Supplementary Material : Mathematica Code for Figure 2 Results
Version 12.3
Paper : Arbitrarily Accurate Analytical Approximations for the Error Function

In[44]:=
```
ClearAll["Global`*"];

(* Define Parameters *)

orderMin = 0;    orderMax = 10;

xMin = 0;   xMax = 4;   yMax = 1;   yMin = 10^-6;
nPoint = 1000;   resolution = (xMax - xMin)/nPoint;

aRatio = 6/11;   iSize = {310, 170};
font1 = {"Courier", FontSize → 9, FontWeight → "Bold"};
pStyleS = {RGBColor[1/4, 0, 0], PointSize[0.005], Thickness[0.004]};
pStyleT = {GrayLevel[0.], PointSize[0.005], Thickness[0.004]};

(* Define Integrand *)

f[x_] := 2/Sqrt[π] * Exp[-x^2];      df[x_, i_] := D[f[tx], {tx, i}] /. tx → x;

Plot[f[x], {x, xMin, xMax}, PlotRange → {{xMin, xMax}, All},
 Frame → True, GridLines → Automatic, FrameTicks → Automatic, AspectRatio → aRatio,
 ImageSize → iSize, PlotStyle → pStyleS, BaseStyle → font1, PlotLabel → "Integrand"]

(* Define Spline Approx: Two Forms *)

p[x_, 0] = 1;
p[x_, k_] := p[x, k] = Expand[(D[p[tx, k - 1], {tx, 1}] /. tx → x) - 2*x*p[x, k - 1]]

Print["  "]
Do[
 Print["order = ", i, "  Definition for p[x,i] = ", p[x, i] ],
 {i, 0, 6, 1}]
Print["   "]
```

$$\text{erfApprox}[n\_, x1\_, x2\_] := \sum_{k=0}^{n}\left(\frac{\text{Factorial}[n]}{\text{Factorial}[n-k]*\text{Factorial}[k+1]}*\frac{\text{Factorial}[2*n+1-k]}{2*\text{Factorial}[2*n+1]}*(x2-x1)^{k+1}*\left(\text{df}[x1,k]+(-1)^k*\text{df}[x2,k]\right)\right)$$

$$\text{erfApproxAlt}[n\_, x1\_, x2\_] := \frac{2}{\sqrt{\pi}}*\sum_{k=0}^{n}\left(\frac{\text{Factorial}[n]}{\text{Factorial}[n-k]*\text{Factorial}[k+1]}*\frac{\text{Factorial}[2*n+1-k]}{2*\text{Factorial}[2*n+1]}*(x2-x1)^{k+1}*\left(p[x1,k]*\text{Exp}\left[-x1^2\right]+(-1)^k*p[x2,k]*\text{Exp}\left[-x2^2\right]\right)\right)$$

$$\text{re}[\text{order}\_, x\_] := 1-\frac{\text{erfApprox}[\text{order}, 0, x]}{\text{Erf}[x]}$$

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(* Explicit Spline Approx. Check on Approx. *)
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$$\text{fE}[0, x\_] := \frac{x}{\sqrt{\pi}}+\frac{e^{-x^2}\,x}{\sqrt{\pi}}$$

$$\text{fE}[1, x\_] := \frac{x}{\sqrt{\pi}}+\frac{x}{\sqrt{\pi}}*\left(1+\frac{x^2}{3}\right)*e^{-x^2}$$

$$\text{fE}[2, x\_] := \frac{x}{\sqrt{\pi}}*\left(1-\frac{x^2}{30}\right)+\frac{x}{\sqrt{\pi}}*\left(1+\frac{11*x^2}{30}+\frac{x^4}{15}\right)*e^{-x^2}$$

$$fE[3, x\_] := \frac{x}{\sqrt{\pi}} * \left(1 - \frac{x^2}{21}\right) + \frac{x}{\sqrt{\pi}} * \left(1 + \frac{8 * x^2}{21} + \frac{17 * x^4}{210} + \frac{x^6}{105}\right) * e^{-x^2}$$

$$fE[4, x\_] := \frac{x}{\sqrt{\pi}} * \left(1 - \frac{x^2}{18} + \frac{x^4}{1260}\right) + \frac{x}{\sqrt{\pi}} * \left(1 + \frac{7 * x^2}{18} + \frac{37 * x^4}{420} + \frac{4 * x^6}{315} + \frac{x^8}{945}\right) * e^{-x^2}$$

$$fE[5, x\_] := \frac{x}{\sqrt{\pi}} * \left(1 - \frac{2 * x^2}{33} + \frac{x^4}{660}\right) + \frac{x}{\sqrt{\pi}} * \left(1 + \frac{13 * x^2}{33} + \frac{61 * x^4}{660} + \frac{67 * x^6}{4620} + \frac{16 * x^8}{10395} + \frac{x^{10}}{10395}\right) * e^{-x^2}$$

```
Print["  "]
Do[
 Print["order = ", i, "     Check on Spline Approx:     ",
  Simplify[erfApprox[i, 0, x] - fE[i, x] ], "         ", Simplify[erfApproxAlt[i, 0, x] - fE[i, x]] ],
 {i, 0, 5, 1}]
Print["  "]

(* RE for Spline Approx. *)

Do[
  data = Table[Abs[N[re[order, xMin + i * resolution]]], {i, 1, nPoint, 1}];
  dataG = Table[{xMin + i * resolution, Abs[N[re[order, xMin + i * resolution]]]}, {i, 1, nPoint, 1}];
  maxRE[order] = Max[data];

  Print[pRE[order] =
    ListLogPlot[dataG, PlotRange → {{xMin, xMax}, {yMin, yMax}}, Joined → True, Frame → True, GridLines → Automatic,
     FrameTicks → Automatic, AspectRatio → aRatio, ImageSize → iSize, PlotStyle → pStyleS, BaseStyle → font1,
     PlotLabel → {"RE: Erf Approx: Order = ", order, " Max RE = ", NumberForm[maxRE[order], 5]}] ],

  {order, orderMin, orderMax, 1}];

tableS = Table[pRE[order], {order, orderMin, orderMax, 1}];
eS = Show[tableS, PlotLabel → "  "]

(* Table of RE bounds for [0,4] *)

dataRE = Table[{order, NumberForm[maxRE[order], 5]}, {order, orderMin, orderMax, 1}];
dataRE = Prepend[dataRE, {SplineOrder, MaxRE}];
Print[StyleForm[TableForm[dataRE], FontSize → 10]];

(* Taylor Series *)

orderMinT = 1;      orderMaxT = 15;
```

$$resolutionT = \frac{xMax - xMin}{100};$$

$$taylorSeries[order\_, x\_] := \frac{2}{\sqrt{\pi}} * \sum_{k=1}^{order} \left(\left(\frac{1 + (-1)^{k+1}}{2}\right) * \frac{(-1)^{(k-1)/2}\, x^k}{k * Factorial\left[\frac{k-1}{2}\right]}\right)$$

$$reT[order\_, x\_] := 1 - \frac{taylorSeries[order, x]}{Erf[x]}$$

```
Print["  "]
Do[
  Print["Taylor series: order = ", i, "      ", Expand[taylorSeries[i, x]] ],
  {i, 1, 9, 2}];

Print["   "];
Do[
 tableRET = Table[{i * resolutionT, Abs[N[reT[orderT, i * resolutionT]]]}, {i, 1, nPoint, 1}];

 pRET[orderT] = ListLogPlot[tableRET, PlotRange → {{xMin, xMax}, {yMin, yMax}}, Joined → False,
   Frame → True, GridLines → Automatic, FrameTicks → Automatic, AspectRatio → aRatio, ImageSize → iSize,
   PlotStyle → pStyleT, BaseStyle → font1, PlotLabel → {"RE in Erf Approx. Order = ", order}],
 {orderT, orderMinT, orderMaxT, 2}]
```

```
tableT = Table[pRET[orderT], {orderT, orderMinT, orderMaxT, 2}];
eT = Show[tableT, PlotLabel → "Taylor Series Approx.  "]

(* Combined Graph: Taylor Series + Spline Approx *)

e1 = Show[eS, eT, PlotLabel → "  "]
```