

Article

A New Sine Family of Generalized Distributions: Statistical Inference with Applications

SidAhmed Benchiha¹, Laxmi Prasad Sapkota² , Aned Al Mutairi³, Vijay Kumar², Rana H. Khashab^{4,*}, Ahmed M. Gemeay⁵ , Mohammed Elgarhy⁶  and Said G. Nassr^{7,*} 

- ¹ Laboratory of Statistics and Stochastic Processes, University of Djillali Liabes, BP 89, Sidi Bel Abbes 22000, Algeria; sidahmed.benchiha@univ-sba.dz
- ² Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur 273001, India; laxmisapkota75@gmail.com (L.P.S.); vkgkp@rediffmail.com (V.K.)
- ³ Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia; aoalmutairi@pnu.edu.sa
- ⁴ Mathematical Sciences Department, College of Applied Sciences, Umm Al-Qura University, Makkah 21961, Saudi Arabia
- ⁵ Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt; ahmed.gemeay@science.tanta.edu.eg
- ⁶ Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef 62521, Egypt; m_elgarhy85@sva.edu.eg
- ⁷ Department of Statistics and Insurance, Faculty of Commerce, Arish University, Al-Arish 45511, Egypt
- * Correspondence: rhkhashab@uqu.edu.sa (R.H.K.); dr.saidstat@gmail.com (S.G.N.)

Abstract: In this article, we extensively study a family of distributions using the trigonometric function. We add an extra parameter to the sine transformation family and name it the alpha-sine-G family of distributions. Some important functional forms and properties of the family are provided in a general form. A specific sub-model alpha-sine Weibull of this family is also introduced using the Weibull distribution as a parent distribution and studied deeply. The statistical properties of this new distribution are investigated and intended parameters are estimated using the maximum likelihood, maximum product of spacings, least square, weighted least square, and minimum distance methods. For further justification of these estimates, a simulation experiment is carried out. Two real data sets are analyzed to show the suggested model's application. The suggested model performed well compares to some existing models considered in the study.

Keywords: sine function; Weibull distribution; moments; estimation methods; hazard function



Citation: Benchiha, S.; Sapkota, L.P.; Al Mutairi, A.; Kumar, V.; Khashab, R.H.; Gemeay, A.M.; Elgarhy, M.; Nassr, S.G. A New Sine Family of Generalized Distributions: Statistical Inference with Applications. *Math. Comput. Appl.* **2023**, *28*, 83. <https://doi.org/10.3390/mca28040083>

Academic Editor: Sandra Ferreira

Received: 13 June 2023

Revised: 5 July 2023

Accepted: 13 July 2023

Published: 16 July 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Statistical distributions are commonly used to study real-world phenomena. The theory of statistical distributions is extensively studied, as are new developments for their application. Several families of distributions have been developed to describe various real-world phenomena. In reality, this new development in distribution theory is a continuing practice. Most probability distributions proposed in the literature have many parameters to make the model more flexible. According to some authors, these estimates are difficult to obtain using numerical resources (see [1]). For modeling real data, it is preferable to create models with few parameters and a high degree of flexibility. To achieve this goal, a group of researchers decided to look for new distributions using trigonometric functions. In the last several years, researchers have been attracted to trigonometric models due to their flexibility and mathematical tractability. Among the various trigonometric G-families, ref. [2] defined the new class of distribution using the sine trigonometric function and

defined the sine-exponential model as its member. The probability density function (PDF) of this family is given by

$$f(x; \varphi) = \frac{\pi}{2} g(x; \varphi) \sin\left\{\frac{\pi}{2} G(x; \varphi)\right\} \quad x \in \mathbb{R}, \quad (1)$$

where $G(x; \varphi)$ is the cumulative distribution function (CDF) of any baseline model and $g(x; \varphi)$ its PDF.

At the same time, the arctangent function defines the arc-tan-G family of distribution [3]. The authors presented the new family of distribution which was used to model Norwegian fire insurance data. This distribution family was proposed for an underlying Pareto distribution and a new distribution called the Pareto arctan distribution, and it was discovered that this distribution provides a good fit when compared to other well-known distributions. Using a similar technique of sine-G, the cosine-G family of distributions was introduced by [4], who also introduced the cosine-Weibull distribution as a member of cosine-G class. Similarly, [5] introduced another sine-G class and studied the sine inverse Weibull distribution as a particular member. Ref. [6] developed the new sine-G family and analyzed the sine-inverse Weibull model in particular. Ref. [7] defined the sine Kumaraswamy-G family of distributions as having two extra parameters. Ref. [8] defined the exponentiated sine-G family and analyzed the particular distribution as the exponentiated sine-Weibull distribution. Further arcsine-G distributions were introduced by [9], and the arcsine exponential distribution with constant and sharp decreasing hazard functions was defined. Another trigonometric function-related probability model introduced by [10] is called the arctan generalized exponential distribution. Using the sine-G family of distribution, [11] developed a new two-parameter model called the sine Burr XII distribution. Hence, we noticed that the simple functions are associated with trigonometric distribution and are mathematically tractable (see [2,5]). Further, we observed that the sine transformation can remarkably enhance the flexibility of $G(x)$ [7]. A new extended cosine-G family of distributions was proposed by [12]. Truncated Cauchy power family of distributions was studied by [13]. Truncated Cauchy power Weibull-G class of distributions was proposed by [14]. The sine half-logistic inverse Rayleigh and sine inverse exponential distributions were discussed in [15,16]. Due to these pleasant features, we are motivated to conduct research on the sine transformation family.

In this study, we developed a new family of trigonometric models using the sine function by introducing an additional scale parameter α , and we called it the alpha sine-G family (AS-G) of distributions.

The remaining sections of this study are organized as follows. The methodology of model development and some key functions of the family of distributions are introduced in Section 2. Some general properties of the AS-G family of distributions (AS-D FD) are presented in Section 3. In Section 4, a particular member of the AS-G family is introduced. A detailed study and application of this model are presented in Section 5. We discuss parametric estimation and simulation experiments in Sections 6 and 7. The applicability of the suggested model is presented in Section 8. Finally, we present the conclusion in Section 9.

2. The New Sine Family of Distributions

2.1. Methodology

To develop a new family of distributions, [17] defined a relation of $G(x; \varphi)$, the cumulative distribution function (CDF) of any baseline distribution and $r(t)$, the PDF of any arbitrary distribution, to obtain the CDF of the new family as

$$F(x; \varphi) = \int_0^{G(x; \varphi)} r(t) dt, \quad (2)$$

where $F(x; \varphi)$ is the CDF of the new class of distributions and φ is the parameter space of baseline distribution. To develop the new sine-G family, Equation (2) can be written as

$$F(x; \varphi) = \int_0^{\frac{\pi}{2} G(x; \varphi)} \cos(t) dt = \sin\left\{\frac{\pi}{2} G(x; \varphi)\right\} \quad x \in \mathbb{R}. \quad (3)$$

Using the structure of Equation (3), we introduce an additional parameter to Equation (3) and the new CDF of AS-G FD can be expressed as

$$F(x; \alpha, \varphi) = \frac{\sin\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}{\sin\left(\frac{\pi\alpha}{2}\right)}; \quad x \in \mathbb{R}, 0 < \alpha < 1. \quad (4)$$

The PDF corresponding to Equation (4) is

$$f(x; \alpha, \varphi) = \frac{\pi\alpha}{2 \sin\left(\frac{\pi\alpha}{2}\right)} g(x; \varphi) \cos\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}; \quad x \in \mathbb{R}, 0 < \alpha < 1, \quad (5)$$

where α is the scale parameter of the AS-G distribution.

Special Case of AS-G FD. When $\alpha = 1$ in the CDF of AS-G FD defined in Equation (4), it is reduced to the sine-G family defined by [2]. Hence, the sine-G family is a special case of AS-G FD.

2.2. Some Important Functional Forms of the New Sine Family of Distributions

In this subsection, we explicitly present some important functions that are necessary for survival analysis, reliability theory, etc.

- **Reliability function:** In probability theory, the reliability function is a function that offers the probability that a system or device will function correctly for a given amount of time, assuming that it has not failed up to that point. Intuitively, the reliability function offers the probability that the device or system will continue to function beyond time x given that it has not failed up to that point. The reliability function for AS-G FD can be expressed as

$$R(x; \alpha, \varphi) = 1 - \frac{\sin\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}{\sin\left(\frac{\pi\alpha}{2}\right)}; \quad x \in \mathbb{R}, 0 < \alpha < 1. \quad (6)$$

- **Hazard function:** In probability theory, the hazard function is a function that describes the rate at which an event occurs given that the event has not yet occurred up to a certain time. The hazard function is often used in survival analysis to model the failure rate of a system over time. The AS-G FD can be defined as

$$h(x) = \frac{\pi\alpha}{2} \frac{g(x; \varphi) \cos\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}{\sin\left(\frac{\pi\alpha}{2}\right) - \sin\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}; \quad x \in \mathbb{R}. \quad (7)$$

- **Odd function:** Odd functions are a useful tool in probability theory for describing certain types of distributions and for simplifying calculations involving them. Here, the odd function for AS-G FD can be expressed as

$$O(x) = \frac{\sin\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}{\sin\left(\frac{\pi\alpha}{2}\right) - \sin\left\{\frac{\pi\alpha}{2} G(x; \varphi)\right\}}; \quad x \in \mathbb{R}. \quad (8)$$

- **Failure rate average (FRA):** The failure rate average function has important applications in reliability engineering and survival analysis, where it is used to model the behavior of systems and estimate their probability of failure over time. It can also be

used to compare different systems' reliability and identify the factors that affect their failure rates.

$$K(x) = -\frac{1}{x} \left[\log \left\{ \sin \left(\frac{\pi\alpha}{2} \right) - \sin \left\{ \frac{\pi\alpha}{2} G(x; \varphi) \right\} \right\} - \log \left\{ \sin \left(\frac{\pi\alpha}{2} \right) \right\} \right]; \quad x \in \mathbb{R}. \quad (9)$$

3. Properties of the New Sine Family of Distributions

3.1. Linear Representation

One can derive useful linear expansions using exponentiated distributions, specifically the exponentiated-G (Exp-G) distribution with power parameter $z > 0$ which has the CDF:

$$G_z(x; \varphi) = [G(x; \varphi)]^z; \quad x \in \mathbb{R}, \quad (10)$$

where $x \in \mathbb{R}$. The corresponding PDF can be expressed as

$$g_z(x; \varphi) = z g(x; \varphi) [G(x; \varphi)]^{(z-1)}, \quad x \in \mathbb{R}. \quad (11)$$

These notations are used in the following discussion. Exponentiated distributions have well-known properties for a wide range of baseline CDF $G(x; \varphi)$ (for more information, see [5,18,19]). The linear representations of $F(x; \varphi)$ and $f(x; \varphi)$ in terms of Exp-G functions are shown in the following result. Using the Taylor expansion for trigonometric function $\sin(x)$, the CDF of AS-G FD can be expressed as

$$F^*(x; \alpha, \varphi) = \sum_{j=0}^{\infty} \Delta_j G^{2j+1}(x; \varphi), \quad (12)$$

where $\Delta_j = \frac{(-1)^j \left(\frac{\pi\alpha}{2}\right)^{2j+1}}{(2j+1)! \sin\left(\frac{\pi\alpha}{2}\right)}$. The PDF corresponding to Equation (12) can be calculated by differentiating it with respect to x ; we obtain

$$f^*(x; \alpha, \varphi) = \sum_{j=0}^{\infty} \Delta_j^* G^{2j}(x; \varphi) g(x; \varphi), \quad (13)$$

where $\Delta_j^* = \Delta_j(2j+1)$.

3.2. Critical Points of the New Sine Family of Distributions

By solving equation $\frac{f(x; \alpha, \varphi)}{dx} = 0$ for x , we can obtain the critical points of $f(x; \alpha, \varphi)$. Let the solution of this equation be x_1 , which can be calculated from

$$\sin\left(\frac{\pi\alpha}{2} G(x_1)\right) [g(x_1)]^2 + \cos\left(\frac{\pi\alpha}{2} G(x_1)\right) g'(x_1) = 0. \quad (14)$$

Similarly, the critical points for hazard function $h(x)$ can be obtained by solving the following equation for solution x_2 :

$$\left\{ \sin\left(\frac{\pi\alpha}{2}\right) - \sin\left(\frac{\pi\alpha}{2} G(x_2)\right) \right\} \left\{ \cos\left(\frac{\pi\alpha}{2} G(x_2)\right) g'(x_2) + \frac{\pi\alpha}{2} \sin\left(\frac{\pi\alpha}{2} G(x_2)\right) [g(x_2)]^2 \right\} = 0. \quad (15)$$

3.3. Quantile Function

The quantile function is useful in statistical analysis and modeling as it provides a way to estimate percentiles and other summary statistics of a probability distribution. Suppose $Q(p)$ is the smallest value of X for which the probability that X is less than or equal to that value is at least p . The quantile function of CDF $F(x; \alpha, \varphi)$ of AS-G FD can be obtained as

$$Q(p; \alpha, \varphi) = G^{-1} \left(\frac{2}{\pi\alpha} \arcsin \left\{ p \sin \left(\frac{\pi\alpha}{2} \right) \right\} \right); \quad p \in (0, 1). \quad (16)$$

Using Equation (16), we can calculate the median, upper and lower quartile, quartile deviation (QD), coefficient of QD, skewness, and kurtosis as presented in Table 1.

Table 1. Various measures based on quantiles.

Statistical Measure	Expression
Median	$G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.5 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)$
Lower Quartile	$G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.25 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)$
Upper Quartile	$G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.75 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)$
QD	$\frac{1}{2}\left[G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.75 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right) - G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.25 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)\right]$
Coefficient of QD	$\frac{\left[G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.75 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right) - G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.25 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)\right]}{\left[G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.75 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right) + G^{-1}\left(\frac{2}{\pi\alpha} \arcsin\left\{0.25 \sin\left(\frac{\pi\alpha}{2}\right)\right\}\right)\right]}$
Skewness [20]	$\frac{Q\left(\frac{3}{4}; \alpha, \varphi\right) - 2Q\left(\frac{1}{2}; \alpha, \varphi\right) + Q\left(\frac{1}{4}; \alpha, \varphi\right)}{Q\left(\frac{3}{4}; \alpha, \varphi\right) - Q\left(\frac{1}{4}; \alpha, \varphi\right)}$
Kurtosis [21]	$\frac{Q\left(\frac{7}{8}; \alpha, \varphi\right) - Q\left(\frac{5}{8}; \alpha, \varphi\right) - Q\left(\frac{1}{8}; \alpha, \varphi\right) + Q\left(\frac{3}{8}; \alpha, \varphi\right)}{Q\left(\frac{3}{4}; \alpha, \varphi\right) - Q\left(\frac{1}{4}; \alpha, \varphi\right)}$

3.4. Moments

In probability theory and statistics, moments of a random variable X are numerical quantities that measure various aspects of its probability distribution. The moments of X are calculated using the values of X and the PDF of X . The K^{th} moment about the origin can be calculated as

$$\mu'_k = \int_{-\infty}^{\infty} x^k f(x) dx. \quad (17)$$

Now, considering the integral and summation terms exist and are interchangeable, using the PDF defined in Equation (5), we can calculate the K^{th} moment as

$$\mu'_k = \sum_{j=0}^{\infty} \Delta_j^* \int_{-\infty}^{\infty} x^k G^{2j}(x; \varphi) g(x; \varphi) dx. \quad (18)$$

Further, the K^{th} moment can also be calculated using the quantile function (for more detail, see [22]) as

$$\mu'_k = \sum_{j=0}^{\infty} \Delta_j^* \int_0^1 x^{2j} \{Q(x; \alpha, \varphi)\}^k dx. \quad (19)$$

3.5. Moment Generating Function

Let $M_X(t)$ be the MGF of X . Using Equation (18), MGF can be defined as

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^m \Delta_j^*}{m!} \int_{-\infty}^{\infty} x^k G^{2j}(x; \varphi) g(x; \varphi) dx. \quad (20)$$

Similarly, using Equation (19), MGF can be expressed as

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^m \Delta_j^*}{m!} \int_0^1 x^{2j} \{Q(x; \alpha, \varphi)\}^k dx. \quad (21)$$

3.6. Mean Residual Life Function

Suppose t is the lifetime of a component or item; then, MRF can be obtained as

$$\mu_{MRF}(t) = \frac{1}{R(t)} \left[E(t) - \sum_{j=0}^{\infty} \Delta_j^* \int_0^t x G^{2j}(x; \varphi) g(x; \varphi) dx \right], \quad (22)$$

where $R(t)$ is the reliability function.

4. Alpha-Sine Weibull Distribution

Model Presentation

In this section, a particular model of AS-G FD is introduced, and we analyze this model briefly. To define the new member, we select $G(x; \varphi)$ as the CDF of Weibull distribution as

$$G(x; \delta, \lambda) = 1 - \exp \left[- \left(\frac{x}{\delta} \right)^\lambda \right]; \quad x \geq 0, \quad \delta, \lambda > 0. \quad (23)$$

The PDF corresponding to CDF (23) can be written as

$$g(x; \delta, \lambda) = \lambda \delta^{-\lambda} x^{\lambda-1} \exp \left[- \left(\frac{x}{\delta} \right)^\lambda \right]; \quad x \geq 0, \quad \delta, \lambda > 0. \quad (24)$$

Substituting Equation (23) in the CDF of AS-G FD defined in Equation (4), we obtain the new member distribution called the AS-Weibull (AS-W) distribution with CDF:

$$F(x; \alpha, \delta, \lambda) = \frac{1}{\sin(\frac{\pi\alpha}{2})} \sin \left\{ \frac{\pi\alpha}{2} \left[1 - \exp \left(- \left(\frac{x}{\delta} \right)^\lambda \right) \right] \right\}; \quad x \geq 0, \quad 0 < \alpha < 1, \delta, \lambda > 0. \quad (25)$$

The PDF of the AS-W distribution can be obtained by differentiating Equation (25) and can be expressed as

$$f(x; \alpha, \delta, \lambda) = \frac{\pi\alpha\lambda\delta^{-\lambda}}{2\sin(\frac{\pi\alpha}{2})} x^{\lambda-1} \exp \left[- \left(\frac{x}{\delta} \right)^\lambda \right] \cos \left\{ \frac{\pi\alpha}{2} \left[1 - \exp \left(- \left(\frac{x}{\delta} \right)^\lambda \right) \right] \right\}. \quad (26)$$

Similarly, the HRF of the AS-W distribution is given by

$$h(x; \alpha, \delta, \lambda) = \frac{\pi}{2} \alpha \lambda \delta^{-\lambda} x^{\lambda-1} \exp \left[- \left(\frac{x}{\delta} \right)^\lambda \right] \frac{\cos \left\{ \frac{\pi\alpha}{2} \left[1 - \exp \left(- \left(\frac{x}{\delta} \right)^\lambda \right) \right] \right\}}{\sin(\frac{\pi\alpha}{2}) - \sin \left\{ \frac{\pi\alpha}{2} \left[1 - \exp \left(- \left(\frac{x}{\delta} \right)^\lambda \right) \right] \right\}}. \quad (27)$$

We demonstrate the various shapes of PDF and HRF for varying two parameters keeping δ and λ constant, respectively; Figures 1 and 2. In Figure 3, we use all three parameters with different combinations. From all these graphical investigations, we find that the suggested model is versatile regarding skewness and kurtosis. Both PDF and HRF can have either

increasing or decreasing or bathtub or inverted bathtub or -j- or reverse-j-shaped curves according to parameter values. Hence, the AS-W model is capable of fitting highly skewed heterogeneous data sets.

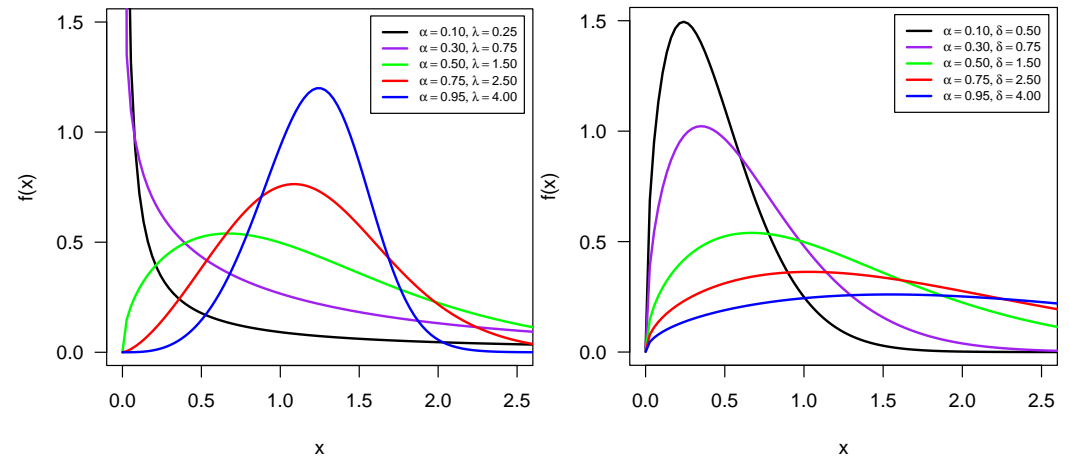


Figure 1. The plots of PDF keeping constant $\delta = 1.5$ (left) and $\lambda = 1.5$ (right).

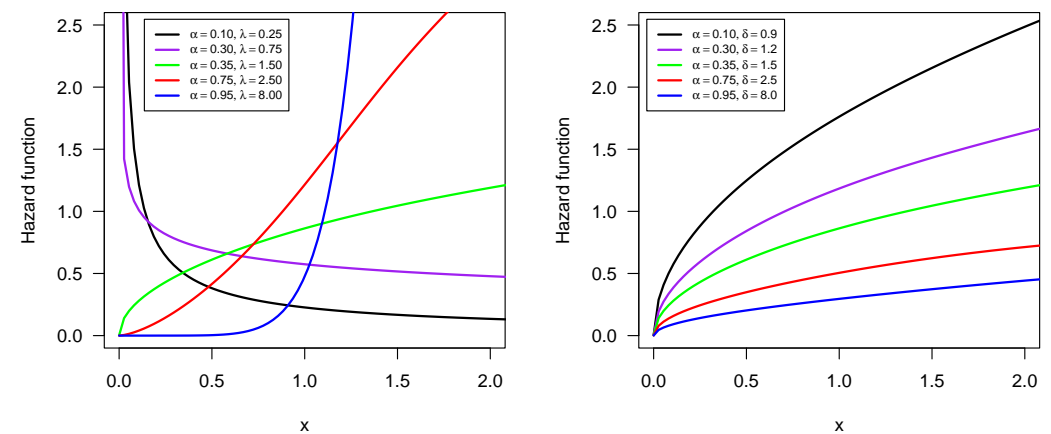


Figure 2. The plots of HRF keeping constant $\delta = 1.5$ (left) and $\lambda = 1.5$ (right).

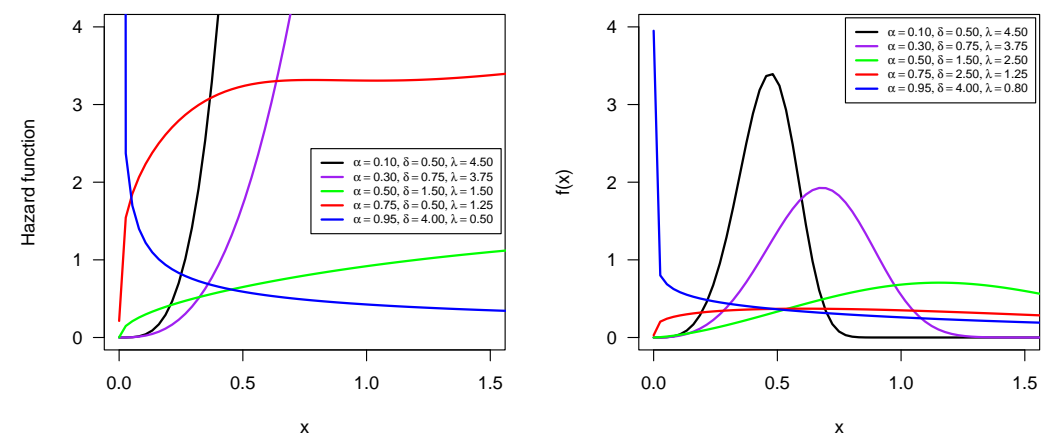


Figure 3. The plots of PDF and HRF with a variation of all three parameters.

5. Properties of the Alpha-Sine Weibull Distribution

5.1. Quantile Function

The QF can be used in statistical analysis and modeling to estimate probability distribution percentiles and other summary statistics. The QF for the AS-W distribution can be expressed as

$$Q(p; \alpha, \delta, \lambda) = \left[-\delta^\lambda \log \left\{ 1 - \frac{2}{\pi\alpha} \arcsin \left\{ p \sin \left(\frac{\pi\alpha}{2} \right) \right\} \right\} \right]^{\frac{1}{\lambda}}; \quad p \in (0, 1). \quad (28)$$

Using Equation (28), we can obtain various statistical measures provided in Table 1. Also, for generating random numbers to the distribution AS-W, we can use the following expression:

$$x = \left[-\delta^\lambda \log \left\{ 1 - \frac{2}{\pi\alpha} \arcsin \left\{ u \sin \left(\frac{\pi\alpha}{2} \right) \right\} \right\} \right]^{\frac{1}{\lambda}}; \quad u \in (0, 1). \quad (29)$$

Using the formulae defined by [20,21] for skewness and kurtosis using quantiles, we plotted the graphs of skewness and kurtosis with various combinations of parameter values presented in Figures 4 and 5.

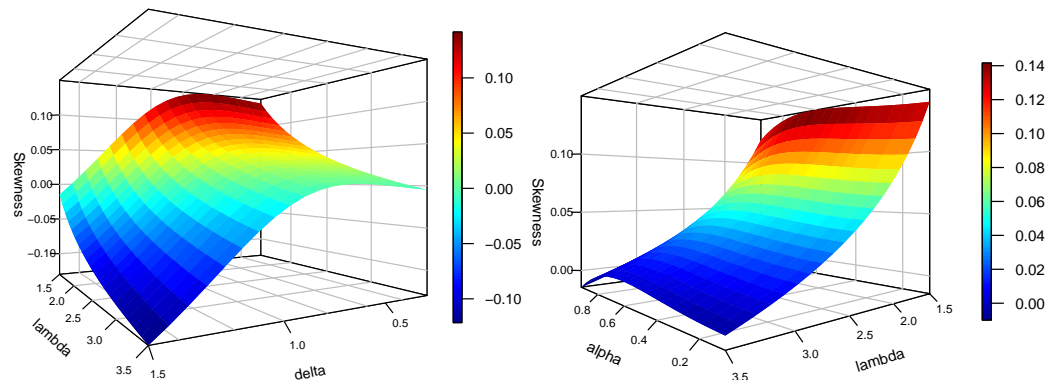


Figure 4. The plots of skewness with constant $\alpha = 0.5$ (left) and constant $\delta = 0.75$ (right).

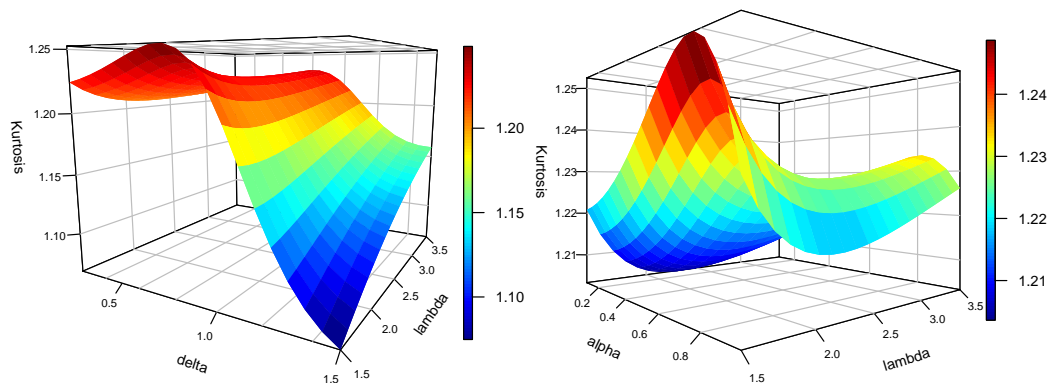


Figure 5. The plots of kurtosis with constant $\alpha = 0.5$ (left) and constant $\delta = 0.75$ (right).

5.2. Linear Expansion of Alpha-Sine Weibull Distribution

Using Equation (12), the expansion of the CDF defined in Equation (25) is given by

$$F^*(x; \alpha, \delta, \lambda) = \sum_{j=0}^{\infty} \Delta_j \left\{ 1 - \exp \left[- \left(\frac{x}{\delta} \right)^\lambda \right] \right\}^{(2j+1)}; \quad x \geq 0, \quad (30)$$

where $\Delta_j = \frac{(-1)^j \left(\frac{\pi\alpha}{2}\right)^{2j+1}}{(2j+1)! \sin\left(\frac{\pi\alpha}{2}\right)}$. Further using the binomial expansion, Equation (30) can be expressed as

$$F^*(x; \alpha, \delta, \lambda) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{jm}^* \exp\left[-m\left(\frac{x}{\delta}\right)^{\lambda}\right]; \quad x \geq 0, \quad (31)$$

where $\Delta_{jm}^* = \Delta_j (-1)^m \binom{2j+1}{m}$. The PDF corresponding to Equation (31) can be written as

$$f^*(x; \alpha, \delta, \lambda) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{jm}^{**} x^{\lambda-1} \exp\left[-(1+m)\left(\frac{x}{\delta}\right)^{\lambda}\right]; \quad x \geq 0, \quad (32)$$

where $\Delta_{jm}^{**} = \Delta_j (-1)^m \binom{2j}{m} \lambda \delta^{-\lambda}$.

5.3. Moments

The K^{th} moment of random variable $X \sim AS - W(\alpha, \delta, \lambda)$ can be obtained by using the following expression:

$$\mu'_k = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{jm}^{**} \frac{\Gamma\left(\frac{k+\lambda}{\lambda}\right)}{\left(\frac{1+m}{\delta^{\lambda}}\right)^{\frac{k+\lambda}{\lambda}}}. \quad (33)$$

5.4. Moment Generating Function of Alpha-Sine Weibull Distribution

The MGF of AS-W for any real number t can be expressed as

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Delta_{jm}^{**} \frac{t^n}{n!} \frac{\Gamma\left(\frac{k+\lambda}{\lambda}\right)}{\left(\frac{1+m}{\delta^{\lambda}}\right)^{\frac{k+\lambda}{\lambda}}}, \quad (34)$$

where $\Gamma(\cdot)$ is the gamma function.

5.5. Mean Waiting Time Function

Let t denote the waiting time or time to failure of an item or event; then, the MWT function can be defined as

$$\mu(t) = t - \frac{1}{F(t; \alpha, \delta, \lambda)} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \lambda^{-1} \Delta_{jm}^{**} \frac{\gamma\left(\frac{1}{\lambda}, \frac{1+m}{\delta^{\lambda}} t^{\lambda}\right)}{\left(\frac{1+m}{\delta^{\lambda}}\right)^{\frac{1}{\lambda}}}, \quad (35)$$

where $\gamma(\cdot)$ is the lower incomplete gamma function.

6. Estimation Methods

In this part of the work, we consider different methods for estimating the parameters of the AS-W distribution.

6.1. Maximum Likelihood Method

Consider a simple random sample $x = (x_1, x_2, \dots, x_m)$ of size m following the AS-W distribution; then, the likelihood function can be presented as

$$\begin{aligned}
L(x; \alpha, \delta, \lambda) &= \prod_{j=1}^m \left(\frac{\pi \alpha \lambda \delta^{-\lambda}}{2 \sin(\frac{\pi \alpha}{2})} \right) e^{-\left(\frac{x_j}{\delta}\right)^\lambda} (x_j)^{\lambda-1} \cos \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_j}{\delta}\right)^\lambda}\right)}{2} \right) \\
&= \left(\frac{\pi \alpha \lambda \delta^{-\lambda}}{2 \sin(\frac{\pi \alpha}{2})} \right)^m e^{-\sum_{j=1}^m \left(\frac{x_j}{\delta}\right)^\lambda} \prod_{j=1}^m (x_j)^{\lambda-1} \cos \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_j}{\delta}\right)^\lambda}\right)}{2} \right).
\end{aligned} \tag{36}$$

Hence, the corresponding log-likelihood function is given as

$$\begin{aligned}
\log L(x; \alpha, \delta, \lambda) &= m \log(\pi) + m \log(\alpha) + m \log(\lambda) - m \lambda \log(\delta) - m \log(2 \sin(\frac{\pi \alpha}{2})) \\
&\quad - \sum_{j=1}^m \left(\frac{x_j}{\delta}\right)^\lambda + (\lambda - 1) \sum_{j=1}^m \log(x_j) + \sum_{j=1}^m \log \left[\cos \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_j}{\delta}\right)^\lambda}\right)}{2} \right) \right].
\end{aligned}$$

The MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\delta}, \hat{\lambda})$ of $\Theta = (\alpha, \delta, \lambda)$ are obtained, respectively, using numerical methods.

6.2. Maximum Product of Spacings Method

Cheng and Amin [23] present this technique as a different method to MLE. It relies on the geometric mean of the spacings, which is

$$v_j(\alpha, \delta, \lambda) = F(x_{(j)} | \alpha, \delta, \lambda) - F(x_{(j-1)} | \alpha, \delta, \lambda), j = 1, \dots, m+1,$$

where $F(t_{(0)} | \alpha, \delta, \lambda) = 0$ and $F(t_{(m+1)} | \alpha, \delta, \lambda) = 1$. We can consider that $\sum_{j=1}^{m+1} v_j(\alpha, \delta, \lambda) = 1$.

The MPS estimators of $\Theta = (\alpha, \delta, \lambda)$ can be solved by increasing the geometric mean of the spacing,

$$\vartheta(\alpha, \delta, \lambda | x) = \left[\prod_{j=1}^{m+1} v_j(\alpha, \delta, \lambda) \right]^{\frac{1}{m+1}}, \tag{37}$$

or similarly by increasing the natural logarithm of the product spacing function of (37) given by

$$\psi(\alpha, \delta, \lambda | x) = \frac{1}{m+1} \sum_{j=1}^{m+1} \log v_j(\alpha, \delta, \lambda).$$

6.3. Least Squares Methods

Our study proposes two variants of least squares, Ordinary Least Squares (OLS) and Weighted Least Squares (WLS).

The OLS estimators can be determined by minimizing

$$\begin{aligned}\Delta(\alpha, \delta, \lambda|x) &= \sum_{j=1}^m \left[F(x_{(j)}|\alpha, \delta, \lambda) - \frac{j}{m+1} \right]^2 \\ &= \sum_{j=1}^m \left[\frac{\sin \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_{(j)}}{\delta} \right)^\lambda} \right)}{2} \right)}{\sin \left(\frac{\pi \alpha}{2} \right)} - \frac{j}{m+1} \right]^2.\end{aligned}$$

However, the WLS estimators can be obtained by minimizing

$$\begin{aligned}\Delta_W(\alpha, \delta, \lambda|x) &= \sum_{j=1}^m \frac{(m+1)^2(m+2)}{j(m-j+1)} \left[F(x_{(j)}|\alpha, \delta, \lambda) - \frac{j}{m+1} \right]^2 \\ &= \sum_{j=1}^m \frac{(m+1)^2(m+2)}{j(m-j+1)} \left[\frac{\sin \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_{(j)}}{\delta} \right)^\lambda} \right)}{2} \right)}{\sin \left(\frac{\pi \alpha}{2} \right)} - \frac{j}{m+1} \right]^2.\end{aligned}$$

6.4. Minimum Distance Methods

Various methods have been proposed based on the minimization of empirical distribution functions and estimated distribution functions. This work uses the Cramer–Von–Mises (CV) and Anderson–Darling (AD) methods. We start with a CV estimator, and we can derive these estimators by minimizing the following functions:

$$\begin{aligned}\zeta(\alpha, \delta, \lambda|x) &= \frac{1}{12m} + \sum_{j=1}^m \left[F(x_{(j)}; \alpha, \delta, \lambda) - \frac{2j-1}{2m} \right]^2 \\ &= \frac{1}{12m} + \sum_{j=1}^m \left[\frac{\sin \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x_{(j)}}{\delta} \right)^\lambda} \right)}{2} \right)}{\sin \left(\frac{\pi \alpha}{2} \right)} - \frac{2j-1}{2m} \right]^2,\end{aligned}$$

moreover, the AD estimators are determined by minimizing

$$\begin{aligned} \zeta(\alpha, \delta, \lambda|x) = & -m - \frac{1}{m} \sum_{j=1}^m (2j-1) \left\{ \log \left[\frac{\sin \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{x(j)}{\delta} \right)^\lambda}{2} \right)}{\sin \left(\frac{\pi \alpha}{2} \right)} \right)}{1 - \frac{\sin \left(\frac{\pi \alpha \left(1 - e^{-\left(\frac{t(m+1-j)}{\delta} \right)^\lambda}{2} \right)}{\sin \left(\frac{\pi \alpha}{2} \right)} \right)} \right] \right\}. \end{aligned} \quad (38)$$

7. Numerical Simulation

To compare unknown parameter estimates of the (AS-W) distribution, a simulation study is conducted with different parameters, and several sample sizes $m = 30, 60, 100, 150, 200$, and 500 are presented. Based on 1000 runs, we compute the average estimate (AE) and mean square error (MSE), which are considered to be the optimality criteria.

From Tables 2–5, the following is clear from the numerical experiments:

- Based on all estimation methods, the average estimate converges to the true values, which shows that these estimators are consistent.
- The AE tends to its initial values as the sample size increase, so we can say that our estimates are unbiased.
- For all methods, whenever the MSEs decrease, the sample size m increases.
- The MLE estimators perform better than all the other methods considered in this work.

Table 2. The AES and MSEs of $\alpha = 0.4, \delta = 2, \lambda = 3$.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
30	$\hat{\alpha}$	0.4282	0.2229	0.5727	0.2056	0.4468	0.2089	0.4707	0.2165	0.3965	0.2279	0.4661	0.2144
	$\hat{\delta}$	2.0909	0.3833	2.2162	0.4673	2.1140	0.2179	2.1152	0.1477	2.0552	0.0475	2.1046	0.1377
	$\hat{\lambda}$	3.1459	0.3697	2.7929	0.2730	2.9844	0.3670	3.0017	0.3144	3.2108	0.4036	3.0294	0.2770
60	$\hat{\alpha}$	0.5516	0.0155	0.5333	0.1673	0.4501	0.1589	0.4570	0.1585	0.3957	0.1626	0.4503	0.1587
	$\hat{\delta}$	2.0105	0.0031	2.1136	0.0490	2.0668	0.0376	2.0637	0.0282	2.0285	0.0191	2.0594	0.0284
	$\hat{\lambda}$	3.0861	0.0258	2.8456	0.1225	2.9515	0.1485	2.9718	0.1242	3.0802	0.1362	2.9820	0.1158
100	$\hat{\alpha}$	0.5429	0.0129	0.5160	0.1593	0.4173	0.1547	0.4282	0.1496	0.3918	0.1458	0.4378	0.1519
	$\hat{\delta}$	2.0054	0.0016	2.0937	0.0366	2.0448	0.0194	2.0446	0.0189	2.0211	0.0134	2.0490	0.0206
	$\hat{\lambda}$	3.0741	0.0222	2.8854	0.0665	2.9601	0.0819	2.9766	0.0678	3.0436	0.0701	2.9799	0.0655

Table 2. Cont.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
150	$\hat{\alpha}$	0.5333	0.0100	0.5197	0.1487	0.4428	0.1360	0.4476	0.1369	0.4175	0.1318	0.4463	0.1386
	$\hat{\delta}$	2.0021	0.0006	2.0819	0.0287	2.0388	0.0154	2.0411	0.0155	2.0230	0.0125	2.0409	0.0157
	$\hat{\lambda}$	3.0633	0.0190	2.9047	0.0439	2.9641	0.0539	2.9732	0.0442	3.0188	0.0451	2.9764	0.0427
200	$\hat{\alpha}$	0.5237	0.0071	0.5464	0.1465	0.4247	0.1372	0.4404	0.1344	0.4234	0.1303	0.4439	0.1322
	$\hat{\delta}$	2.0003	0.0001	2.0910	0.0290	2.0323	0.0129	2.0366	0.0137	2.0256	0.0121	2.0366	0.0135
	$\hat{\lambda}$	3.0531	0.0159	2.9242	0.0342	2.9802	0.0411	2.9868	0.0325	3.0209	0.0342	2.9892	0.0327
500	$\hat{\alpha}$	0.5027	0.0008	0.4988	0.1292	0.4153	0.1220	0.4110	0.1198	0.3952	0.1173	0.4115	0.1188
	$\hat{\delta}$	2.0000	0.0000	2.0624	0.0207	2.0232	0.0096	2.0205	0.0095	2.0122	0.0081	2.0200	0.0094
	$\hat{\lambda}$	3.0297	0.0089	2.9533	0.0145	2.9794	0.0166	2.9861	0.0138	3.0008	0.0137	2.9871	0.0136

Table 3. The AES and MSEs of $\alpha = 0.4, \delta = 1, \lambda = 1.5$.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
30	$\hat{\alpha}$	0.2849	0.1535	0.4189	0.1723	0.3544	0.1660	0.3813	0.1785	0.3276	0.1644	0.3751	0.1642
	$\hat{\delta}$	1.0271	0.0322	1.1228	0.2186	1.0844	0.3253	1.0866	0.0735	1.0361	0.0235	1.0658	0.0312
	$\hat{\lambda}$	1.5828	0.0631	1.4216	0.0535	1.4915	0.0732	1.5027	0.0642	1.6071	0.0870	1.5224	0.0585
60	$\hat{\alpha}$	0.4456	0.0091	0.4559	0.1658	0.3692	0.1467	0.3912	0.1513	0.3620	0.1387	0.3953	0.1506
	$\hat{\delta}$	1.0211	0.0049	1.1065	0.0401	1.0575	0.0192	1.0652	0.0222	1.0406	0.0167	1.0647	0.0230
	$\hat{\lambda}$	1.5492	0.0098	1.4396	0.0255	1.4911	0.0357	1.4999	0.0295	1.5527	0.0348	1.5045	0.0271
100	$\hat{\alpha}$	0.4443	0.0066	0.4371	0.1590	0.3933	0.1348	0.3974	0.1376	0.3694	0.1267	0.4026	0.1380
	$\hat{\delta}$	1.0214	0.0032	1.0922	0.0333	1.0584	0.0173	1.0603	0.0187	1.0403	0.0139	1.0611	0.0192
	$\hat{\lambda}$	1.5381	0.0057	1.4635	0.0143	1.4995	0.0197	1.5059	0.0160	1.5389	0.0182	1.5087	0.0157
150	$\hat{\alpha}$	0.4456	0.0068	0.4645	0.1582	0.3779	0.1307	0.3865	0.1320	0.3649	0.1234	0.3895	0.1330
	$\hat{\delta}$	1.0136	0.0020	1.0971	0.0329	1.0486	0.0138	1.0509	0.0149	1.0363	0.0116	1.0520	0.0157
	$\hat{\lambda}$	1.5297	0.0045	1.4626	0.0106	1.4918	0.0140	1.4976	0.0114	1.5198	0.0122	1.4988	0.0110
200	$\hat{\alpha}$	0.4410	0.0061	0.4412	0.1528	0.3754	0.1280	0.3748	0.1271	0.3583	0.1200	0.3764	0.1296
	$\hat{\delta}$	1.0118	0.0018	1.0898	0.0296	1.0505	0.0129	1.0492	0.0137	1.0379	0.0111	1.0508	0.0146
	$\hat{\lambda}$	1.5245	0.0037	1.4695	0.0080	1.4901	0.0101	1.4969	0.0082	1.5137	0.0086	1.4972	0.0080
500	$\hat{\alpha}$	0.4380	0.0057	0.4484	0.1420	0.3764	0.1161	0.3778	0.1151	0.3629	0.1094	0.3789	0.1153
	$\hat{\delta}$	1.0012	0.0002	1.0815	0.0264	1.0413	0.0110	1.0416	0.0121	1.0330	0.0098	1.0418	0.0122
	$\hat{\lambda}$	1.5115	0.0017	1.4802	0.0034	1.4932	0.0039	1.4960	0.0032	1.5034	0.0032	1.4964	0.0032

Table 4. The AES and MSEs of $\alpha = 0.6, \delta = 2, \lambda = 1$.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
30	$\hat{\alpha}$	0.6597	0.0195	0.5390	0.2054	0.5081	0.1784	0.5212	0.1858	0.4882	0.1821	0.5191	0.1895
	$\hat{\delta}$	2.0642	0.1340	2.6170	8.4422	2.3168	2.3140	2.3242	1.8946	2.2091	2.7088	2.2942	1.1570
	$\hat{\lambda}$	1.0537	0.0140	0.9369	0.0276	0.9900	0.0380	0.9979	0.0330	1.0665	0.0433	1.0081	0.0285
60	$\hat{\alpha}$	0.6494	0.0099	0.5570	0.1611	0.4988	0.1622	0.5099	0.1605	0.4692	0.1636	0.5016	0.1645
	$\hat{\delta}$	2.0634	0.0127	2.2948	0.4264	2.1652	0.3995	2.1645	0.2667	2.0678	0.1926	2.1487	0.2479
	$\hat{\lambda}$	1.0276	0.0055	0.9435	0.0135	0.9803	0.0164	0.9865	0.0139	1.0219	0.0152	0.9883	0.0125
100	$\hat{\alpha}$	0.6442	0.0088	0.5808	0.1450	0.5099	0.1456	0.5104	0.1474	0.4700	0.1500	0.5034	0.1487
	$\hat{\delta}$	2.0510	0.0102	2.2622	0.3124	2.1219	0.2023	2.1178	0.1835	2.0400	0.1417	2.1039	0.1735
	$\hat{\lambda}$	1.0250	0.0050	0.9625	0.0075	0.9858	0.0086	0.9913	0.0073	1.0135	0.0078	0.9944	0.0070

Table 4. Cont.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
150	$\hat{\alpha}$	0.6378	0.0076	0.5631	0.1385	0.4894	0.1464	0.4867	0.1480	0.4572	0.1500	0.4871	0.1483
	$\hat{\delta}$	2.0552	0.0110	2.2113	0.2558	2.0807	0.1505	2.0728	0.1324	2.0186	0.1090	2.0729	0.1341
	$\hat{\lambda}$	1.0140	0.0028	0.9649	0.0054	0.9829	0.0063	0.9873	0.0051	1.0024	0.0051	0.9883	0.0050
200	$\hat{\alpha}$	0.6272	0.0054	0.5662	0.1418	0.4872	0.1449	0.4817	0.1453	0.4546	0.1477	0.4877	0.1450
	$\hat{\delta}$	2.0500	0.0100	2.2230	0.2529	2.0804	0.1423	2.0684	0.1301	2.0218	0.1096	2.0768	0.1367
	$\hat{\lambda}$	1.0108	0.0022	0.9718	0.0041	0.9895	0.0048	0.9929	0.0039	1.0045	0.0040	0.9931	0.0038
500	$\hat{\alpha}$	0.608	0.0016	0.5878	0.1189	0.4732	0.1353	0.4847	0.1241	0.4688	0.1234	0.4885	0.1239
	$\hat{\delta}$	2.000	0.0000	2.2199	0.2269	2.0446	0.0999	2.0459	0.0965	2.0175	0.0843	2.0511	0.1001
	$\hat{\lambda}$	1.000	0.0000	0.9808	0.0018	0.9936	0.0018	0.9949	0.0015	1.0000	0.0015	0.9949	0.0015

Table 5. The AES and MSEs of $\alpha = 0.7, \delta = 3, \lambda = 2.5$.

Sample Size		MLE		MPS		LSE		WLS		CVE		ADE	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
30	$\hat{\alpha}$	0.7483	0.0274	0.7345	0.1700	0.6787	0.1822	0.6691	0.1730	0.5878	0.2083	0.6681	0.1844
	$\hat{\delta}$	3.0782	0.0368	3.5572	3.0963	3.4365	2.7615	3.3212	1.4609	3.1723	0.7521	3.3279	1.4390
	$\hat{\lambda}$	2.5959	0.0441	2.2631	0.2335	2.4062	0.2695	2.4325	0.2375	2.6046	0.2761	2.4583	0.2202
60	$\hat{\alpha}$	0.7210	0.0063	0.6983	0.1259	0.6762	0.1393	0.6420	0.1358	0.5789	0.1566	0.6587	0.1363
	$\hat{\delta}$	3.0507	0.0152	3.2034	0.1991	3.1919	0.3088	3.1130	0.1402	3.0399	0.1163	3.1341	0.1548
	$\hat{\lambda}$	2.5813	0.0244	2.3468	0.1024	2.4330	0.1317	2.4604	0.1028	2.5513	0.1109	2.4620	0.0987
100	$\hat{\alpha}$	0.7054	0.0016	0.6886	0.1179	0.6692	0.1197	0.6293	0.1264	0.5736	0.1445	0.6494	0.1217
	$\hat{\delta}$	3.0354	0.0106	3.1517	0.1021	3.1244	0.1366	3.0739	0.0787	3.0172	0.0689	3.0901	0.0833
	$\hat{\lambda}$	2.5756	0.0227	2.3837	0.0631	2.4531	0.0757	2.4717	0.0618	2.5291	0.0638	2.4711	0.0590
150	$\hat{\alpha}$	0.7027	0.0008	0.6884	0.1119	0.6650	0.1186	0.6020	0.1345	0.5654	0.1444	0.6330	0.1253
	$\hat{\delta}$	3.0246	0.0074	3.1333	0.0846	3.1112	0.1146	3.0444	0.0621	3.0061	0.0571	3.0696	0.0679
	$\hat{\lambda}$	2.5552	0.0166	2.3985	0.0393	2.4481	0.0500	2.4685	0.0377	2.5075	0.0375	2.4649	0.0371
200	$\hat{\alpha}$	0.7006	0.0002	0.6686	0.1123	0.6532	0.1147	0.5924	0.1309	0.5587	0.1392	0.6294	0.1191
	$\hat{\delta}$	3.0168	0.0050	3.1048	0.0724	3.0849	0.0863	3.0256	0.0514	2.9926	0.0492	3.0551	0.0578
	$\hat{\lambda}$	2.5450	0.0135	2.4154	0.0282	2.4574	0.0366	2.4756	0.0275	2.5059	0.0279	2.4706	0.0272
500	$\hat{\alpha}$	0.7000	0.0000	0.6829	0.0887	0.6505	0.0995	0.5927	0.1110	0.5671	0.1176	0.6345	0.0976
	$\hat{\delta}$	3.0000	0.0000	3.0887	0.0561	3.0578	0.0586	3.0039	0.0359	2.9809	0.0339	3.0369	0.0428
	$\hat{\lambda}$	2.5126	0.0038	2.4351	0.0132	2.4620	0.0144	2.4740	0.0107	2.4876	0.0101	2.4687	0.0112

8. Applications

In this part of the work, we provide two application datasets to show the effectiveness and flexibility of the AS-W distribution. Different statistic measures for the two data sets are presented in Table 6.

Table 6. Summary statistics for the selected datasets.

Datasets	Minimum	One Quntile	Median	Mean	Three Quntile	Maximum	Skew	Kurt
Dataset 1	0.070	1.170	2.490	3.494	5.840	13.300	1.152	3.890
Dataset 2	2.998	21.187	51.385	55.123	75.435	138.500	0.555	2.108

8.1. First Data Set

The first data set represents the total annual rainfall (in inches) during the month of January from 1880 to 1916 recorded at Los Angeles Civic Center; ref. [24] provided the values. The data are reported in Table 7.

Table 7. The total annual rainfall.

1.33	1.43	1.01	1.62	3.15	1.05	7.72	0.2	6.03	0.25	7.83	0.25	0.88	6.29	0.94
5.84	3.23	3.7	1.26	2.64	1.17	2.49	1.62	2.1	0.14	2.57	3.85	7.02	5.04	7.27
1.53	6.7	0.07	2.01	10.35	5.42	13.3								

8.2. Second DataSet

The second data set is the failure times of eight components at three different temperatures, 100, 120, 140, introduced by [25]. The value data are described Table 8.

Table 8. The values of the failure times of eight components at three different temperatures.

14.712	32.644	61.979	65.521	105.50	114.60	120.40
138.50	8.610	11.741	54.535	55.047	58.928	63.391
105.18	113.02	2.998	5.016	15.628	23.040	27.851
37.843	38.050	48.226				

The AS-W distribution is fitted to these two datasets and compared with the following:

- Sine-inverse Weibull [4]:

$$F(x, \alpha, \theta) = \sin\left\{\frac{\pi}{2}e^{(-\alpha x^{-\theta})}\right\}.$$

$$f(x; \alpha, \theta) = \frac{\pi}{2}\alpha\theta x^{-\theta-1}e^{(-\alpha x^{-\theta})}\cos\left\{\frac{\pi}{2}e^{(-\alpha x^{-\theta})}\right\} \quad x > 0, \alpha, \theta > 0.$$

- The inverse Weibull distribution [26]:

$$F(x, \tau, \theta) = e^{-\left(\frac{\theta}{x}\right)^\tau}.$$

$$f(x; \tau, \theta) = f(x) = \frac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} \quad x > 0, \tau, \theta > 0.$$

- Weighted generalized quasi Lindley distribution (WGQLD) [27]:

$$F(x, \alpha, \theta) = 1 - \frac{24 + 6\alpha^2[2 + x\theta(2 + x\theta)] + 6\alpha[6 + x\theta(6 + x\theta(3 + x\theta))] + x\theta[24 + x\theta(12 + x\theta(4 + x\theta))]}{12(1 + \alpha)(2 + \alpha)}e^{-\theta x}.$$

$$f(x; \alpha, \theta) = \frac{\theta^3 x^2 \cdot (\theta^2 x^2 + 6\alpha\theta x + 6\alpha^2)e^{-\theta x}}{12(\alpha + 1)(\alpha + 2)} \quad x > 0, \alpha, \theta > 0.$$

- Sine Burr XII distribution [11]:

$$F(x) = \sin\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1 + x^a)^b}\right]\right\} : a, b, x > 0.$$

$$f(x) = \frac{\pi}{2} \frac{abx^{a+1}}{(1 + x^a)^{b+1}} \cos\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1 + x^a)^b}\right]\right\}, a, b, x > 0.$$

The MLEs, SEs and corresponding log-likelihood $l(\cdot)$ values for AS-G FD model for both datasets are provided in Table 9. For the decision about the best fitting of the competing model, we computed several criteria measures such as the Akaike information criteria (AIC), the consistent Akaike information criteria (CAIC), the Bayesian information criteria (BIC), and the Hannan–Quinn information criteria (HQIC).

Table 9. The MLEs, SEs and corresponding log-likelihood $l(\cdot)$ values for the AS-G FD model.

Datasets	Estimate	SE	$l(x; \cdot)$
Dataset 1	$\hat{\alpha} = 0.0003$	1.1977	−83.265
	$\hat{\lambda} = 1.0495$	0.1381	
	$\hat{\delta} = 3.55905$	0.5862	
Dataset 2	$\hat{\alpha} = 0.002$	1.083	−119.119
	$\hat{\lambda} = 59.518$	9.820	
	$\hat{\delta} = 1.300$	0.216	

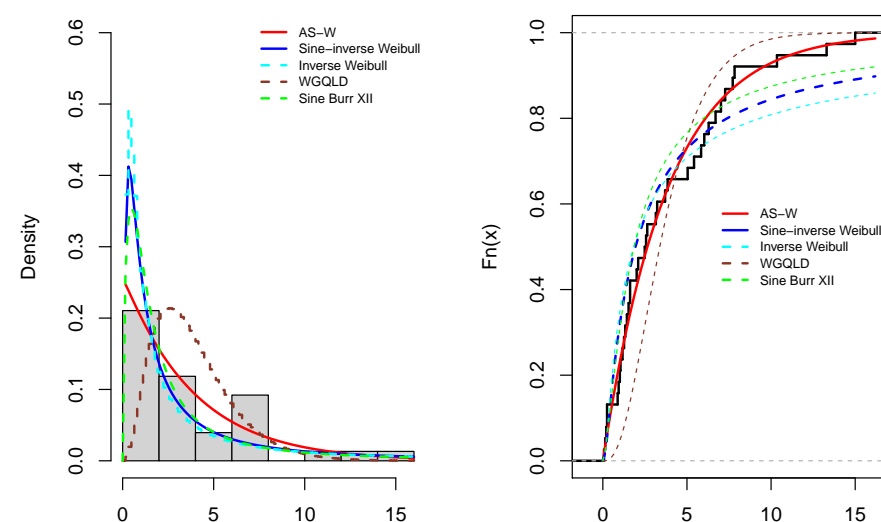
From the results given in Tables 10 and 11, we noted that the AS-W model provides a better fit with the minimum value of AIC, AICc, BIC, HQIC, and KS and the largest p -values compared with other models considered in this work. Figures 6 and 7 support this assertion. Box plot and TTT plot along with the PP-Plot for the two real data sets are, respectively, presented in Figures 8 and 9.

Table 10. The goodness of fit tests for Dataset 1.

Model	AIC	AICc	BIC	HQIC	K-S	p -Value
AS-W	172.5304	173.2577	177.3632	174.2342	0.0907	0.9212
Sine-inverse Weibull	184.3137	184.6666	187.5355	185.4495	0.15862	0.3096
Inverse Weibull	190.8537	191.2066	194.0755	191.9896	0.1897	0.1394
WGQLD	206.7907	207.1436	210.0125	207.9265	0.2682	0.0097
Sine Burr XII	181.3963	181.7493	184.6181	182.5322	0.1423	0.4417

Table 11. The goodness of fit tests for Dataset 2.

Model	AIC	AICc	BIC	HQIC	K-S	p -Value
AS-W	244.239	245.439	247.7732	245.1767	0.1271	0.7875
Sine-inverse Weibull	251.187	251.7585	253.5431	251.8121	0.1546	0.5622
Inverse Weibull	255.0592	255.6306	257.4153	255.6843	0.1778	0.3884
WGQLD	252.8124	253.3839	255.1686	253.4375	0.1950	0.2824
Sine Burr XII	284.8518	285.4232	287.2079	285.4768	0.3609	0.0026

**Figure 6.** Plots of estimated probability density functions and cumulative distribution functions for Dataset 1.

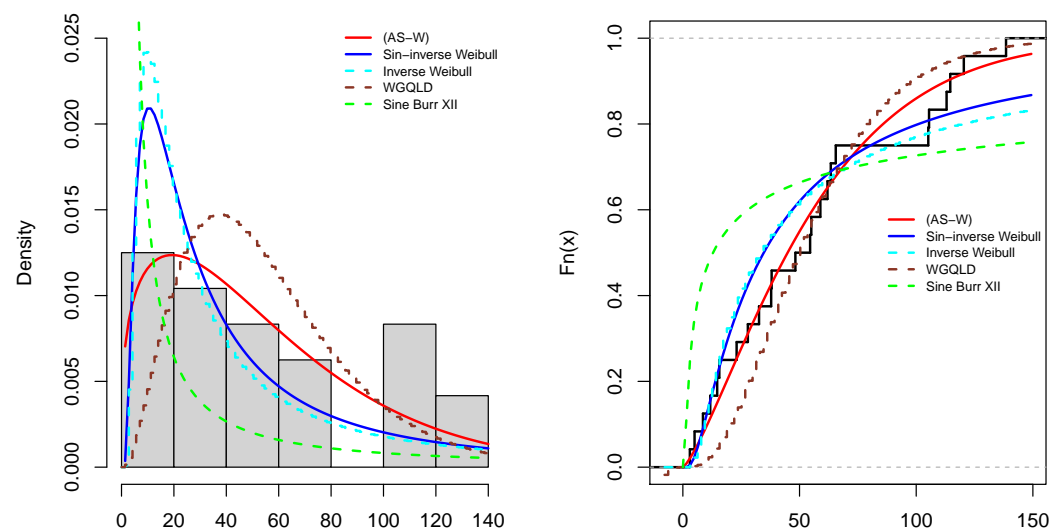


Figure 7. Plots of estimated probability density functions and cumulative distribution functions for Dataset 2.

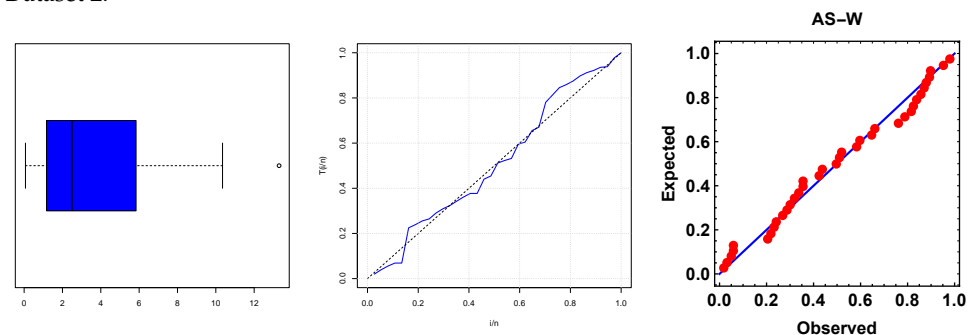


Figure 8. Box, TTT, and PP plots for the first real data set.

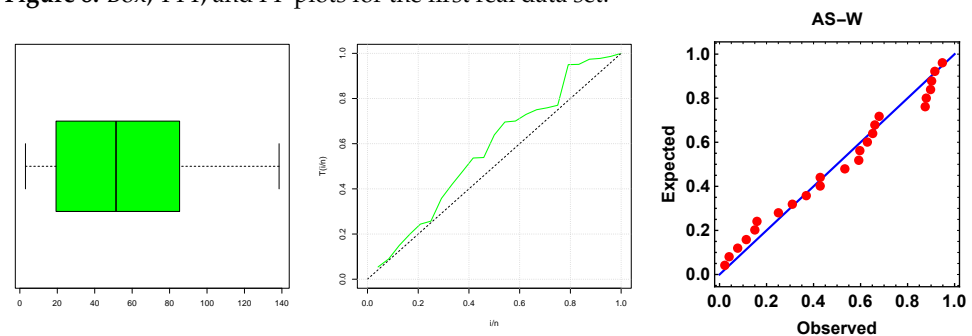


Figure 9. Box, TTT, and PP plots for the second real data set.

9. Conclusions

We intensely study a new family of distributions with a trigonometric function. We introduce an extra parameter to the sine transformation family and name it the alpha-sine-G family of distributions. Some important functional forms and properties of the family are provided in a general form. A specific three-parameter sub-model alpha-sine Weibull of this family is also introduced using Weibull distribution as a parent distribution; it is studied deeply. The statistical properties of this new distribution are investigated. From the graphical investigations of the PDF and HRF shapes, we find that the suggested model is versatile regarding skewness and kurtosis. Both the PDF and HRF can have either increasing or decreasing or bathtub or inverted bathtub or -j- or reverse-j-shaped curves according to the parameter values. Hence, the AS-W model is also capable of fitting highly skewed heterogeneous data sets. We obtain the estimates of AS-W parameters using several methods, including MLE, MPS, OLS, WLS, CV, and AD. A simulation experiment is

carried out to justify these estimates further and finds that AEs nearly converge to the true values of the parameter, and MSEs are approaching zero as the sample size increases. We study two real data applications and demonstrate that the AS-W distribution is consistently the best model among all its competitors. Hence, we expect that the suggested family of distributions can be used to generate new flexible models for modeling real data, even heterogeneous data from different fields of application. For future works, many authors can use the new suggested family of distributions to generate new continuous statistical models, such as alpha-sine-power Lomax, alpha-sine-power Topp Leone and alpha-sine-power Lindley distributions.

Author Contributions: Conceptualization, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; methodology, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; software, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; validation, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; formal analysis, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; investigation, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; resources, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; data curation, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; writing—original draft preparation, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; writing—review and editing, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; visualization, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; supervision, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; project administration, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N.; funding acquisition, S.B., L.P.S., A.A.M., V.K., R.H.K., A.M.G., M.E. and S.G.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research has no funds.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Tarantola, A. *Inverse Problem Theory and Methods for Model Parameter Estimation*; SIAM: Philadelphia, PA, USA, 2005.
2. Kumar, D.; Singh, U.; Singh, S.K. A new distribution using sine function-its application to bladder cancer patients data. *J. Stat. Appl. Probab.* **2015**, *4*, 417.
3. Gómez-Déniz, E.; Calderín-Ojeda, E. Modelling insurance data with the Pareto ArcTan distribution. *ASTIN Bull. J. IAA* **2015**, *45*, 639–660. [\[CrossRef\]](#)
4. Souza, L.; Junior, W.R.O.; de Brito, C.C.R.; Ferreira, T.A.E.; Soares, L.G.M. General properties for the Cos-G class of distributions with applications. *Eurasian Bull. Math.* **2019**, *2*, 63–79.
5. Souza, L.; Junior, W.; De Brito, C.; Chesneau, C.; Ferreira, T.; Soares, L. On the Sin-G class of distributions: Theory, model and application. *J. Math. Model.* **2019**, *7*, 357–379.
6. Mahmood, Z.; Chesneau, C.; Tahir, M.H. A new sine-G family of distributions: Properties and applications. *Bull. Comput. Appl. Math.* **2019**, *7*, 53–81.
7. Chesneau, C.; Jamal, F. The sine Kumaraswamy-G family of distributions. *J. Math. Ext.* **2020**, *15*, 1–26.
8. Muhammad, M.; Alshanbari, H.M.; Alanzi, A.R.A.; Liu, L.; Sami, W.; Chesneau, C.; Jamal, F. A new generator of probability models: The exponentiated sine-G family for lifetime studies. *Entropy* **2021**, *23*, 1394. [\[CrossRef\]](#)
9. Rahman, M.M. Arcsine-G Family of Distributions. *J. Stat. Appl. Probab. Lett.* **2021**, *8*, 169–179.
10. Chaudhary, A.K.; Sapkota, L.P.; Kumar, V. Some properties and applications of arctan generalized exponential distribution. *Int. J. Innov. Res. Sci. Eng. Technol. IJIRSET* **2021**, *10*, 456–468.
11. Isa, A.M.; Ali, B.A.; Zannah, U. Sine Burr XII Distribution: Properties and Application to Real Data Sets. *Arid. Zone J. Basic Appl. Res.* **2022**, *1*, 48–58. [\[CrossRef\]](#)
12. Muhammad, M.; Bantan, R.; Liu, L.; Chesneau, C.; Tahir, M.; Jamal, F.; Elgarhy, M. A New Extended Cosine-G Distributions for Lifetime Studies. *Mathematics* **2021**, *9*, 2758. [\[CrossRef\]](#)
13. Aldahlan, M.; Jamal, F.; Chesneau, C.; Elgarhy, M.; Elbatal, I. The truncated Cauchy power family of distributions with inference and applications. *Entropy* **2020**, *22*, 346. [\[CrossRef\]](#)
14. Alotaibi, N.; Elbatal, I.; Almetwally, E.; Alyami, S.; Al-Moisheer, A.; Elgarhy, M. Truncated Cauchy Power Weibull-G Class of Distributions: Bayesian and Non-Bayesian Inference Modelling for COVID-19 and Carbon Fiber Data. *Mathematics* **2022**, *10*, 1565. [\[CrossRef\]](#)
15. Shrahili, M.; Elbatal, I.; Elgarhy, M. Sine Half-Logistic Inverse Rayleigh Distribution: Properties, Estimation, and Applications in Biomedical Data. *J. Math.* **2021**, *2021*, 4220479. [\[CrossRef\]](#)
16. Shrahili, M.; Elbatal, I.; Almutiry, W.; Elgarhy, M. Estimation of Sine Inverse Exponential Model under Censored Schemes. *J. Math.* **2021**, *2021*, 7330385. [\[CrossRef\]](#)

17. Eugene, N.; Lee, C.; Famoye, F. Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* **2002**, *31*, 497–512. [[CrossRef](#)]
18. Nadarajah, S.; Gupta, A.K. The exponentiated gamma distribution with application to drought data. *Calcutta Stat. Assoc. Bull.* **2007**, *59*, 29–54. [[CrossRef](#)]
19. Lemonte, A.J.; Barreto-Souza, W.; Cordeiro, G.M. The exponentiated Kumaraswamy distribution and its log-transform. *Braz. J. Probab. Stat.* **2013**, *27*, 31–53. [[CrossRef](#)]
20. Kenney, J.F.; Keeping, E.S. *Mathematics of Statistics*; D. Van Nostrand Company, Inc.: New York, NY, USA, 1962.
21. Moors, J.J.A. A quantile alternative for kurtosis. *J. R. Stat. Soc. Ser. D Stat.* **1988**, *37*, 25–32. [[CrossRef](#)]
22. Balakrishnan, N.; Cohen, A.C. *Order Statistics and Inference: Estimation Methods*; Academic Press: San Diego, CA, USA, 1991.
23. Cheng, R.C.H.; Amin, N.A.K. Estimating parameters in continuous univariate distributions with a shifted origin. *J. R. Stat. Soc. Ser. B Methodol.* **1983**, *45*, 394–403. [[CrossRef](#)]
24. Selim, M.A. Estimation and prediction for Nadarajah-Haghighi distribution based on record values. *Pak. J. Stat.* **2018**, *34*, 77–90. [[CrossRef](#)]
25. Murthy, D.P.; Xie, M.; Jiang, R. *Weibull Models*; John Wiley & Sons: Hoboken, NJ, USA, 2004.
26. Keller, A.Z.; Goblin, M.T.; Farnworth, N.R. Reliability analysis of commercial vehicle engines. *Reliab. Eng.* **1985**, *10*, 15–25. [[CrossRef](#)]
27. Benchiha, S.; Al-Omari, A.I.; Alotaibi, N.; Shrahili, M. Weighted generalized quasi Lindley distribution: Different methods of estimation, applications for COVID-19 and engineering data. *AIMS Math.* **2021**, *6*, 11850–11878. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.