$$\|x(b) - x_0(b)\| \leq b a_{db} \bar{\varphi} \sup_{0 \leq \tau \leq b} \|T_0(b, \tau)\| \left(1 + h a_{db} \sup_{0 \leq \tau \leq b} \|T_0(b, \tau)\| \right) \quad (69)$$

Case a: $\rho = 0$ and $\left\| \begin{pmatrix} \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \\ \int_0^\infty e^{A_{00}(t-\tau)} \tilde{A}_{021}(\tau) d\tau \end{pmatrix}_{22} \right\| < 1$, which is guaranteed if $K^{-1} > \tilde{a}_{021}^I = \int_0^\infty |\tilde{A}_{021}(\tau)| d\tau$ (Proposition 2). Then, one has, from (60), (68) and (69), the following:

$$\begin{cases} x(a) = x_0(a) \text{ if } a = 0 \\ \|x(a) - x_0(a)\| \leq a a_{da} \bar{\varphi} K \left(1 - \left\| \int_0^a e^{A_{00}(a-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} \left(1 + h a_{da} K \left(1 - \left\| \int_0^a e^{A_{00}(a-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} \right) \text{ if } a > 0 \end{cases} \quad (70)$$

$$\|x(b) - x_0(b)\| \leq b a_{db} \bar{\varphi} K \left(1 - \left\| \int_0^b e^{A_{00}(b-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} \left(1 + h a_{db} K \left(1 - \left\| \int_0^b e^{A_{00}(b-\tau)} \tilde{A}_0(\tau) d\tau \right\| \right)^{-1} \right) \quad (71)$$

If, furthermore, the stronger constraint $\tilde{a}_{021}^I = \int_0^\infty |\tilde{A}_{021}(\tau)| d\tau < K^{-1}$ holds, then two stronger sufficiency-type conditions than (70) and (71), which in turn guarantee them, are

$$\begin{cases} x(a) = x_0(a) \text{ if } a = 0 \\ \|x(a) - x_0(a)\| \leq a a_{da} \bar{\varphi} K \left(1 - K \tilde{a}_{021}^I \right)^{-1} \left(1 + h a_{da} K \left(1 - K \tilde{a}_{021}^I \right)^{-1} \right) \text{ if } a > 0 \end{cases} \quad (72)$$

and

$$\|x(b) - x_0(b)\| \leq b a_{db} \bar{\varphi} K \left(1 - K \tilde{a}_{021}^I \right)^{-1} \left(1 + h a_{db} K \left(1 - K \tilde{a}_{021}^I \right)^{-1} \right) \quad (73)$$

Case b: $\rho < 0$ and $\tilde{a}_0 < |\rho|/K$

One uses (44), (48), (55) and the left-hand sides of (25) and (26). Then,

$$\begin{cases} x(a) = x_0(a) \text{ if } a = 0 \\ \|x(a) - x_0(a)\| \leq \frac{K|\rho| a_{da} \bar{\varphi} a}{|\rho| - K \tilde{a}_0} \left(1 + \frac{K|\rho|}{|\rho| - K \tilde{a}_0} (1 + a_{da} a) \right) \text{ if } a > 0 \end{cases} \quad (74)$$

and

$$\|x(b) - x_0(b)\| \leq \frac{K|\rho|a_{db}\bar{\varphi}b}{|\rho| - K\tilde{a}_0} \left(1 + \frac{K|\rho|}{|\rho| - K\tilde{a}_0} (1 + a_{db}b)\right) \quad (75)$$

Case c: $\rho \in \mathbf{R}$

One uses (48) and the left-hand sides of (25) and (26). Then,

$$\begin{cases} x(a) = x_0(a) \text{ if } a = 0 \\ \|x(a) - x_0(a)\| \leq Ke^{(\rho+K\tilde{a}_0)a} a_{da}\bar{\varphi}a \left(1 + Ke^{(\rho+K\tilde{a}_0)a} (1 + a_{da}a)\right) \text{ if } a > 0 \end{cases} \quad (76)$$

and

$$\|x(b) - x_0(b)\| \leq Ke^{(\rho+K\tilde{a}_0)b} a_{db}\bar{\varphi}b \left(1 + Ke^{(\rho+K\tilde{a}_0)b} (1 + a_{db}b)\right) \quad (77)$$

Note that Case c also includes the situation of possible instability of A_{00} ; that is, the case of positive stability abscissa. As a result, (76) and (77) apply for any real ρ but they are not very tight for $\rho \leq 0$ compared to (72)–(75).

Note that there exist $\tilde{\lambda}_0, \tilde{\lambda}_0^I, \lambda_{da}, \lambda_{db}, \lambda_{\bar{\varphi}} \in \mathbf{R}_{0+}$, depending on the given ε , such that

$$\tilde{a}_0 \leq \tilde{\lambda}_0\varepsilon; \tilde{a}_{021}^I \leq \lambda_{021}^I\varepsilon; a_{da} \leq \lambda_{da}\varepsilon; a_{db} \leq \lambda_{db}\varepsilon; \bar{\varphi} \leq \lambda_{\bar{\varphi}}\varepsilon \quad (78)$$

The worst-case errors at the two-point boundary points are kept below any prescribed arbitrary threshold $\varepsilon > 0$ under the following conditions.

Then,

(1) If $\rho = 0$ and $a > 0$ (Case a of critical stability of A_{00}), then one has from (70) and (71), that

$$a\lambda_{da}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_0a)^{-1} \left(1 + h\lambda_{da}K(1 - K\tilde{\lambda}_0a)^{-1}\right) \leq 1 \Rightarrow \|x(a) - x_0(a)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0 < 1/Ka \quad (79)$$

$$b\lambda_{db}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_0b)^{-1} \left(1 + h\lambda_{db}K(1 - K\tilde{\lambda}_0b)^{-1}\right) \leq 1 \Rightarrow \|x(b) - x_0(b)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0 < 1/Kb \quad (80)$$

and, from (72) and (73),

$$a\lambda_{da}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_0^I)^{-1} \left(1 + h\lambda_{da}K(1 - K\tilde{\lambda}_0^I)^{-1}\right) \leq 1 \Rightarrow \|x(a) - x_0(a)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0^I < 1/K \quad (81)$$

$$b\lambda_{db}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_0^I)^{-1} \left(1 + h\lambda_{db}K(1 - K\tilde{\lambda}_0^I)^{-1}\right) \leq 1 \Rightarrow \|x(b) - x_0(b)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0^I < 1/K \quad (82)$$

The constraint $\tilde{a}_{021}^I = \int_0^\infty |\tilde{A}_{021}(\tau)| d\tau < K^{-1} \leq \lambda_{021}^I\varepsilon$ can be replaced by the weakest ones:

$$\tilde{a}_{0a21}^I = \int_0^a |\tilde{A}_{021}(\tau)| d\tau < K^{-1} \leq \lambda_{0a}^I\varepsilon; \tilde{a}_{0b21}^I = \int_0^b |\tilde{A}_{021}(\tau)| d\tau < K^{-1} \leq \lambda_{0b}^I\varepsilon \quad (83)$$

Thus, (81) and (82) might be replaced with

$$a\lambda_{da}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_{0a}^I)^{-1} \left(1 + h\lambda_{da}K(1 - K\tilde{\lambda}_{0a}^I)^{-1}\right) \leq 1 \Rightarrow \|x(a) - x_0(a)\| \leq \varepsilon \text{ if } \tilde{\lambda}_{0a}^I < 1/K \quad (84)$$

$$b\lambda_{db}\lambda_{\bar{\varphi}}K(1 - K\tilde{\lambda}_{0b}^I)^{-1} \left(1 + h\lambda_{db}K(1 - K\tilde{\lambda}_{0b}^I)^{-1}\right) \leq 1 \Rightarrow \|x(b) - x_0(b)\| \leq \varepsilon \text{ if } \tilde{\lambda}_{0b}^I < 1/K \quad (85)$$

(2) If $a > 0$, $\rho < 0$ and $\tilde{a}_0 < |\rho|/K$ (Case b of stability of A_{00}) then one has from (74) and (75) that

$$\frac{K|\rho|\lambda_{da}\lambda_{\bar{\varphi}}}{|\rho| - K\tilde{\lambda}_0} \left(1 + \frac{K|\rho|}{|\rho| - K\tilde{\lambda}_0} (1 + \lambda_{da}a)\right) \leq 1 \Rightarrow \|x(a) - x_0(a)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0 < |\rho|/K \quad (86)$$

$$\frac{K|\rho|\lambda_{db}\lambda_{\bar{\varphi}}}{|\rho| - K\tilde{\lambda}_0} \left(1 + \frac{K|\rho|}{|\rho| - K\tilde{\lambda}_0} (1 + \lambda_{db}a) \right) \leq 1 \Rightarrow \|x(b) - x_0(b)\| \leq \varepsilon \text{ if } \tilde{\lambda}_0 < |\rho|/K \quad (87)$$

(3) If $\rho \in \mathbf{R}$ (Case c including the potential stability, critical stability or instability of A_{00}), then one has from (76) and (77) that

$$Ke^{(\rho+K\tilde{\lambda}_0)a}\lambda_{da}\lambda_{\bar{\varphi}}a \left(1 + Ke^{(\rho+K\tilde{\lambda}_0)a}(1 + \lambda_{da}a) \right) \Rightarrow \|x(a) - x_0(a)\| \leq \varepsilon \quad (88)$$

$$Ke^{(\rho+K\tilde{\lambda}_0)b}\lambda_{db}\lambda_{\bar{\varphi}}b \left(1 + Ke^{(\rho+K\tilde{\lambda}_0)b}(1 + \lambda_{db}b) \right) \Rightarrow \|x(b) - x_0(b)\| \leq \varepsilon \quad (89)$$

6. Concluding Remarks

This paper has discussed the relations between two linear time-varying second-order differential equations with two-point boundary values. One of them is undelayed and it is a Sturm–Liouville system on a finite time interval, while the other one is a delayed perturbed current version of the first one. It is proved that the delayed differential system loses, in general, the Sturm–Liouville property, due to the lack of tracking of the prescribed two-point boundary conditions fixed for the undelayed nominal Sturm–Liouville differential system. Both the undelayed nominal equation and the delay-perturbed differential one are described equivalently by second-order differential systems, each of them formed by two first-order differential equations. The reason for this equivalent description is to facilitate a joint calculation of the respective solutions and their first-order time derivatives through time on the interval of interest via the involvement and “ad hoc” use of their respective evolution operators. In this context, the formal results are developed for the equivalent second-order differential systems of first-order equations, based on their respective evolution operators. This allows one to easily visualize how the nominal Sturm–Liouville system is able to track the given admissible two-point boundary conditions, while the delayed perturbed one loses that property, in general.

The admissible two-point boundary conditions given for the nominal differential equation translate in a natural way into four two-point boundary values at the initial and final values on the time interval of interest of the respective solutions and its first time-derivatives with respect to time. The above-mentioned second differential system is seen as a perturbation of the nominal one, since it is subject to a constant delay in its dynamics. This is the reason for the loss of the Sturm–Liouville property, since the four single two-point boundary values cannot be jointly kept, in general, identical for both systems. The undelayed Sturm–Liouville differential system plays the role of a nominal differential system, while the delayed one is viewed as a perturbation of the above one. Finally, the errors of the two-point boundary values between those of the nominal differential system and those of the perturbed current one are evaluated, and mutually compared, in terms of worst-case norm errors for small deviations of the perturbed system related to the nominal one. For that purpose, one takes advantage of the proposed equivalent descriptions of both differential equations via second-order differential systems, via their respective evolution operators. It is foreseen to extend these results in future works to systems with the presence of multiple internal delays in their dynamics and to forced systems with delays in their forcing terms.

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