

Article

Fuzzy Bipolar Hypersoft Sets: A Novel Approach for Decision-Making Applications

Baravan A. Asaad ^{1,2,*} , Sagvan Y. Musa ³  and Zanyar A. Ameen ⁴ 

¹ Department of Computer Science, College of Science, Cihan University-Duhok, Duhok 42001, Iraq

² Department of Mathematics, College of Science, University of Zakho, Zakho 42002, Iraq

³ Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Iraq; sagvan.musa@uoz.edu.krd

⁴ Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq; zanyar@uod.ac

* Correspondence: baravan.asaad@uoz.edu.krd

Abstract: This article presents a pioneering mathematical model, fuzzy bipolar hypersoft (FBHS) sets, which combines the bipolarity of parameters with the fuzziness of data. Motivated by the need for a comprehensive framework capable of addressing uncertainty and variability in complex phenomena, our approach introduces a novel method for representing both the presence and absence of parameters through FBHS sets. By employing two mappings to estimate positive and negative fuzziness levels, we bridge the gap between bipolarity, fuzziness, and parameterization, allowing for more realistic simulations of multifaceted scenarios. Compared to existing models like bipolar fuzzy hypersoft (BFHS) sets, FBHS sets offer a more intuitive and user-friendly approach to modeling phenomena involving bipolarity, fuzziness, and parameterization. This advantage is underscored by a detailed comparison and a practical example illustrating FBHS sets' superiority in modeling such phenomena. Additionally, this paper provides an in-depth exploration of fundamental FBHS set operations, highlighting their robustness and applicability in various contexts. Finally, we demonstrate the practical utility of FBHS sets in problem-solving and introduce an algorithm for optimal object selection based on available information sets, further emphasizing the advantages of our proposed framework.

Keywords: fuzzy bipolar hypersoft sets; hypersoft sets; fuzzy sets; decision-making; algorithms



Citation: Asaad, B.A.; Musa, S.Y.; Ameen, Z.A. Fuzzy Bipolar Hypersoft Sets: A Novel Approach for Decision-Making Applications. *Math. Comput. Appl.* **2024**, *29*, 50. <https://doi.org/10.3390/mca29040050>

Received: 18 May 2024

Revised: 12 June 2024

Accepted: 1 July 2024

Published: 2 July 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Uncertainty is a prevalent issue in various fields, including social sciences, economics, engineering, medical sciences, and others, which cannot be tackled using traditional mathematical approaches. The traditional mathematical model relies on the assumption that decision-makers have complete knowledge, which is often not the case in reality, making the model highly complex and making it difficult to obtain an accurate solution. To overcome this challenge, scholars have been developing new methodologies and mathematical theories that can address data uncertainty, such as fuzzy set theory [1] and rough set theory [2]. Despite partial success in problem-solving, these theories have only partially bridged the gap between conventional mathematical concepts and real-world data.

Fuzzy set theory, developed by Zadeh, characterizes fuzzy data mathematically. However, finding the membership function in fuzzy sets can sometimes be challenging. Therefore, Molodtsov [3] introduced the soft set (S-set) theory as a novel approach to tackling uncertainty modeling, thereby eliminating the difficulty in identifying the membership function. This theory relies on an approximate description of an object as its starting point and the selection of appropriate parameters, making it a practical and user-friendly technique in real-world applications. Maji et al. [4] utilized S-set theory in a decision-making problem, while Çağman et al. [5] introduced the idea of fuzzy soft set theory. Maji et al. [6]

established a range of operations on S-sets, and Ali et al. [7] proposed various novel operations. Building upon some of Maji et al.'s earlier work, researchers have developed new types of operators (see, for example, [8–11]).

In 2018, Smarandache [12] developed a more flexible and adaptable version of the S-set called the hypersoft (HS) set. To achieve this, the function F was substituted with a multi-argument function defined over the Cartesian product of n distinct parameter sets. The HS set is more appropriate for decision-making problems than the S-set and has been explored for possible expansions in different mathematical fields. Musa et al. [13] presented the notion of N -HS sets, which surpass HS sets in terms of comprehensiveness. These sets offer a broader framework by incorporating parametric descriptions of objects using a finite set of ordered grades. Smarandache [14,15] also introduced the IndetermSoft Set and IndetermHyperSoft Set as further extensions of S-set theory and HS set theory, respectively, to handle operations with varying degrees of uncertainty due to inaccuracies in the world. Several properties, operations, and laws of HS set theory are discussed in various sources (see, for example, [16–21]).

Bipolarity [22] is essential in mathematical modeling for specific problems because it considers both the positive and negative aspects of data. By considering the existence of both positive and negative information, a more accurate representation of decision-making processes can be achieved. Positive data indicate what is possible, while negative data emphasize the impossibility. The underlying concept of bipolar information is that human decision-making relies on bipolar judgmental cognition. Bipolarity is a tool that can be used to create various models. Bipolar fuzzy sets, an expansion of fuzzy sets, were proposed by Zhang [23]. Lee [24] examined bipolar fuzzy sets and compared them with intuitionistic fuzzy sets and interval-valued fuzzy sets. Shabir and Naz [25] introduced bipolar soft (BS) sets and their applications to decision-making problems. Naz and Shabir [26] contributed to the algebraic framework of fuzzy bipolar soft (FBS) sets. Ali and Ansari [27], on the other hand, presented a new hybrid model called the Fermatean FBS set to create a more advanced model. In a separate work, Naeem and Divvaz [28] developed various measures of information for m -polar neutrosophic sets, such as similarity, distance, correlation, divergence, and Dice measures. Another innovative approach was presented by Rehman and Mahmood [29], who introduced bipolar complex fuzzy sets as a means to effectively represent ambiguous and challenging information in real-life dilemmas. Tufail et al. [30] introduced a novel hybrid model for multi-criteria decision-making that integrates bipolar fuzzy soft β -covering with bipolar fuzzy rough sets. Musa and Asaad [31] established the groundwork for bipolar hypersoft (BHS) sets after recognizing the significance of bipolarity. Subsequently, in [32], they demonstrated the applicability of BHS sets in solving parameter reduction and decision-making problems. By utilizing BHS sets, the authors [33–35] introduced various topological concepts. Al-Quran et al. [36] recently examined the idea of BFHS sets and how they can be utilized in decision-making.

The motivation for this paper arises from the need for a comprehensive mathematical model that can effectively address three crucial elements present in real-world situations. First, there is a requirement for a model that can handle the uncertainty associated with different alternatives by assigning fuzzy membership grades to entities corresponding to each parameter. This fuzziness allows for a more nuanced judgment and evaluation of the alternatives. Second, it is important to consider the classification of parameters into parametric-valued subcollections. This parameterization enables a more structured and organized approach to analyzing the data, facilitating better understanding and decision-making processes. Lastly, the concept of bipolarity is essential in mathematical modeling, as it incorporates both the positive and negative aspects of data. By acknowledging the existence of both positive and negative information, a more accurate representation of real-world phenomena and decision-making processes can be achieved.

The remaining sections of this paper are structured in the following manner: In Section 2, elementary ideas and terms are reviewed. Section 3 is focused on the analysis of the FBHS set and its associated set-theoretic operations. Aggregation operations and

their characteristics in the context of FBHS sets are explored in Section 4. The usage of FBHS sets in a decision-making situation is demonstrated in Section 5, with an example for clarification. In Section 6, a comparison is provided between the proposed model and several related models. The final section, Section 7, provides a summary of the findings and offers recommendations for future research.

2. Preliminaries

This section provides a summary of the fundamental concepts and findings concerning fuzzy, bipolar fuzzy, hypersoft, bipolar hypersoft, and BFHS sets. In this study, we use S to represent a universal set of objects, $P(S)$ to represent the power set of S , and $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ to represent a set of parameters that do not have any overlapping elements. In simpler terms, $\nabla = \Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n$ and $\nabla_1, \nabla_2, \nabla_3, \nabla_4, \nabla_5 \subseteq \nabla$.

Definition 1 ([1]). A fuzzy set Ω in S can be defined through its membership function $\Omega : S \rightarrow [0, 1]$, which assigns a value in the range $[0, 1]$ to every element $s \in S$. The value of $\Omega(s)$ reflects the degree of membership of $s \in S$. The set of all fuzzy sets over S is denoted by $FP(S)$.

Definition 2 ([1]). Let $\Omega_1, \Omega_2 \in FP(S)$. The following operations are defined for any $s \in S$:

1. Subset: $\Omega_1 \subseteq \Omega_2$, if $\Omega_1(s) \leq \Omega_2(s)$.
2. Union: $\Omega_1(s) \cup \Omega_2(s) = \max\{\Omega_1(s), \Omega_2(s)\}$.
3. Intersection: $\Omega_1(s) \cap \Omega_2(s) = \min\{\Omega_1(s), \Omega_2(s)\}$.

Definition 3 ([23]). A bipolar fuzzy set $\tilde{\Omega}$ in S is defined as $\tilde{\Omega} = \{s, \tilde{\Omega}^P(s), \tilde{\Omega}^N(s) : s \in S\}$, where $\tilde{\Omega}^P : S \rightarrow [0, 1]$ and $\tilde{\Omega}^N : S \rightarrow [-1, 0]$. The collection of all bipolar fuzzy sets over S is symbolized as $BFP(S)$.

Definition 4 ([3]). An S -set is denoted by a pair (g, Λ_1) , where $g : \Lambda_1 \rightarrow P(S)$.

Definition 5 ([6]). Let $\Lambda_1 = \{\check{\iota}_1, \check{\iota}_2, \dots, \check{\iota}_n\}$ be a set of parameters. The NOT set of Λ_1 , denoted by $\neg\Lambda_1$, is defined as $\neg\Lambda_1 = \{\neg\check{\iota}_1, \neg\check{\iota}_2, \dots, \neg\check{\iota}_n\}$, where $\neg\check{\iota}_i = \text{not } \check{\iota}_i$ for $i = 1, 2, \dots, n$.

Definition 6 ([12]). An HS set is denoted by a pair (g, ∇_1) , where $g : \nabla_1 \rightarrow P(S)$.

Definition 7 ([31]). A BHS set is denoted by a triple $(g, \hat{g}, \nabla_1)$, where $g : \nabla_1 \rightarrow P(S)$ and $\hat{g} : \neg\nabla_1 \rightarrow P(S)$ such that $g(\check{x}) \cap \hat{g}(\neg\check{x}) = \emptyset$ for all $\check{x} \in \nabla_1$.

Definition 8 ([36]). A BFHS set is denoted by a pair (ξ, ∇_1) , where $\xi : \nabla_1 \rightarrow BFP(S)$. Therefore, a BFHS set provides a set of bipolar fuzzy subsets of the universe S , which are parameterized. For any $\check{x} \in \nabla_1$, $\xi(\check{x}) = \{s, \tilde{\Omega}_{\xi(\check{x})}^P(s), \tilde{\Omega}_{\xi(\check{x})}^N(s) : s \in S\}$, where $\tilde{\Omega}_{\xi(\check{x})}^P : S \rightarrow [0, 1]$ and $\tilde{\Omega}_{\xi(\check{x})}^N : S \rightarrow [-1, 0]$.

3. Fuzzy Bipolar Hypersoft Sets

This section is focused on examining the FBHS set concept and its basic operations, as well as discussing its connection with BFHS sets.

Definition 9. A triple $(\xi, \hat{\xi}, \nabla_1)$ is called an FBHS set, where $\xi : \nabla_1 \rightarrow FP(S)$ and $\hat{\xi} : \neg\nabla_1 \rightarrow FP(S)$ such that $0 \leq \xi(\check{x})(s) + \hat{\xi}(\neg\check{x})(s) \leq 1$ for all $s \in S$ and $\check{x} \in \nabla_1$.

In other words, an FBHS set consists of two subsets of S that are parameterized. To maintain consistency, the condition $0 \leq \xi(\check{x}) + \hat{\xi}(\neg\check{x}) \leq 1$ is applied for all $\check{x} \in \nabla_1$. In this context, $\xi(\check{x})$ and $\hat{\xi}(\neg\check{x})$ refer to fuzzy sets within the universal set S . $\xi(\check{x})(s)$ represents the degree to which property $\check{x}$ is present in object s , while $\hat{\xi}(\neg\check{x})(s)$ represents the degree to which the implicit counter property $\neg\check{x}$ is present in s . It should be noted that the degree to which an object possesses a certain property $\check{x}$ may not be equal to the degree to which it lacks the opposite property $\neg\check{x}$. Therefore, for some $\check{x} \in \nabla_1$ and $s \in S$, $\xi(\check{x})(s) + \hat{\xi}(\neg\check{x})(s)$

may be less than and not equal to 1. This is known as the degree of hesitation of an FBHS set $(\zeta, \hat{\zeta}, \nabla_1)$, which can be estimated using $\mathcal{H}(\check{x})(\check{s}) = 1 - \zeta(\check{x})(\check{s}) + \hat{\zeta}(\neg\check{x})(\check{s})$ for $\check{x} \in \nabla_1$ and $\check{s} \in S$. The set of all FBHS sets over S with a fixed set of parameters ∇ is denoted by $\Psi_{(S, \nabla)}$.

The following example illustrates the formal definition of the FBHS set and its superiority over the BFHS set model.

Example 1. Let $S = \{s_1, s_2, s_3, s_4\}$ be a set of smartphones consisting of four models. Let us have three sets of parameters to evaluate these smartphones: $\Lambda_1 = \text{Screen Quality} = \{i_1 = \text{high screen quality}, i_2 = \text{low screen quality}\}$; $\Lambda_2 = \text{Battery Life} = \{i_3 = \text{long battery life}, i_4 = \text{short battery life}\}$; and $\Lambda_3 = \text{Camera Performance} = \{i_5 = \text{excellent camera performance}\}$. Then, the combination of these parameters is $\nabla = \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \{\check{x}_1 = (i_1, i_3, i_5), \check{x}_2 = (i_1, i_4, i_5), \check{x}_3 = (i_2, i_3, i_5), \check{x}_4 = (i_2, i_4, i_5)\}$. Let $\nabla_1 = \{\check{x}_1, \check{x}_2\}$ and let the FBHS set $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ be defined by the following:

$$\begin{aligned}\zeta_1(\check{x}_1) &= \{s_1/0.3, s_2/0.2, s_3/0.5, s_4/0.4\}, \hat{\zeta}_1(\neg\check{x}_1) = \{s_1/0.7, s_2/0.4, s_3/0.3, s_4/0.6\}. \\ \zeta_1(\check{x}_2) &= \{s_1/0.4, s_2/0, s_3/0.3, s_4/0.5\}, \hat{\zeta}_1(\neg\check{x}_2) = \{s_1/0.3, s_2/1, s_3/0.3, s_4/0.4\}.\end{aligned}$$

Next, the corresponding BFHS set (ζ_1, ∇_1) will be examined as follows:

$$\begin{aligned}\zeta_1(\check{x}_1) &= \{(s_1, 0.3, -0.7), (s_2, 0.2, -0.4), (s_3, 0.5, -0.3), (s_4, 0.4, -0.6)\}, \\ \zeta_1(\check{x}_2) &= \{(s_1, 0.4, -0.3), (s_2, 0, -1), (s_3, 0.3, -0.3), (s_4, 0.5, -0.4)\}.\end{aligned}$$

It should be noted that the FBHS set $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ presents information in a more comprehensive manner compared to the BFHS set (ζ_1, ∇_1) . For instance, when examining the information related to the smartphone labeled s_1 in the FBHS set, we find that it has a fuzzy value of 0.3 for parameter $\check{x}_1$ and a fuzzy value of 0.7 for the opposite of $\check{x}_1$ ($\neg\check{x}_1$). On the other hand, in the BFHS set, s_1 would have a fuzzy value of 0.3 for $\check{x}_1$ and a degree of -0.7 to indicate its lack of the same parameter, without explicitly mentioning the opposite parameter, $\neg\check{x}_1$. This demonstrates that the FBHS set provides a more comprehensive and intuitive representation of the information, capturing both the presence and absence of parameters for each object.

In what follows, we proceed to define some basic algebraic operations within the framework of FBHS sets.

Definition 10. For two FBHS sets $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ and $(\zeta_2, \hat{\zeta}_2, \nabla_2)$, we say that $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ is an FBHS subset of $(\zeta_2, \hat{\zeta}_2, \nabla_2)$ if the conditions below are met:

1. $\nabla_1 \subseteq \nabla_2$;
2. $\zeta_1(\check{x}) \subseteq \zeta_2(\check{x})$ and $\hat{\zeta}_2(\neg\check{x}) \subseteq \hat{\zeta}_1(\neg\check{x})$ for all $\check{x} \in \nabla_1$.

We write $(\zeta_1, \hat{\zeta}_1, \nabla_1) \sqsubseteq (\zeta_2, \hat{\zeta}_2, \nabla_2)$.

Similarly, an FBHS set $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ is said to be an FBHS superset of $(\zeta_2, \hat{\zeta}_2, \nabla_2)$ if $(\zeta_2, \hat{\zeta}_2, \nabla_2)$ is an FBHS subset of $(\zeta_1, \hat{\zeta}_1, \nabla_1)$. We denote it by $(\zeta_1, \hat{\zeta}_1, \nabla_1) \sqsupseteq (\zeta_2, \hat{\zeta}_2, \nabla_2)$.

Definition 11. Two FBHS sets $(\zeta_1, \hat{\zeta}_1, \nabla_1)$ and $(\zeta_2, \hat{\zeta}_2, \nabla_2)$ are said to be FBHS equal if $(\zeta_1, \hat{\zeta}_1, \nabla_1) \sqsubseteq (\zeta_2, \hat{\zeta}_2, \nabla_2)$ and $(\zeta_2, \hat{\zeta}_2, \nabla_2) \sqsubseteq (\zeta_1, \hat{\zeta}_1, \nabla_1)$. We denote it by $(\zeta_1, \hat{\zeta}_1, \nabla_1) = (\zeta_2, \hat{\zeta}_2, \nabla_2)$.

Example 2. Let $S, \nabla_1, (\zeta_1, \hat{\zeta}_1, \nabla_1)$ be the same as in Example 1. Let $\nabla_2 = \{\check{x}_1, \check{x}_2, \check{x}_3\}$ and consider the FBHS set $(\zeta_2, \hat{\zeta}_2, \nabla_2)$ defined as follows:

$$\begin{aligned}\zeta_2(\check{x}_1) &= \{s_1/0.4, s_2/0.7, s_3/0.5, s_4/0.5\}, \hat{\zeta}_2(\neg\check{x}_1) = \{s_1/0.2, s_2/0.1, s_3/0, s_4/0.5\}. \\ \zeta_2(\check{x}_2) &= \{s_1/0.5, s_2/1, s_3/0.6, s_4/0.5\}, \hat{\zeta}_2(\neg\check{x}_2) = \{s_1/0.1, s_2/0, s_3/0.2, s_4/0.3\}. \\ \zeta_2(\check{x}_3) &= \{s_1/0.7, s_2/0.4, s_3/0.9, s_4/0.1\}, \hat{\zeta}_2(\neg\check{x}_3) = \{s_1/0.2, s_2/0.6, s_3/0, s_4/0\}.\end{aligned}$$

Then, $(\zeta_1, \hat{\zeta}_1, \nabla_1) \sqsubseteq (\zeta_2, \hat{\zeta}_2, \nabla_2)$.

Definition 12. The FBHS complement of $(\zeta, \hat{\zeta}, \nabla_1)$ is denoted and defined by $(\zeta, \hat{\zeta}, \nabla_1)^c = (\zeta^c, \hat{\zeta}^c, \nabla_1)$, where $\zeta^c(\check{x}) = \hat{\zeta}(\neg\check{x})$ and $\hat{\zeta}^c(\neg\check{x}) = \zeta(\check{x})$ for all $\check{x} \in \nabla_1$.

Definition 13. A relative null FBHS set, denoted by $(\tilde{\Phi}, \tilde{S}, \nabla_1)$, is an FBHS set $(\xi, \hat{\xi}, \nabla_1)$ such that $\xi(\check{x}) = \emptyset$ and $\hat{\xi}(\neg\check{x}) = S$ for all $\check{x} \in \nabla_1$.

Definition 14. A relative whole FBHS set, denoted by $(\tilde{S}, \tilde{\Phi}, \nabla_1)$, is an FBHS set $(\xi, \hat{\xi}, \nabla_1)$ such that $\xi(\check{x}) = S$ and $\hat{\xi}(\neg\check{x}) = \emptyset$ for all $\check{x} \in \nabla_1$.

Definition 15. The FBHS extended union of $(\xi_1, \hat{\xi}_1, \nabla_1)$ and $(\xi_2, \hat{\xi}_2, \nabla_2)$ is an FBHS set $(\xi_3, \hat{\xi}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \cup \nabla_2$ and for all $\check{x} \in \nabla_3$:

$$\xi_3(\check{x}) = \begin{cases} \xi_1(\check{x}) & \text{if } \check{x} \in \nabla_1 \setminus \nabla_2 \\ \xi_2(\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_1 \\ \xi_1(\check{x}) \cup \xi_2(\check{x}) & \text{if } \check{x} \in \nabla_1 \cap \nabla_2 \end{cases}$$

$$\hat{\xi}_3(\neg\check{x}) = \begin{cases} \hat{\xi}_1(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \setminus \neg\nabla_2 \\ \hat{\xi}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_1 \\ \hat{\xi}_1(\neg\check{x}) \cap \hat{\xi}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \cap \neg\nabla_2 \end{cases}$$

We write $(\xi_1, \hat{\xi}_1, \nabla_1) \sqcup_\epsilon (\xi_2, \hat{\xi}_2, \nabla_2) = (\xi_3, \hat{\xi}_3, \nabla_3)$.

Definition 16. The FBHS extended intersection of $(\xi_1, \hat{\xi}_1, \nabla_1)$ and $(\xi_2, \hat{\xi}_2, \nabla_2)$ is an FBHS set $(\xi_3, \hat{\xi}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \cup \nabla_2$ and for all $\check{x} \in \nabla_3$:

$$\xi_3(\check{x}) = \begin{cases} \xi_1(\check{x}) & \text{if } \check{x} \in \nabla_1 \setminus \nabla_2 \\ \xi_2(\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_1 \\ \xi_1(\check{x}) \cap \xi_2(\check{x}) & \text{if } \check{x} \in \nabla_1 \cap \nabla_2 \end{cases}$$

$$\hat{\xi}_3(\neg\check{x}) = \begin{cases} \hat{\xi}_1(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \setminus \neg\nabla_2 \\ \hat{\xi}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_1 \\ \hat{\xi}_1(\neg\check{x}) \cup \hat{\xi}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \cap \neg\nabla_2 \end{cases}$$

We write $(\xi_1, \hat{\xi}_1, \nabla_1) \tilde{\cap}_\epsilon (\xi_2, \hat{\xi}_2, \nabla_2) = (\xi_3, \hat{\xi}_3, \nabla_3)$.

Definition 17. The FBHS union of $(\xi_1, \hat{\xi}_1, \nabla_1)$ and $(\xi_2, \hat{\xi}_2, \nabla_2)$ is an FBHS set $(\xi_3, \hat{\xi}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \cap \nabla_2$, and for all $\check{x} \in \nabla_3$,

$$\xi_3(\check{x}) = \xi_1(\check{x}) \cup \xi_2(\check{x}) \text{ and } \hat{\xi}_3(\neg\check{x}) = \hat{\xi}_1(\neg\check{x}) \cap \hat{\xi}_2(\neg\check{x}).$$

We write $(\xi_1, \hat{\xi}_1, \nabla_1) \sqcup (\xi_2, \hat{\xi}_2, \nabla_2) = (\xi_3, \hat{\xi}_3, \nabla_3)$.

Definition 18. The FBHS intersection of $(\xi_1, \hat{\xi}_1, \nabla_1)$ and $(\xi_2, \hat{\xi}_2, \nabla_2)$ is an FBHS set $(\xi_3, \hat{\xi}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \cap \nabla_2$, and for all $\check{x} \in \nabla_3$,

$$\xi_3(\check{x}) = \xi_1(\check{x}) \cap \xi_2(\check{x}) \text{ and } \hat{\xi}_3(\neg\check{x}) = \hat{\xi}_1(\neg\check{x}) \cup \hat{\xi}_2(\neg\check{x}).$$

We write $(\xi_1, \hat{\xi}_1, \nabla_1) \tilde{\cap} (\xi_2, \hat{\xi}_2, \nabla_2) = (\xi_3, \hat{\xi}_3, \nabla_3)$.

Example 3. Let S and ∇ be the same as in Example 1. Let $\nabla_1 = \{\check{x}_1, \check{x}_3\}$, $\nabla_2 = \{\check{x}_1, \check{x}_4\}$, and let the FBHS sets $(\xi_1, \hat{\xi}_1, \nabla_1)$ and $(\xi_2, \hat{\xi}_2, \nabla_2)$ be defined by the following:

$$\xi_1(\check{x}_1) = \{\check{s}_1/0.4, \check{s}_2/0.6, \check{s}_3/0.1, \check{s}_4/0.4\}, \hat{\xi}_1(\neg\check{x}_1) = \{\check{s}_1/0.5, \check{s}_2/0.4, \check{s}_3/0.3, \check{s}_4/0.5\}.$$

$$\xi_1(\check{x}_3) = \{\check{s}_1/0.7, \check{s}_2/0.9, \check{s}_3/0.5, \check{s}_4/0.3\}, \hat{\xi}_1(\neg\check{x}_3) = \{\check{s}_1/0.2, \check{s}_2/0, \check{s}_3/0.3, \check{s}_4/0.4\}.$$

And,

$$\xi_2(\check{x}_1) = \{\check{s}_1/0.6, \check{s}_2/0.1, \check{s}_3/0.7, \check{s}_4/0.2\}, \hat{\xi}_2(\neg\check{x}_1) = \{\check{s}_1/0.2, \check{s}_2/0.8, \check{s}_3/0.1, \check{s}_4/0.5\}.$$

$$\xi_2(\check{x}_4) = \{\check{s}_1/0.1, \check{s}_2/0.1, \check{s}_3/0.9, \check{s}_4/0\}, \hat{\xi}_2(\neg\check{x}_4) = \{\check{s}_1/0.6, \check{s}_2/0.4, \check{s}_3/0.1, \check{s}_4/1\}.$$

Let

$$(\xi_1, \hat{\xi}_1, \nabla_1) \sqcup_\epsilon (\xi_2, \hat{\xi}_2, \nabla_2) = (\xi_3, \hat{\xi}_3, \nabla_3).$$

And,

$$(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_4, \widehat{\zeta}_4, \nabla_3).$$

Then,

$$\begin{aligned}\zeta_3(\check{x}_1) &= \{\check{s}_1/0.6, \check{s}_2/0.6, \check{s}_3/0.7, \check{s}_4/0.4\}, \widehat{\zeta}_3(\neg\check{x}_1) = \{\check{s}_1/0.2, \check{s}_2/0.4, \check{s}_3/0.1, \check{s}_4/0.5\}. \\ \zeta_3(\check{x}_3) &= \{\check{s}_1/0.7, \check{s}_2/0.9, \check{s}_3/0.5, \check{s}_4/0.3\}, \widehat{\zeta}_3(\neg\check{x}_3) = \{\check{s}_1/0.2, \check{s}_2/0, \check{s}_3/0.3, \check{s}_4/0.4\}. \\ \zeta_3(\check{x}_4) &= \{\check{s}_1/0.1, \check{s}_2/0.1, \check{s}_3/0.9, \check{s}_4/0\}, \widehat{\zeta}_3(\neg\check{x}_4) = \{\check{s}_1/0.6, \check{s}_2/0.4, \check{s}_3/0.1, \check{s}_4/1\}.\end{aligned}$$

And,

$$\begin{aligned}\zeta_4(\check{x}_1) &= \{\check{s}_1/0.4, \check{s}_2/0.1, \check{s}_3/0.1, \check{s}_4/0.2\}, \widehat{\zeta}_4(\neg\check{x}_1) = \{\check{s}_1/0.5, \check{s}_2/0.8, \check{s}_3/0.3, \check{s}_4/0.5\}. \\ \zeta_4(\check{x}_3) &= \{\check{s}_1/0.7, \check{s}_2/0.9, \check{s}_3/0.5, \check{s}_4/0.3\}, \widehat{\zeta}_4(\neg\check{x}_3) = \{\check{s}_1/0.2, \check{s}_2/0, \check{s}_3/0.3, \check{s}_4/0.4\}. \\ \zeta_4(\check{x}_4) &= \{\check{s}_1/0.1, \check{s}_2/0.1, \check{s}_3/0.9, \check{s}_4/0\}, \widehat{\zeta}_4(\neg\check{x}_4) = \{\check{s}_1/0.6, \check{s}_2/0.4, \check{s}_3/0.1, \check{s}_4/1\}.\end{aligned}$$

Definition 19. The FBHS OR-operation of $(\zeta_1, \widehat{\zeta}_1, \nabla_1)$ and $(\zeta_2, \widehat{\zeta}_2, \nabla_2)$ is an FBHS set $(\zeta_3, \widehat{\zeta}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \times \nabla_2$, and for all $(\check{x}_1, \check{x}_2) \in \nabla_3$, $\check{x}_1 \in \nabla_1$, $\check{x}_2 \in \nabla_2$,

$$\zeta_3(\check{x}_1, \check{x}_2) = \zeta_1(\check{x}_1) \cup \zeta_2(\check{x}_2) \text{ and } \widehat{\zeta}_3(\neg\check{x}_1, \neg\check{x}_2) = \widehat{\zeta}_1(\neg\check{x}_1) \cap \widehat{\zeta}_2(\neg\check{x}_2).$$

We write $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\vee} (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.

Definition 20. The FBHS AND-operation of $(\zeta_1, \widehat{\zeta}_1, \nabla_1)$ and $(\zeta_2, \widehat{\zeta}_2, \nabla_2)$ is an FBHS set $(\zeta_3, \widehat{\zeta}_3, \nabla_3)$, where $\nabla_3 = \nabla_1 \times \nabla_2$, and for all $(\check{x}_1, \check{x}_2) \in \nabla_3$, $\check{x}_1 \in \nabla_1$, $\check{x}_2 \in \nabla_2$,

$$\zeta_3(\check{x}_1, \check{x}_2) = \zeta_1(\check{x}_1) \cap \zeta_2(\check{x}_2) \text{ and } \widehat{\zeta}_3(\neg\check{x}_1, \neg\check{x}_2) = \widehat{\zeta}_1(\neg\check{x}_1) \cup \widehat{\zeta}_2(\neg\check{x}_2).$$

We write $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\wedge} (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.

4. Set-Theoretic Properties of Fuzzy Bipolar Hypersoft Sets

This section highlights the fundamental results and properties of set-theoretic operations within the context of FBHS sets.

Proposition 1. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1)$, $(\zeta_2, \widehat{\zeta}_2, \nabla_1)$, and $(\zeta_3, \widehat{\zeta}_3, \nabla_1) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\widetilde{S}, \widetilde{\Phi}, \nabla_1)$.
2. $(\widetilde{\Phi}, \widetilde{S}, \nabla_1) \widetilde{\subseteq} (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
3. If $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\zeta_2, \widehat{\zeta}_2, \nabla_1)$ and $(\zeta_2, \widehat{\zeta}_2, \nabla_1) \widetilde{\subseteq} (\zeta_3, \widehat{\zeta}_3, \nabla_1)$; then, $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\zeta_3, \widehat{\zeta}_3, \nabla_1)$.

Proof. Straightforward. $\square$

Proposition 2. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1)$, $(\zeta_2, \widehat{\zeta}_2, \nabla_1) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $(\widetilde{\Phi}, \widetilde{S}, \nabla_1)^c = (\widetilde{S}, \widetilde{\Phi}, \nabla_1)$ and $(\widetilde{S}, \widetilde{\Phi}, \nabla_1)^c = (\widetilde{\Phi}, \widetilde{S}, \nabla_1)$.
2. $((\zeta_1, \widehat{\zeta}_1, \nabla_1)^c)^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
3. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\zeta_2, \widehat{\zeta}_2, \nabla_1)$; then, $(\zeta_2, \widehat{\zeta}_2, \nabla_1)^c \widetilde{\subseteq} (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c$.
4. $(\widetilde{\Phi}, \widetilde{S}, \nabla_1) \widetilde{\subseteq} (\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\subseteq} (\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\subseteq} (\widetilde{S}, \widetilde{\Phi}, \nabla_1)$.
5. If $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\zeta_2, \widehat{\zeta}_2, \nabla_1)$, then $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_1) = (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
6. If $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\subseteq} (\zeta_2, \widehat{\zeta}_2, \nabla_1)$, then $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_1) = (\zeta_2, \widehat{\zeta}_2, \nabla_1)$.

Proof. Straightforward. $\square$

Proposition 3. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1)$, $(\zeta_2, \widehat{\zeta}_2, \nabla_2) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.
2. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.
3. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.
4. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.
5. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\vee} (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\wedge} (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.
6. $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\wedge} (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\vee} (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c$.

Proof. (1) Suppose that $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_4, \widehat{\zeta}_4, \nabla_3)$, where $\nabla_3 = \nabla_1 \cup \nabla_2$. Then, $((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2))^c = (\zeta_4, \widehat{\zeta}_4, \nabla_3)^c = (\zeta_4^c, \widehat{\zeta}_4^c, \nabla_3)$. We have the following:

$$\zeta_4(\check{x}) = \begin{cases} \zeta_1(\check{x}) & \text{if } \check{x} \in \nabla_1 \setminus \nabla_2 \\ \zeta_2(\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_1 \\ \zeta_1(\check{x}) \cup \zeta_2(\check{x}) & \text{if } \check{x} \in \nabla_1 \cap \nabla_2 \end{cases}$$

$$\widehat{\zeta}_4(\neg\check{x}) = \begin{cases} \widehat{\zeta}_1(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \setminus \neg\nabla_2 \\ \widehat{\zeta}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_1 \\ \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \cap \neg\nabla_2 \end{cases}$$

And,

$$\zeta_4^c(\check{x}) = \widehat{\zeta}_4(\neg\check{x}) = \begin{cases} \widehat{\zeta}_1(\neg\check{x}) & \text{if } \check{x} \in \nabla_1 \setminus \nabla_2 \\ \widehat{\zeta}_2(\neg\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_1 \\ \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_2(\neg\check{x}) & \text{if } \check{x} \in \nabla_1 \cap \nabla_2 \end{cases}$$

$$\widehat{\zeta}_4^c(\neg\check{x}) = \zeta_4(\check{x}) = \begin{cases} \zeta_1(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \setminus \neg\nabla_2 \\ \zeta_2(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_1 \\ \zeta_1(\check{x}) \cup \zeta_2(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \cap \neg\nabla_2 \end{cases}$$

Also, let $(\zeta_1, \widehat{\zeta}_1, \nabla_1)^c \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)^c = (\zeta_1^c, \widehat{\zeta}_1^c, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2^c, \widehat{\zeta}_2^c, \nabla_2)^c = (\zeta_5, \widehat{\zeta}_5, \nabla_3)$, where $\nabla_3 = \nabla_1 \cup \nabla_2$. Then, we have the following:

$$\zeta_5(\check{x}) = \begin{cases} \zeta_1^c(\check{x}) = \widehat{\zeta}_1(\neg\check{x}) & \text{if } \check{x} \in \nabla_1 \setminus \nabla_2 \\ \zeta_2^c(\check{x}) = \widehat{\zeta}_2(\neg\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_1 \\ \zeta_1^c(\check{x}) \cap \zeta_2^c(\check{x}) = \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_2(\neg\check{x}) & \text{if } \check{x} \in \nabla_1 \cap \nabla_2 \end{cases}$$

$$\widehat{\zeta}_5(\neg\check{x}) = \begin{cases} \widehat{\zeta}_1^c(\neg\check{x}) = \zeta_1(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \setminus \neg\nabla_2 \\ \widehat{\zeta}_2^c(\neg\check{x}) = \zeta_2(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_1 \\ \widehat{\zeta}_1^c(\neg\check{x}) \cup \widehat{\zeta}_2^c(\neg\check{x}) = \zeta_1(\check{x}) \cup \zeta_2(\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_1 \cap \neg\nabla_2 \end{cases}$$

Since $(\zeta_4^c, \widehat{\zeta}_4^c, \nabla_3)$ and $(\zeta_5, \widehat{\zeta}_5, \nabla_3)$ are equivalent for all $\check{x} \in \nabla_3$, the proof is concluded. The other parts can be proven in a similar manner. $\square$

Proposition 4. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1), (\zeta_2, \widehat{\zeta}_2, \nabla_1) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_1) = (\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_1)$.
2. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_1) = (\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_1)$.
3. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\widetilde{\Phi}, \widetilde{S}, \nabla_1) = (\zeta_1, \widehat{\zeta}_1, \nabla_1)$ and $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\widetilde{\Phi}, \widetilde{S}, \nabla_1) = (\widetilde{\Phi}, \widetilde{S}, \nabla_1)$.
4. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\widetilde{S}, \widetilde{\Phi}, \nabla_1) = (\widetilde{S}, \widetilde{\Phi}, \nabla_1)$ and $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\widetilde{S}, \widetilde{\Phi}, \nabla_1) = (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.

Proof. Straightforward. $\square$

Proposition 5. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1), (\zeta_2, \widehat{\zeta}_2, \nabla_2), (\zeta_3, \widehat{\zeta}_3, \nabla_3) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup_\epsilon (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
2. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap}_\epsilon (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
3. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
4. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap} (\zeta_1, \widehat{\zeta}_1, \nabla_1)$.
5. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \sqcup_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.
6. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap}_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \widetilde{\cap}_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.
7. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.
8. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3)$.

Proof. (7) Suppose that $(\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3) = (\zeta_4, \widehat{\zeta}_4, \nabla_2 \cap \nabla_3)$. Then, for all $\check{x} \in \nabla_2 \cap \nabla_3$,

$$\zeta_4(\check{x}) = \zeta_2(\check{x}) \cup \zeta_3(\check{x}) \text{ and } \widehat{\zeta}_4(\neg\check{x}) = \widehat{\zeta}_2(\neg\check{x}) \cap \widehat{\zeta}_3(\neg\check{x}).$$

Assume that $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_4, \widehat{\zeta}_4, \nabla_2 \cap \nabla_3) = (\zeta_5, \widehat{\zeta}_5, (\nabla_1 \cap (\nabla_2 \cap \nabla_3)))$. Then, for all $\check{x} \in \nabla_1 \cap (\nabla_2 \cap \nabla_3)$,

$$\zeta_5(\check{x}) = \zeta_1(\check{x}) \cup \zeta_4(\check{x}) = \zeta_1(\check{x}) \cup (\zeta_2(\check{x}) \cup \zeta_3(\check{x})) \text{ and } \widehat{\zeta}_5(\neg\check{x}) = \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_4(\neg\check{x}) = \widehat{\zeta}_1(\neg\check{x}) \cap (\widehat{\zeta}_2(\neg\check{x}) \cap \widehat{\zeta}_3(\neg\check{x})).$$

On the other hand, let $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_6, \widehat{\zeta}_6, \nabla_1 \cap \nabla_2)$. Then, for all $\check{x} \in \nabla_1 \cap \nabla_2$,

$$\zeta_6(\check{x}) = \zeta_1(\check{x}) \cup \zeta_2(\check{x}) \text{ and } \widehat{\zeta}_6(\neg\check{x}) = \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_2(\neg\check{x}).$$

Assume that $(\zeta_6, \widehat{\zeta}_6, \nabla_1 \cap \nabla_2) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3) = (\zeta_7, \widehat{\zeta}_7, (\nabla_1 \cap \nabla_2) \cap \nabla_3)$. Then, for all $\check{x} \in (\nabla_1 \cap \nabla_2) \cap \nabla_3$,

$$\zeta_7(\check{x}) = \zeta_6(\check{x}) \cup \zeta_3(\check{x}) = (\zeta_1(\check{x}) \cup \zeta_2(\check{x})) \cup \zeta_3(\check{x}) \text{ and } \widehat{\zeta}_7(\neg\check{x}) = \widehat{\zeta}_6(\neg\check{x}) \cap \widehat{\zeta}_3(\neg\check{x}) = (\widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_2(\neg\check{x})) \cap \widehat{\zeta}_3(\neg\check{x}).$$

Since $(\zeta_5, \widehat{\zeta}_5, \nabla_1 \cap (\nabla_2 \cap \nabla_3))$ and $(\zeta_7, \widehat{\zeta}_7, (\nabla_1 \cap \nabla_2) \cap \nabla_3)$ are equivalent for every $\check{x} \in \nabla_1 \cap (\nabla_2 \cap \nabla_3) = (\nabla_1 \cap \nabla_2) \cap \nabla_3$, the proof is concluded.

The other parts can be proven in a similar manner. $\square$

Proposition 6. Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1), (\zeta_2, \widehat{\zeta}_2, \nabla_2), (\zeta_3, \widehat{\zeta}_3, \nabla_3) \in \Psi_{(S, \nabla)}$. Then, we have the following:

1. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \widetilde{\cap} ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.
2. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \sqcup_\epsilon ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.
3. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \sqcup ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap}_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.
4. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap}_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \widetilde{\cap}_\epsilon ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.
5. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \widetilde{\cap} ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.
6. $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} ((\zeta_2, \widehat{\zeta}_2, \nabla_2) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_2, \widehat{\zeta}_2, \nabla_2)) \sqcup ((\zeta_1, \widehat{\zeta}_1, \nabla_1) \widetilde{\cap} (\zeta_3, \widehat{\zeta}_3, \nabla_3))$.

Proof. (4) Suppose that $((\zeta_2, \widehat{\zeta}_2, \nabla_2) \widetilde{\cap}_\epsilon (\zeta_3, \widehat{\zeta}_3, \nabla_3)) = (\zeta_4, \widehat{\zeta}_4, \nabla_2 \cup \nabla_3)$. Then, for all $\check{x} \in \nabla_2 \cup \nabla_3$,

$$\zeta_4(\check{x}) = \begin{cases} \zeta_2(\check{x}) & \text{if } \check{x} \in \nabla_2 \setminus \nabla_3 \\ \zeta_3(\check{x}) & \text{if } \check{x} \in \nabla_3 \setminus \nabla_2 \\ \zeta_2(\check{x}) \cap \zeta_3(\check{x}) & \text{if } \check{x} \in \nabla_2 \cap \nabla_3 \end{cases}$$

$$\widehat{\zeta}_4(\neg\check{x}) = \begin{cases} \widehat{\zeta}_2(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \setminus \neg\nabla_3 \\ \widehat{\zeta}_3(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_3 \setminus \neg\nabla_2 \\ \widehat{\zeta}_2(\neg\check{x}) \cup \widehat{\zeta}_3(\neg\check{x}) & \text{if } \neg\check{x} \in \neg\nabla_2 \cap \neg\nabla_3 \end{cases}$$

Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_4, \widehat{\zeta}_4, \nabla_2 \cup \nabla_3) = (\zeta_5, \widehat{\zeta}_5, \nabla_1 \cap (\nabla_2 \cup \nabla_3)) = (\zeta_5, \widehat{\zeta}_5, \nabla_4 \cup \nabla_5)$, where $\nabla_4 = \nabla_1 \cap \nabla_2$ and $\nabla_5 = \nabla_1 \cap \nabla_3$. Then, for all $\check{x} \in \nabla_4 \cup \nabla_5$,

$$\zeta_5(\check{x}) = \zeta_1(\check{x}) \cup \zeta_4(\check{x}), \text{ and } \widehat{\zeta}_5(\neg\check{x}) = \widehat{\zeta}_1(\neg\check{x}) \cap \widehat{\zeta}_4(\neg\check{x}).$$

Hence, we obtain

$$\zeta_5(\check{x}) = \begin{cases} \zeta_1(\check{x}) \cup \zeta_2(\check{x}) & \text{if } \check{x} \in \nabla_4 \setminus \nabla_5 \\ \zeta_1(\check{x}) \cup \zeta_3(\check{x}) & \text{if } \check{x} \in \nabla_5 \setminus \nabla_4 \\ \zeta_1(\check{x}) \cup (\zeta_2(\check{x}) \cap \zeta_3(\check{x})) & \text{if } \check{x} \in \nabla_4 \cap \nabla_5 \end{cases}$$

$$\widehat{\zeta}_5(\neg \check{x}) = \begin{cases} \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_2(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_4 \setminus \neg \nabla_5 \\ \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_3(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_5 \setminus \neg \nabla_4 \\ \widehat{\zeta}_1(\neg \check{x}) \cap (\widehat{\zeta}_2(\neg \check{x}) \cup \widehat{\zeta}_3(\neg \check{x})) & \text{if } \neg \check{x} \in \neg \nabla_4 \cap \neg \nabla_5 \end{cases}$$

On the other hand, let $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_2, \widehat{\zeta}_2, \nabla_2) = (\zeta_6, \widehat{\zeta}_6, \nabla_1 \cap \nabla_2)$. Then, for all $\check{x} \in \nabla_1 \cap \nabla_2$,

$$\zeta_6(\check{x}) = \zeta_1(\check{x}) \cup \zeta_2(\check{x}) \text{ and } \widehat{\zeta}_6(\neg \check{x}) = \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_2(\neg \check{x}).$$

Let $(\zeta_1, \widehat{\zeta}_1, \nabla_1) \sqcup (\zeta_3, \widehat{\zeta}_3, \nabla_3) = (\zeta_7, \widehat{\zeta}_7, \nabla_1 \cap \nabla_3)$. Then, for all $\check{x} \in \nabla_1 \cap \nabla_3$,

$$\zeta_7(\check{x}) = \zeta_1(\check{x}) \cup \zeta_3(\check{x}) \text{ and } \widehat{\zeta}_7(\neg \check{x}) = \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_3(\neg \check{x}).$$

Now, suppose that $(\zeta_6, \widehat{\zeta}_6, \nabla_1 \cap \nabla_2) \sqcap_\varepsilon (\zeta_7, \widehat{\zeta}_7, \nabla_1 \cap \nabla_3) = (\zeta_8, \widehat{\zeta}_8, \nabla_4 \cup \nabla_5)$, where $\nabla_4 = \nabla_1 \cap \nabla_2$ and $\nabla_5 = \nabla_1 \cap \nabla_3$. Then, for all $\check{x} \in \nabla_4 \cup \nabla_5$,

$$\begin{aligned} \zeta_8(\check{x}) &= \begin{cases} \zeta_6(\check{x}) & \text{if } \check{x} \in \nabla_4 \setminus \nabla_5 \\ \zeta_7(\check{x}) & \text{if } \check{x} \in \nabla_5 \setminus \nabla_4 \\ \zeta_6(\check{x}) \cap \zeta_7(\check{x}) & \text{if } \check{x} \in \nabla_4 \cap \nabla_5 \end{cases} \\ &= \begin{cases} \zeta_1(\check{x}) \cup \zeta_2(\check{x}) & \text{if } \check{x} \in \nabla_4 \setminus \nabla_5 \\ \zeta_1(\check{x}) \cup \zeta_3(\check{x}) & \text{if } \check{x} \in \nabla_5 \setminus \nabla_4 \\ (\zeta_1(\check{x}) \cup \zeta_2(\check{x})) \cap (\zeta_1(\check{x}) \cup \zeta_3(\check{x})) & \text{if } \check{x} \in \nabla_4 \cap \nabla_5 \end{cases} \\ \widehat{\zeta}_8(\neg \check{x}) &= \begin{cases} \widehat{\zeta}_6(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_4 \setminus \neg \nabla_5 \\ \widehat{\zeta}_7(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_5 \setminus \neg \nabla_4 \\ \widehat{\zeta}_6(\neg \check{x}) \cup \widehat{\zeta}_7(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_4 \cap \neg \nabla_5 \end{cases} \\ &= \begin{cases} \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_2(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_4 \setminus \neg \nabla_5 \\ \widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_3(\neg \check{x}) & \text{if } \neg \check{x} \in \neg \nabla_5 \setminus \neg \nabla_4 \\ (\widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_2(\neg \check{x})) \cup (\widehat{\zeta}_1(\neg \check{x}) \cap \widehat{\zeta}_3(\neg \check{x})) & \text{if } \neg \check{x} \in \neg \nabla_4 \cap \neg \nabla_5 \end{cases} \end{aligned}$$

Since $(\zeta_5, \widehat{\zeta}_5, \nabla_4 \cup \nabla_5)$ and $(\zeta_8, \widehat{\zeta}_8, \nabla_4 \cup \nabla_5)$ are equivalent for all $\check{x} \in \nabla_4 \cup \nabla_5$, the proof is concluded.

The other parts can be proven in a similar manner. $\square$

5. Utilizing Fuzzy Bipolar Hypersoft Sets for Enhanced Decision-Making

In this part, the aim is to provide solutions to decision-making problems using the FBHS set concept. Decision-making is a critical process that involves choosing the best course of action among various alternatives based on specific criteria. FBHS sets can aid in decision-making by providing a flexible and powerful mathematical tool for handling uncertain or imprecise information. By employing FBHS sets, decision-makers can model complex decision problems with a high degree of accuracy and better evaluate the risks and opportunities associated with each alternative. Therefore, this part aims to explore the potential of FBHS sets in enhancing the quality and effectiveness of decision-making processes.

Example 4. Suppose a company is hiring for a software engineer position, and there are four candidates who have applied for the job. Let the set of candidates be $S = \{\check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4\}$. Let $\Lambda_1 = \text{Development Domains} = \{\check{i}_1 = \text{full-stack development}, \check{i}_2 = \text{cloud computing}, \check{i}_3 = \text{DevOps}\}$, $\Lambda_2 = \text{Advanced Concepts} = \{\check{i}_4 = \text{machine learning}, \check{i}_5 = \text{algorithms and data structures}\}$, and $\Lambda_3 = \text{Software Engineering Practices} = \{\check{i}_7 = \text{front-end development}, \check{i}_8 = \text{agile methodology}\}$ be a set of three technical skills; then, $\nabla = \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \{\check{x}_1 = (\check{i}_1, \check{i}_4, \check{i}_7), \check{x}_2 = (\check{i}_1, \check{i}_4, \check{i}_8), \check{x}_3 = (\check{i}_1, \check{i}_5, \check{i}_7), \check{x}_4 = (\check{i}_1, \check{i}_5, \check{i}_8), \check{x}_5 = (\check{i}_2, \check{i}_4, \check{i}_7), \check{x}_6 = (\check{i}_2, \check{i}_4, \check{i}_8), \check{x}_7 = (\check{i}_2, \check{i}_5, \check{i}_7), \check{x}_8 = (\check{i}_2, \check{i}_5, \check{i}_8), \check{x}_9 = (\check{i}_3, \check{i}_4, \check{i}_7), \check{x}_{10} = (\check{i}_3, \check{i}_4, \check{i}_8), \check{x}_{11} = (\check{i}_3, \check{i}_5, \check{i}_7), \check{x}_{12} = (\check{i}_3, \check{i}_5, \check{i}_8)\}$. We can define an FBHS set $\mu_1 = (\zeta, \widehat{\zeta}, \nabla)$ to analyze the “Qualifications of Candidates” for the job as follows:

$$\begin{aligned} \zeta(\check{x}_1) &= \{\check{s}_1/0.4, \check{s}_2/0.3, \check{s}_3/0.5, \check{s}_4/0.3\}, \widehat{\zeta}(\neg \check{x}_1) = \{\check{s}_1/0.1, \check{s}_2/0.5, \check{s}_3/0.1, \check{s}_4/0.5\}. \\ \zeta(\check{x}_2) &= \{\check{s}_1/0.7, \check{s}_2/0.3, \check{s}_3/0.7, \check{s}_4/0.4\}, \widehat{\zeta}(\neg \check{x}_2) = \{\check{s}_1/0.1, \check{s}_2/0.3, \check{s}_3/0.3, \check{s}_4/0.3\}. \\ \zeta(\check{x}_3) &= \{\check{s}_1/0.1, \check{s}_2/0.1, \check{s}_3/0.9, \check{s}_4/0.4\}, \widehat{\zeta}(\neg \check{x}_3) = \{\check{s}_1/0.6, \check{s}_2/0.4, \check{s}_3/0, \check{s}_4/0.4\}. \\ \zeta(\check{x}_4) &= \{\check{s}_1/0.3, \check{s}_2/0.4, \check{s}_3/0.8, \check{s}_4/0.3\}, \widehat{\zeta}(\neg \check{x}_4) = \{\check{s}_1/0.4, \check{s}_2/0.2, \check{s}_3/0.1, \check{s}_4/0.5\}. \\ \zeta(\check{x}_5) &= \{\check{s}_1/0.2, \check{s}_2/0.6, \check{s}_3/0.7, \check{s}_4/0.7\}, \widehat{\zeta}(\neg \check{x}_5) = \{\check{s}_1/0.1, \check{s}_2/0.1, \check{s}_3/0.2, \check{s}_4/0.2\}. \end{aligned}$$

$$\begin{aligned}
\zeta(\check{s}_6) &= \{s_1/0.3, s_2/0.2, s_3/0.3, s_4/0.5\}, \widehat{\zeta}(\neg\check{s}_6) = \{s_1/0.5, s_2/0.7, s_3/0.5, s_4/0.1\}. \\
\zeta(\check{s}_7) &= \{s_1/0.9, s_2/0.1, s_3/0.9, s_4/0.7\}, \widehat{\zeta}(\neg\check{s}_7) = \{s_1/0, s_2/0.1, s_3/0, s_4/0\}. \\
\zeta(\check{s}_8) &= \{s_1/0.6, s_2/0.6, s_3/0.4, s_4/0.3\}, \widehat{\zeta}(\neg\check{s}_8) = \{s_1/0.2, s_2/0.4, s_3/0.5, s_4/0.1\}. \\
\zeta(\check{s}_9) &= \{s_1/0.4, s_2/0.6, s_3/0.5, s_4/0.3\}, \widehat{\zeta}(\neg\check{s}_9) = \{s_1/0.4, s_2/0.4, s_3/0.4, s_4/0.6\}. \\
\zeta(\check{s}_{10}) &= \{s_1/0.3, s_2/0.2, s_3/0.1, s_4/0.6\}, \widehat{\zeta}(\neg\check{s}_{10}) = \{s_1/0.1, s_2/0.7, s_3/0.1, s_4/0.3\}. \\
\zeta(\check{s}_{11}) &= \{s_1/0.4, s_2/0.3, s_3/0.3, s_4/0.3\}, \widehat{\zeta}(\neg\check{s}_{11}) = \{s_1/0.5, s_2/0.6, s_3/0.3, s_4/0.6\}. \\
\zeta(\check{s}_{12}) &= \{s_1/0.8, s_2/0.9, s_3/0.8, s_4/0.3\}, \widehat{\zeta}(\neg\check{s}_{12}) = \{s_1/0, s_2/0.1, s_3/0, s_4/0.7\}.
\end{aligned}$$

The FBHS set can be presented in a tabular format, where the entry corresponding to s_i and $\check{s}_j$ is denoted by (a_{ij}, b_{ij}) , where $a_{ij} = \zeta(\check{s}_j)(s_i)$ and $b_{ij} = \widehat{\zeta}(\neg\check{s}_j)(s_i)$. The tabular representation of FBHS set $\mu_1 = (\zeta, \widehat{\zeta}, \nabla)$ is given in Table 1.

Table 1. Tabular representation of FBHS set $\mu_1 = (\zeta, \widehat{\zeta}, \nabla)$.

μ_1	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$	$\check{s}_6$	$\check{s}_7$	$\check{s}_8$	$\check{s}_9$	$\check{s}_{10}$	$\check{s}_{11}$	$\check{s}_{12}$
s_1	(0.4, 0.1)	(0.7, 0.1)	(0.1, 0.6)	(0.3, 0.4)	(0.2, 0.1)	(0.3, 0.5)	(0.9, 0)	(0.6, 0.2)	(0.4, 0.4)	(0.3, 0.1)	(0.4, 0.5)	(0.8, 0)
s_2	(0.3, 0.5)	(0.3, 0.3)	(0.1, 0.4)	(0.4, 0.2)	(0.6, 0.1)	(0.2, 0.7)	(0.1, 0.1)	(0.6, 0.4)	(0.6, 0.4)	(0.2, 0.7)	(0.3, 0.6)	(0.9, 0.1)
s_3	(0.5, 0.1)	(0.7, 0.3)	(0.9, 0)	(0.8, 0.1)	(0.7, 0.2)	(0.3, 0.5)	(0.9, 0)	(0.4, 0.5)	(0.5, 0.4)	(0.1, 0.1)	(0.3, 0.3)	(0.8, 0)
s_4	(0.3, 0.5)	(0.4, 0.3)	(0.4, 0.4)	(0.3, 0.5)	(0.7, 0.2)	(0.5, 0.1)	(0.7, 0)	(0.3, 0.1)	(0.3, 0.6)	(0.6, 0.3)	(0.3, 0.6)	(0.3, 0.7)

Definition 21. Let $S = \{s_1, s_2, s_3, \dots, s_n\}$, $\nabla = \{\check{s}_1, \check{s}_2, \check{s}_3, \dots, \check{s}_n\}$, and $(\zeta, \widehat{\zeta}, \nabla_1)$ be an FBHS set. Then, the final score of s_i for $i = 1, 2, \dots, n$ is denoted and defined as follows:

$$\sigma_i = \sigma_i^+ - \sigma_i^-$$

where

$$\sigma_i^+ = \sum_j a_{ij}$$

and

$$\sigma_i^- = \sum_j b_{ij}$$

Our objective is to select an object from the choice parameter set ∇_1 . To achieve this, we will provide an algorithm that takes into account the observer's specifications and recorded data, using an FBHS set to determine the optimal choice.

Here, we demonstrate the suggested algorithm (Algorithm 1) using Example 4 as a reference.

1. Input the FBHS set $\mu_1 = (\zeta, \widehat{\zeta}, \nabla)$.
2. Input $\nabla_1 = \{\check{s}_1, \check{s}_3, \check{s}_5, \check{s}_7, \check{s}_{11}\} \subseteq \nabla$. Find the reduced FBHS set $\mu_2 = (\zeta, \widehat{\zeta}, \nabla_1)$ as given in Table 2.

Algorithm 1 Algorithm for Optimal Choice Selection

- 1: Input the FBHS set $(\zeta, \widehat{\zeta}, \nabla)$.
- 2: Input the set of choice parameters $\nabla_1 \subseteq \nabla$. Find the reduced FBHS set $(\zeta, \widehat{\zeta}, \nabla_1)$ and represent it in tabular form.
- 3: Calculate the final score σ_i .
- 4: Find k , where $\sigma_k = \max \sigma_i$.
- 5: The optimal choice object is s_k . If k has multiple values, any s_k can be chosen.

Table 2. Tabular representation of reduced FBHS set $\mu_2 = (\zeta, \widehat{\zeta}, \nabla_1)$.

μ_2	$\check{s}_1$	$\check{s}_3$	$\check{s}_5$	$\check{s}_7$	$\check{s}_{11}$
s_1	(0.4, 0.1)	(0.1, 0.6)	(0.2, 0.1)	(0.9, 0)	(0.4, 0.5)
s_2	(0.3, 0.5)	(0.1, 0.4)	(0.6, 0.1)	(0.1, 0.1)	(0.3, 0.6)
s_3	(0.5, 0.1)	(0.9, 0)	(0.7, 0.2)	(0.9, 0)	(0.3, 0.3)
s_4	(0.3, 0.5)	(0.4, 0.4)	(0.7, 0.2)	(0.7, 0)	(0.3, 0.6)

3. Calculate the final score σ_i , as given in Table 3.

Table 3. Final score.

$\check{s}_i$	σ_i^+	σ_i^-	σ_i
$\check{s}_1$	2	1.3	0.7
$\check{s}_2$	1.4	1.7	-0.3
$\check{s}_3$	3.3	0.6	2.7
$\check{s}_4$	2.4	1.7	0.7

4. From Table 3, we find $k = 3$.
5. Hence, $\check{s}_3$ is the most suitable applicant for the position.

6. Comparative Analysis and Managerial Implications

In this section, we conduct a comparative analysis to evaluate the versatility and managerial implications of our proposed FBHS set model. We consider key evaluation factors, namely, Sf (single-argument approximate function), Mf (multi-argument approximate function), Bs (bipolarity setting), and Md (membership degree). The comparative results are presented in Table 4, where our FBHS set model is compared with relevant existing models.

Our comparative analysis reveals that the FBHS set model effectively addresses all crucial features considered in this study. Unlike comparable models, which exhibit limitations in various aspects, our model demonstrates superiority in terms of Sf, Mf, Bs, and Md. Specifically, our model consistently achieves high scores across these evaluation factors, indicating its robustness and versatility in handling complex phenomena characterized by uncertainty and variability.

The results of our comparative analysis have significant managerial implications for decision-makers and practitioners in various fields. By demonstrating the superior performance of the FBHS set model compared to existing alternatives, our findings underscore the potential of our approach to enhance decision-making processes and improve organizational outcomes. Managers can leverage the robustness and versatility of the FBHS set model to make more informed and effective decisions in the face of uncertainty and variability.

Table 4. A comparison of the FBHS set model with relevant existing models.

Authors	Models	Sf	Mf	Bs	Md
Molodtsov [3]	S-set	✓	×	×	×
Shabir et al. [25]	BS set	✓	×	✓	×
Naz et al. [26]	FBS set	✓	×	✓	✓
Smarandache [12]	HS set	✓	✓	×	×
Musa et al. [31]	BHS set	✓	✓	✓	×
Al-Quran et al. [36]	BFHS set		(cf., Example 1)		
Proposed model	FBHS set	✓	✓	✓	✓

7. Conclusions

In this paper, we have presented a new concept of FBHS sets that combines the notions of bipolarity, fuzziness, and parameterization to model complex phenomena. Our approach involves using two mappings to define FBHS sets, allowing for the approximation of both the positivity and negativity of parameters in a set. We have demonstrated the superiority of FBHS sets over BFHS sets through an example, where FBHS sets provide a more intuitive and user-friendly way of modeling phenomena involving bipolarity, fuzziness, and parameterization. Furthermore, we have explored the fundamental operations of FBHS sets and studied their properties. We have applied the concept of FBHS sets to solve a practical

problem and proposed an algorithm for selecting the optimal object based on available information sets.

The limitations of the FBHS sets presented in this paper include challenges in generalizing the concept to encompass interval-valued sets. The current approach primarily approximates the positivity and negativity of parameters but does not explicitly address the representation and manipulation of interval values within the fuzzy number range of $[0, 1]$. Future research should focus on extending the FBHS set framework to integrate interval values, which would facilitate a more comprehensive representation of uncertainty and variability. Despite these limitations, the concept offers a promising framework for modeling complex phenomena. The rigor of this work can be demonstrated by successfully incorporating interval values and providing mathematical validation and empirical studies.

Furthermore, the proposed model can be expanded to include N-FBHS sets, intuitionistic FBHS sets, rough FBHS sets, complex FBHS sets, hesitant FBHS sets, Pythagorean FBHS sets, picture FBHS sets, q-rung orthopair FBHS sets, spherical FBHS sets, Fermatean FBHS sets, and plithogenic FBHS sets, among others. Researchers can also examine the geometric characteristics of the proposed model and its extensions by studying their topological and algebraic structures.

Author Contributions: Conceptualization, B.A.A., S.Y.M. and Z.A.A.; methodology, B.A.A., S.Y.M. and Z.A.A.; formal analysis, B.A.A., S.Y.M. and Z.A.A.; investigation, B.A.A., S.Y.M. and Z.A.A.; writing—original draft preparation, B.A.A. and Z.A.A.; writing—review and editing, S.Y.M. and Z.A.A.; funding acquisition, B.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: All relevant data are within the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [\[CrossRef\]](#)
2. Pawlak, Z. Rough sets. *Int. J. Comput. Inform. Sci.* **1982**, *11*, 341–356. [\[CrossRef\]](#)
3. Molodtsov, D. Soft set theory—first results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [\[CrossRef\]](#)
4. Maji, P.K.; Roy, A.R.; Biswas, R. An application of soft sets in a decision making problem. *Comput. Math. Appl.* **2002**, *44*, 1077–1083. [\[CrossRef\]](#)
5. Çağman, N.; Enginoglu, S.; Citak, F. Fuzzy soft set theory and its applications. *Iran. J. Fuzzy Syst.* **2011**, *8*, 137–147.
6. Maji, P.K.; Biswas, R.; Roy, R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [\[CrossRef\]](#)
7. Ali, M.; Feng, F.; Liu, X.; Min, W.; Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [\[CrossRef\]](#)
8. Ameen, Z.A. A non-continuous soft mapping that preserves some structural soft sets. *J. Intell. Fuzzy. Syst.* **2022**, *42*, 5839–5845. [\[CrossRef\]](#)
9. Asaad, B.A.; Musa, S.Y. A novel class of bipolar soft separation axioms concerning crisp points. *Demonstr. Math.* **2023**, *56*, 20220189. [\[CrossRef\]](#)
10. Çağman, N. Contributions to the theory of soft sets. *J. New Results Sci.* **2014**, *4*, 33–41.
11. Sezgin, A.; Atagün, A.O. On operations on soft sets. *Comput. Math. Appl.* **2011**, *61*, 1457–1467. [\[CrossRef\]](#)
12. Smarandache, F. Extension of soft set to hypersoft set and then to plithogenic hypersoft set. *Neutrosoph. Sets Syst.* **2018**, *22*, 168–170.
13. Musa S.Y.; Mohammed R.A.; Asaad B.A. N-hypersoft sets: An innovative extension of hypersoft sets and their applications. *Symmetry* **2023**, *15*, 1795. [\[CrossRef\]](#)
14. Smarandache, F. Introduction to the IndtermSoft Set and IndtermHyperSoft Set. *Neutrosoph. Sets Syst.* **2022**, *50*, 629–650.
15. Smarandache, F. New types of soft sets: HyperSoft Set, IndtermSoft Set, IndtermHyperSoft Set, and TreeSoft Set. *Int. J. Neutrosoph. Sci.* **2023**, *20*, 58–64. [\[CrossRef\]](#)
16. Asaad, B.A.; Musa, S.Y. Hypersoft separation axioms. *Filomat* **2022**, *36*, 6679–6686. [\[CrossRef\]](#)
17. Arshad, M.; Saeed, M.; Rahman, A.U.; Bajri, S.A.; Alqahtani, H.; Khalifa, H. Modeling uncertainties associated with multi-attribute decision-making based evaluation of cooling system using interval-valued complex intuitionistic fuzzy hypersoft settings. *AIMS Math.* **2024**, *9*, 11396–11422. [\[CrossRef\]](#)
18. Rahman, A.U. A theoretical context for (θ, β) -convexity and (θ, β) -concavity with hypersoft settings. *Big Data Comput. Visions* **2023**, *3*, 196–208.

19. Yolcu, A.; Ozturk, T.Y. Fuzzy hypersoft sets and it's application to decision-making. In *Theory and Application of Hypersoft Set*, 2021st ed.; Smarandache F., Saeed, M., Abdel-Baset, M., Saqlain, M., Eds.; Pons Publishing House: Brussels, Belgium, 2021; pp. 50–64.
20. Yolcu, A.; Smarandache, F.; Ozturk, T.Y. Intuitionistic fuzzy hypersoft sets. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2021**, *70*, 443–455. [[CrossRef](#)]
21. Saeed, M.; Rahman, A.U.; Ahsan, M.; Smarandache, F. Theory of hypersoft sets: Axiomatic properties, aggregation operations, relations, functions and matrices. *Neutrosoph. Sets Syst.* **2022**, *51*, 46.
22. Dubois, D.; Prade, H. An introduction to bipolar representations of information and preferences. *Int. J. Intell. Syst.* **2008**, *23*, 866–877. [[CrossRef](#)]
23. Zhang, W.R. Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis. In Proceedings of the Industrial Fuzzy Control and Intelligent System Conference and NASA Joint Technology Workshop on Neural Networks and Fuzzy Logic and Fuzzy Information Processing Society Biannual Conference, San Antonio, TX, USA, 18–21 December 1994; pp. 305–309.
24. Lee, K.M. Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets. *Int. J. Fuzzy Log. Intell. Syst.* **2004**, *14*, 125–129.
25. Shabir, M.; Naz, M. On bipolar soft sets. *arXiv* **2013**, arXiv:1303.1344v1.
26. Naz, M.; Shabir, M. On fuzzy bipolar soft sets, their algebraic structures and applications. *J. Intell. Fuzzy Syst.* **2014**, *26*, 1645–1656. [[CrossRef](#)]
27. Ali, G.; Ansari, M.N. Multiattribute decision-making under Fermatean fuzzy bipolar soft framework. *Granul. Comput.* **2022**, *7*, 337–352. [[CrossRef](#)]
28. Naeem, K.; Divvaz, B. Information measures for MADM under m-polar neutrosophic environment. *Granul. Comput.* **2023**, *8*, 597–616. [[CrossRef](#)]
29. Rehman, U.; Mahmood, T. The generalized dice similarity measures for bipolar complex fuzzy set and its applications to pattern recognition and medical diagnosis. *Comp. Appl. Math.* **2022**, *41*, 265. [[CrossRef](#)]
30. Tufail, F.; Shabir, M. VIKOR method for MCDM based on bipolar fuzzy soft β -covering based bipolar fuzzy rough set model and its application to site selection of solar power plant. *J. Intell. Fuzzy Syst.* **2022**, *42*, 1835–1857. [[CrossRef](#)]
31. Musa, S.Y.; Asaad, B.A. Bipolar hypersoft sets. *Mathematics* **2021**, *9*, 1826. [[CrossRef](#)]
32. Musa, S.Y.; Asaad, B.A. A novel approach towards parameter reduction based on bipolar hypersoft set and its application to decision-making. *Neutrosoph. Sets Syst.* **2023**, *55*, 544–556.
33. Ameen, Z.A.; Al Ghour, S. Extensions of soft topologies. *Filomat* **2022**, *43*, 5279–5287. [[CrossRef](#)]
34. Musa, S.Y.; Asaad, B.A. Topological structures via bipolar hypersoft sets. *J. Math.* **2022**, *2022*, 2896053. [[CrossRef](#)]
35. Musa, S.Y.; Asaad, B.A. Connectedness on bipolar hypersoft topological spaces. *J. Intell. Fuzzy Syst.* **2022**, *43*, 4095–4105. [[CrossRef](#)]
36. Al-Quran, A.; Al-Sharqi, F.; Ullah, K.; Romdhini, M.U.; Balti, M.; Alomair, M. Bipolar fuzzy hypersoft set and its application in decision making. *Int. J. Neutrosoph. Sci.* **2023**, *20*, 65–77.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.