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The Complementary q -Lidstone Interpolating Polynomials and Applications

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Abstract: In this paper, we introduce the complementary q -Lidstone interpolating polynomial of degree $2n$, which involves interpolating data at the odd-order q -derivatives. For this polynomial, we will provide a q -Peano representation of the error function. Next, we use these results to prove the existence of solutions of the complementary q -Lidstone boundary value problems. Some examples are included.

Keywords: q -Lidstone polynomials; q -Peano's theorem; complementary q -Lidstone; boundary value problems

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1. Introduction

In 1929, Lidstone [1] introduced a generalization of Taylor's series that approximates a given function in a neighborhood of two points instead of one. Recently, Ismail and Mansour [2] introduced a q -analog of the Lidstone expansion theorem. They proved that, under certain conditions, an entire function $f(z)$ can be expanded with respect to the points 0 and 1 in terms of the q -Lidstone polynomials $A_n(z)$ and $B_n(z)$:

$$f(z) = \sum_{n=0}^{\infty} \left[A_n(z) D_{q^{-1}}^{2n} f(1) - B_n(z) D_{q^{-1}}^{2n} f(0) \right].$$

Here, $A_n(z) = \eta_{q^{-1}}^1 B_n(z)$ and:

$$B_n(z) = \frac{2^{2n+1}}{[2n+1]!} B_{2n+1}(z/2; q),$$

where $\eta_{q^{-1}}^y z^n$ denotes the q -translation operator defined by:

$$\eta_{q^{-1}}^y z^n = q^{\frac{n(n-1)}{2}} z^n (-y/z; q^{-1})_n = y^n (-z/y; q)_n,$$

and $B_n(z; q)$ is the q -analogue of the Bernoulli polynomials, which is defined by the generating function:

$$\frac{t E_q(z t)}{E_q(t/2) e_q(t/2) - 1} = \sum_{n=0}^{\infty} B_n(z; q) \frac{t^n}{[n]!}, \quad (1)$$

$E_q(z)$ and $e_q(z)$ are the q -exponential functions defined by Jackson, cf., e.g., [3,4],

$$E_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{z^j}{[j]!}; \quad z \in \mathbb{C} \quad \text{and} \quad e_q(z) := \sum_{j=0}^{\infty} \frac{z^j}{[j]!}; \quad |z| < 1.$$

The q -Lidstone polynomials $A_n(z)$ and $B_n(z)$ of degree $(2n + 1)$ and satisfy:

$$\begin{aligned} A_0(z) &= z \text{ and } B_0(z) = z - 1, \\ A_n(0) &= A_n(1) = B_n(0) = B_n(1) = 0, \text{ for } n \in \mathbb{N}, \\ D_{q^{-1}}^2 A_n(z) &= A_{n-1}(z) \text{ and } D_{q^{-1}}^2 B_n(z) = B_{n-1}(z). \end{aligned} \quad (2)$$

Throughout this paper, unless otherwise stated, q is a positive number less than one. The sets A_q, A_q^* are defined by:

$$A_q := \{q^n : n \in \mathbb{N}_0\}, \quad A_q^* := A_q \cup \{0\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. If X is the set A_q or A_q^* , then for $n > 1$, we use $C_q^n(X)$ to denote the space of all continuous functions with continuous q -derivatives up to order $n - 1$ on X . We shall follow the notations and terminology in [3,5].

In [6], we studied the boundary value problems, which consist of an even order q -differential equation and the q -Lidstone boundary conditions. This paper extends this technique to solve the following problem:

$$(-1)^n D_{q^{-1}}^{(2n+1)} g(z) = \phi(z, g(z)), \quad z \in A_q^*, \quad n \in \mathbb{N}, \quad (3)$$

subject to the boundary conditions:

$$g(0) = \tilde{\beta}, \quad D_{q^{-1}}^{2j-1} g(0) = q\tilde{\beta}_j, \quad D_{q^{-1}}^{2j-1} g(1) = q\tilde{\gamma}_j, \quad 1 \leq j \leq n, \quad (4)$$

where $\tilde{\beta}, \tilde{\beta}_j, \tilde{\gamma}_j \in \mathbb{C}$, ϕ is a continuous real function defined on the set $A_q^* \times \mathbb{R}^{j+1}$, $0 \leq j \leq 2n$ and:

$$g := (g_0, g_1, \dots, g_j) = (g, D_{q^{-1}} g, \dots, D_{q^{-1}}^j g) \in C_{q^{-1}}^{2n+1}(A_q^*).$$

We will give a q -analog of the complementary Lidstone interpolation, which was introduced in [7] and drawn on by Agarwal, Pinelas, and Wong in [8]. More precisely, we introduce and construct explicitly the complementary q -Lidstone interpolating polynomial of degree $2n$, which involves interpolating data at the odd-order derivatives. Furthermore, we will provide a q -Peano representation of the error function. These results are of fundamental importance in every aspect of numerical mathematics, in the theory of q -differential equations such as maximum principles, q -boundary value problems, oscillation theory, disconjugacy, and disfocality.

This article is organized as follows. In the next section, we give the formula of the q -Lidstone interpolating polynomial $Q_n(z; q)$ of degree $(2n - 1)$ and provide a q -Peano representation of the error function. In Section 3, we introduce and construct explicitly the complementary q -Lidstone interpolating polynomial $P_n(z; q)$ of degree $2n$, which involves interpolating data at the odd-order derivatives. In Section 4, we are interested in the existence of solutions of the complementary q -Lidstone boundary value problems (3) and (4), and we will give some illustrative examples. General conclusions of this work are summarized in Section 5.

2. Some Basic Results on the Interpolating Polynomial

We begin by some results from [6]:

Lemma 1. Let $g \in C^{2n}(A_q^*)$. Then

$$g(x) = \sum_{m=0}^{n-1} \left[D_{q^{-1}}^{2m} g(1) A_m(x) - D_{q^{-1}}^{2m} g(0) B_m(x) \right] + \int_0^1 G_n(x, qt) D_{q^{-1}}^{2n} g(q^2 t) d_q t,$$

where A_m and B_m are q -Lidstone polynomials of degree $2m + 1$, and:

$$G(x, t) = \begin{cases} -qt(1-x), & 0 \leq t < x \leq 1; \\ -qx(1-t), & 0 \leq x < t \leq 1 \end{cases} \quad (5)$$

$$\begin{aligned} G_1(x, qt) &:= G(x, qt) \\ G_n(x, qt) &= \int_0^1 G(x, qy) G_{n-1}(qy, qt) d_q y \quad (n = 2, 3, \dots) \\ &= \int_0^1 \dots \int_0^1 G(x, qt_1) G(qt_1, qt_2) \dots G(qt_{n-1}, qt) d_q t_1 d_q t_2 \dots d_q t_{n-1}. \end{aligned} \quad (6)$$

Remark 1. For $n \in \mathbb{N}$, the function $G_n(z, qs)$ satisfies:

$$0 \leq (-1)^n G_n(z, qs) = |G_n(z, qs)|.$$

As in the classical field of approximation theory [9], we consider the q -Lidstone interpolating polynomial $Q_n(z; q)$, $z \in A_q^*$, of degree $2n - 1$ satisfying the q -Lidstone conditions:

$$D_{q^{-1}}^{2j} Q_n(0; q) = \beta_j, \quad D_{q^{-1}}^{2j} Q_n(1; q) = \gamma_j, \quad (\beta_j, \gamma_j \in \mathbb{C}, j = 0, 1, \dots, n-1).$$

A representation of the q -Lidstone interpolating polynomial $Q_n(z; q)$ is given by the following:

Lemma 2. The q -Lidstone interpolating polynomial $Q_n(z; q)$ can be expressed as:

$$Q_n(z; q) = \sum_{m=0}^{n-1} \left[\gamma_m A_m(z) - \beta_m B_m(z) \right].$$

Proof. It is clear that $Q_n(z; q)$ is a polynomial of degree at most $(2n - 1)$. From (2), we have:

$$\begin{aligned} (D_{q^{-1}}^{2j} Q_n)(z; q) &= \sum_{m=j}^{n-1} \left[\gamma_m D_{q^{-1}}^{2j} A_m(z) - \beta_m D_{q^{-1}}^{2j} B_m(z) \right] \\ &= \sum_{m=j}^{n-1} \left[\gamma_m A_{m-j}(z) - \beta_m B_{m-j}(z) \right] \\ &= \sum_{m=0}^{n-j-1} \left[\gamma_{m+j} A_m(z) - \beta_{m+j} B_m(z) \right]. \end{aligned}$$

It follows that:

$$(D_{q^{-1}}^{2j} Q_n)(0; q) = -\beta_j B_0(z) \Big|_{z=0} = \beta_j, \quad 0 \leq j \leq n-1,$$

and:

$$(D_{q^{-1}}^{2j} Q_n)(1; q) = \gamma_j A_0(z) \Big|_{z=1} = \gamma_j, \quad 0 \leq j \leq n-1.$$

□

Let $(D_{q^{-1}}^{2j} f)(0) = \beta_j$ and $(D_{q^{-1}}^{2j} f)(1) = \gamma_j$ ($0 \leq j \leq n-1$) where $f \in C^{2n}(A_q^*)$. In such a case, $Q_n(z; q)$ is called the q -Lidstone interpolating polynomial of the function $f(z)$. For the associated error:

$$R(z; q) = f(z) - Q_n(z; q) \quad (7)$$

we provide a q -Peano representation. Therefore, in the following, we recall a q -Peano kernel theorem from [10], which is an important role in our results.

We use the notation \mathcal{P}_n to denote the space of polynomials of degree n , and we consider functions of class $C_q^{n+1}(A_q^*)$.

Define the two variables polynomials $\phi_n(z, t)$, $z, t \in \mathbb{C}$, to be:

$$\phi_0(z, t) := 1, \quad \phi_n(z, t) := \begin{cases} z^n \left(\frac{t}{z}; q\right)_n, & z \neq 0; \\ (-1)^n q^{\frac{n(n-1)}{2}} t^n, & z = 0. \end{cases} \quad (8)$$

Theorem 1. (q -Peano kernel theorem) Let L be a linear functional defined over $C_q^{n+1}(A_q^*)$. If $L(p) = 0$ for all polynomials $p(z)$ of degree n , then for all $f \in C_q^{n+1}(A_q^*)$:

$$L(f) = \int_0^1 (D_q^{n+1} f)(t) K_n(z, qt) d_q t,$$

where:

$$K_n(z, qt) = \frac{1}{[n]_q!} L_z(\phi_n^+(z, qt));$$

here, L_z means the linear functional L applied to $\phi_n^+(z, qt)$ as a function of z , and:

$$\phi_n^+(z, qt) := \begin{cases} \phi_n(z, qt), & z \geq t; \\ 0, & z < t. \end{cases}$$

Let z_0, z_1, \dots, z_n be distinct points in A_q^* . We denote by $I_k(z)$, $k = 0, 1, \dots, n$, to the polynomials that are defined on A_q^* and satisfy the following condition:

$$\sum_{k=0}^n I_k(z) = 1, \quad \sum_{k=0}^n r(z_k) I_k(z) = r(z), \quad \text{for } r \in \mathcal{P}_n. \quad (9)$$

Lemma 3. (see [10]) Suppose z_0, z_1, \dots, z_n are distinct points in A_q^* . Define the corresponding error functional by:

$$L_z(f) = f(z) - \sum_{k=0}^n f(z_k) I_k(z). \quad (10)$$

Then:

$$[n]_q! L(f) = \sum_{k=0}^n I_k(z) \int_{t=z_k}^z \phi_n(z_k, t) (D_q^{n+1} f)(t) d_q t.$$

Now, we prove the main result.

Theorem 2. Let $f \in C^{2n}(A_q^*)$. Then:

$$R(z; q) = \int_0^1 G_n(z, qt) (D_{q^{-1}}^{2n} f)(q^2 t) d_q t; \quad (11)$$

here, G_n has a q -Peano representation:

$$G_n(z, qt) = - \begin{cases} \sum_{i=0}^{n-1} \frac{q^{2(n-i)^2+3i-n}}{[2n-2i-1]_q!} A_i(z)(q^{1+2i}t; q)_{2n-2i-1}, & z < t; \\ \sum_{i=0}^{n-1} \frac{q^{2(2n-1)(n-i)}}{[2n-2i-1]_q!} B_i(z)(-t)^{2n-2i-1}, & z \geq t. \end{cases}$$

Proof. According to Lemma 2, the q -Lidstone interpolating polynomial of the function f can be expressed as:

$$Q_n(z; q) = \sum_{i=0}^{n-1} \left[(D_{q^{-1}}^{2i} f)(1) A_i(z) - (D_{q^{-1}}^{2i} f)(0) B_i(z) \right],$$

where the associated error:

$$\begin{aligned} R(z; q) &= f(z) - Q_n(z; q) \\ &= f(z) - \sum_{i=0}^{n-1} \left[(D_{q^{-1}}^{2i} f)(1) A_i(z) - (D_{q^{-1}}^{2i} f)(0) B_i(z) \right]. \end{aligned} \quad (12)$$

Therefore, from Lemma 1, we obtain (11).

Now, we apply Theorem 1. Note that, the reminder $L(f)$ defined by:

$$L(f) := R(z; q) = \int_0^1 G_n(z, qt) (D_{q^{-1}}^{2n} f)(q^{2n} t) d_q t,$$

where:

$$G_n(z, qt) := K_{2n-1}(z, qt) = \frac{q^{n(2n-1)}}{[2n-1]_q!} L_z(\phi_{2n-1}^+(z, qt)).$$

By Equation (12), we obtain:

$$\begin{aligned} L_z(\phi_{2n-1}^+(z, qt)) &= \\ \phi_{2n-1}^+(z, qt) &- \sum_{i=0}^{n-1} \left[(D_{q^{-1}}^{2i} \phi_{2n-1}^+)(1) A_i(z) - (D_{q^{-1}}^{2i} \phi_{2n-1}^+)(0) B_i(z) \right]. \end{aligned}$$

We can verify that:

$$\begin{aligned} (D_{q^{-1}}^{2i} \phi_{2n-1})(z, qt) &= \frac{[2n-1]_q!}{[2n-2i-1]_q!} q^{i-2i^2} \phi_{2n-2i-1}(q^{-2i}z, qt) \\ &= \begin{cases} \frac{[2n-1]_q!}{[2n-2i-1]_q!} q^{i-2i^2} (zq^{-2i})_{2n-2i-1} (q^{1+2i}t/z; q)_{2n-2i-1}, & z \neq 0; \\ \frac{[2n-1]_q!}{[2n-2i-1]_q!} q^{i-2i^2+(2n-2i-1)(n-i)} (-t)^{2n-2i-1}, & z = 0. \end{cases} \end{aligned}$$

Therefore, by Lemma 3, we conclude that G_n has a q -Peano representation:

$$G_n(z, qt) = - \begin{cases} \sum_{i=0}^{n-1} \frac{q^{2(n-i)^2+3i-n}}{[2n-2i-1]_q!} A_i(z)(q^{1+2i}t; q)_{2n-2i-1}, & z < t; \\ \sum_{i=0}^{n-1} \frac{q^{2(2n-1)(n-i)}}{[2n-2i-1]_q!} B_i(z)(-t)^{2n-2i-1}, & z \geq t. \end{cases}$$

□

3. The Complementary q -Lidstone Interpolating Polynomials

In this section, we consider the complementary q -Lidstone interpolating polynomial $P_n(z; q)$ in A_q^* , which is of degree $2n$ and satisfies the conditions:

$$P_n(0; q) = \tilde{\beta}, \quad (D_{q^{-1}}^{2i-1} P_n)(0; q) = \tilde{\beta}_i, \quad (D_{q^{-1}}^{2i-1} P_n)(1; q) = \tilde{\gamma}_i,$$

where $\tilde{\beta}, \tilde{\beta}_i, \tilde{\gamma}_i \in \mathbb{C}$, $1 \leq i \leq m$.

In the next result, we denote by $\nu_m(z)$ and $\tau_m(z)$ ($m \geq 0$) the first q^{-1} -derivatives of $A_m(z)$ and $B_m(z)$, respectively. That is,

$$D_{q^{-1}} A_m(z) = \nu_m(z) \text{ and } D_{q^{-1}} B_m(z) = \tau_m(z), \quad m \geq 0.$$

Then, it immediately follows that:

1. $\nu_0(z) = 1 = \tau_0(z)$;
2. $D_{q^{-1}} \nu_m(z) = A_{m-1}(z)$;
3. $D_{q^{-1}} \tau_m(z) = B_{m-1}(z)$;
4. $\int_0^q \nu_m(s) d_qs = \int_0^q \tau_m(s) d_qs = 0$, $m \geq 1$;
5. $\int_0^z \tau_{m-1}(s) d_qs = D_{q^{-1}} \tau_m(z)$, $m \geq 1$.

Theorem 3. Let $g \in C^{2n+1}(A_q^*)$ and $P_n(z; q)$ be the complementary q -Lidstone interpolating polynomial of degree $2n$ of the function $g(z)$. Then:

$$g(z) = P_n(z; q) + \tilde{R}(z; q), \quad (13)$$

where:

$$\begin{aligned} P_n(z; q) = & g(0) + \frac{1}{q} \sum_{m=1}^n \left[(D_{q^{-1}}^{2m-1} g)(1)(\nu_m(z) - \nu_m(0)) \right] + \\ & \frac{1}{q} \sum_{m=1}^n \left[(D_{q^{-1}}^{2m-1} g)(0)(\tau_m(0) - \tau_m(z)) \right], \end{aligned} \quad (14)$$

and $\tilde{R}(z; q)$ is the residue term:

$$\tilde{R}(z; q) = \int_0^1 H_n(z, qs) (D_{q^{-1}}^{2n+1} g)(q^2 s) d_qs. \quad (15)$$

Furthermore, the kernel $H_n(z, qs)$ has the q -Peano representation:

$$\begin{aligned} H_n(z, qs) = & \int_0^{qz} G_n(t, qs) d_q t \\ = & \frac{1}{q} \sum_{m=1}^n \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} \left(\nu_m(0) - \nu_m(z) \right) (q^{2m-1} s; q)_{2n-2m+1}; \quad z < s, \end{aligned}$$

and for $z \geq s$,

$$H_n(z, qs) = \frac{q^{4n^2-1} s^{2n}}{[2n]_q!} + \frac{1}{q} \sum_{m=1}^n \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} \left(\tau_m(z) - \tau_m(0) \right) s^{2n-2m+1}.$$

Proof. Let $f = D_{q^{-1}}g$. Integrate both sides of (7) from zero to qz , to obtain:

$$\begin{aligned} \int_0^{qz} D_{q^{-1}}g(t) d_q t &= g(z) - g(0) \\ &= \sum_{m=0}^{n-1} \left[(D_{q^{-1}}^{2m+1}g)(1) \int_0^{qz} A_m(t) d_q t - (D_{q^{-1}}^{2m+1}g)(0) \int_0^{qz} B_m(t) d_q t \right] \\ &\quad + \int_0^{qz} \left(\int_0^1 G_n(t, qs) (D_{q^{-1}}^{2n+1}g)(q^2 s) d_q s \right) d_q t. \end{aligned}$$

From (2), we have:

$$\begin{aligned} \int_0^{qz} A_m(t) d_q t &= \int_0^{qz} D_{q^{-1}}^2 A_{m+1}(t) d_q t \\ &= \int_0^{qz} \frac{1}{q} D_q^2 A_{m+1}(t/q^2) d_q t \\ &= \frac{1}{q} \left[D_{q^{-1}} A_{m+1}(z) - D_{q^{-1}} A_{m+1}(0) \right] \\ &= \frac{1}{q} \left[v_{m+1}(z) - v_{m+1}(0) \right], \quad m \in \mathbb{N}_0. \end{aligned}$$

Similarly, we can verify that:

$$\int_0^{qz} B_m(t) d_q t = \frac{1}{q} \left[\tau_{m+1}(z) - \tau_{m+1}(0) \right].$$

It follows

$$\begin{aligned} g(z) &= \int_0^{qz} \left(\int_0^1 G_n(t, qs) (D_{q^{-1}}^{2n+1}g)(q^2 s) d_q s \right) d_q t + g(0) \\ &\quad + \frac{1}{q} \sum_{m=1}^n \left[(D_{q^{-1}}^{2m-1}g)(1)(v_m(z) - v_m(0)) + (D_{q^{-1}}^{2m-1}g)(0)(\tau_m(0) - \tau_m(z)) \right], \end{aligned}$$

and then, we get Equation (13), where:

$$\begin{aligned} P_n(z; q) &= g(0) + \frac{1}{q} \sum_{m=1}^n \left[(D_{q^{-1}}^{2m-1}g)(1)(v_m(z) - v_m(0)) + \right. \\ &\quad \left. (D_{q^{-1}}^{2m-1}g)(0)(\tau_m(0) - \tau_m(z)) \right], \\ \tilde{R}(z; q) &= \int_0^1 \left(\int_0^{qz} G_n(t, qs) d_q t \right) (D_{q^{-1}}^{2n+1}g)(q^2 s) d_q s \\ &= \int_0^1 H_n(z, qs) (D_{q^{-1}}^{2n+1}g)(q^2 s) d_q s. \end{aligned} \tag{16}$$

By using Theorem 2, for $z < s$, we obtain:

$$\begin{aligned} H_n(z, qs) &= \int_0^{qz} G_n(t, qs) d_q t \\ &= - \sum_{m=0}^{n-1} \frac{q^{2(n-m)^2+3m-n}}{[2n-2m-1]_q!} \left(\int_0^{qz} A_m(t) d_q t \right) (q^{1+2m}s; q)_{2n-2m-1} \\ &= - \frac{1}{q} \sum_{m=1}^n \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} \left(v_m(z) - v_m(0) \right) (q^{2m-1}s; q)_{2n-2m+1}. \end{aligned}$$

Similarly, for $z \geq s$, we have:

$$\begin{aligned} H_n(z, qs) &= \int_0^{qz} G_n(t, qs) d_q t = \int_0^{qs} G_n(t, qs) d_q t + \int_{qs}^{qz} G_n(t, qs) d_q t \\ &= -\frac{1}{q} \sum_{m=1}^n \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} \left(v_m(s) - v_m(0) \right) (q^{2m-1}s; q)_{2n-2m+1} \\ &\quad + \frac{1}{q} \sum_{m=1}^n \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} \left(\tau_m(z) - \tau_m(s) \right) (s)^{2n-2m+1}. \end{aligned} \quad (17)$$

Finally, we will take $g(z) = \frac{q^{2n^2+n-1}}{[2n]_q!} \phi_{2n}(z, t)$, where $\phi_{2n}(z, t)$ is the polynomial function of degree $2n$ defined in (8). Then, after some calculations, we verify that:

$$D_{q^{-1}, z}^{2m-1} g(z) = \frac{q^{2n^2+n-1}}{[2n-2m+1]_q!} q^{(m-1)(1-2m)} \phi_{2n-2m+1}(zq^{1-2m}, s).$$

Hence, we obtain:

$$\begin{aligned} (D_{q^{-1}}^{2m-1} g)(1) &= \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} (q^{2m-1}s; q)_{2n-2m+1}, \\ (D_{q^{-1}}^{2m-1} g)(0) &= \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} (-s)^{2n-2m+1}, \\ g(0) &= \frac{q^{4n^2-1}}{[2n]_q!} s^{2n}. \end{aligned}$$

By using (14), we get:

$$\begin{aligned} &\frac{q^{2n^2+n-1}}{[2n]_q!} \phi_{2n}(z, t) = \\ &\frac{q^{4n^2-1}}{[2n]_q!} s^{2n} + \frac{1}{q} \sum_{m=1}^n \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} \left(\tau_m(0) - \tau_m(z) \right) (-s)^{2n-2m+1} \\ &+ \frac{1}{q} \sum_{m=1}^n \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} \left(v_m(z) - v_m(0) \right) (q^{2m-1}s; q)_{2n-2m+1}. \end{aligned}$$

Therefore, for $z = s$, we have:

$$\begin{aligned} &\frac{1}{q} \sum_{m=1}^n \frac{q^{2(n-m+1)^2+3(m-1)-n}}{[2n-2m+1]_q!} \left(v_m(s) - v_m(0) \right) (q^{2m-1}s; q)_{2n-2m+1} = \\ &\frac{1}{q} \sum_{m=1}^n \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} \left(\tau_m(0) - \tau_m(s) \right) (s)^{2n-2m+1} - \frac{q^{4n^2-1}}{[2n]_q!} s^{2n}. \end{aligned} \quad (18)$$

Combining (17) and (18), for $z \geq s$, we get:

$$H_n(z, qs) = \frac{q^{4n^2-1}s^{2n}}{[2n]_q!} + \frac{1}{q} \sum_{m=1}^n \frac{q^{2(2n-1)(n-m+1)}}{[2n-2m+1]_q!} \left(\tau_m(z) - \tau_m(0) \right) s^{2n-2m+1}.$$

This completes the proof. \square

Remark 2. by using Remark 1, we obtain:

$$0 \leq (-1)^n H_n(z, qs) = |H_n(z, qs)|. \quad (19)$$

Lemma 4. (see [6]) For $z \in A_q^*$, there exist some constants C_n , such that:

$$\begin{aligned} \int_0^1 |G_n(z, q^n s)| d_q t &\leq \frac{(1-q)^{2n}}{q^{n(n-3/2)}} C_n, \\ \int_0^1 |D_{q^{-1}, z} G_n(z, q^n s)| d_q t &\leq \frac{(1-q)^{2(n-1)}}{q^{(n-1)(n-5/2)}} \left(q + \frac{q^2}{1+q}\right) C_n. \end{aligned} \quad (20)$$

Lemma 5. There exist some constants $C_{2n+1, k}$ such that:

$$\left| (D_{q^{-1}}^k g)(z) - (D_{q^{-1}}^k P_n)(z; q) \right| \leq C_{2n+1, k} \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right|,$$

for $k = 0, 1, \dots, 2n$.

Proof. From (13) and (15), we get:

$$\begin{aligned} |g(z) - P_n(z; q)| &= \left| \int_0^1 H_n(z, qs) (D_{q^{-1}}^{2n+1} g)(q^2 s) d_q s \right| \\ &\leq \max_{z \in A_q^*} \left(\int_0^1 |H_n(z, qs)| d_q s \right) \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right|. \end{aligned} \quad (21)$$

Note that:

$$\int_0^1 |H_n(z, qs)| d_q s \leq \int_0^1 \int_0^{qz} |G_n(t, qs)| d_q t d_q s, \quad (22)$$

and from Lemma 4, we conclude that the double q -integral on the right-hand side of (22) is absolutely convergent. Therefore, we can interchange the order of the q -integrations to obtain:

$$\begin{aligned} \int_0^1 |H_n(z, qs)| d_q s &\leq \int_0^{qz} \left(\int_0^1 |G_n(t, qs)| d_q s \right) d_q t \\ &\leq \frac{q(1-q)^{2n}}{q^{n(n-\frac{3}{2})}} C_n z. \end{aligned}$$

Since $z \in A_q^*$, we obtain:

$$\int_0^1 |H_n(z, qs)| d_q s \leq C_{2n+1, 0}, \quad (23)$$

where $C_{2n+1, 0} = \frac{q(1-q)^{2n}}{q^{n(n-\frac{3}{2})}} C_n$. Combining (21) with (23), we get the result for $k = 0$.

Again, by using (13) and (15), we get:

$$D_{q^{-1}}^{(k)} g(z) - D_{q^{-1}}^{(k)} P_n(z; q) = \int_0^1 D_{q^{-1}}^{k-1} G_n(z, qs) (D_{q^{-1}}^{(2n+1)} g)(q^2 s) d_q s.$$

Hence, using (6), (16), and Lemma 4, we obtain:

$$\begin{aligned} & \left| D_{q^{-1}}^{2k-1} g(z) - D_{q^{-1}}^{2k-1} P_n(z; q) \right| \\ & \leq \max_{z \in A_q^*} \left(\int_0^1 \left| D_{q^{-1}}^{2k-2} G_n(z, qs) \right| d_qs \right) \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right| \\ & = \max_{z \in A_q^*} \left(\int_0^1 \left| G_{n-k+1}(z, qs) \right| d_qs \right) \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right| \\ & \leq \frac{(1-q)^{2(n-k+1)}}{q^{(n-k+1)(n-k-\frac{1}{2})}} C_n \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right| \\ & = C_{2n+1, 2k-1} \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right|, \quad 1 \leq k \leq n. \end{aligned}$$

Similarly, we conclude that:

$$\begin{aligned} & \left| D_{q^{-1}}^{2k} g(z) - D_{q^{-1}}^{2k} P_n(z; q) \right| \\ & \leq \max_{z \in A_q^*} \left(\int_0^1 \left| D_{q^{-1}}^{2k-1} G_n(z, qs) \right| d_qs \right) \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right| \\ & \leq \frac{(1-q)^{2(n-k)}}{q^{(n-k)(n-k-\frac{3}{2})}} \left(q + \frac{q^2}{1+q} \right) C_n \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right| \\ & = C_{2n+1, 2k} \max_{z \in A_q^*} \left| (D_{q^{-1}}^{2n+1} g)(q^2 z) \right|, \quad 1 \leq k \leq n. \end{aligned}$$

This completes the proof. \square

4. Applications

In this section, we present the necessary and sufficient conditions for the existence of solutions of the complementary q -Lidstone boundary value problem (3) and (4).

The proof depends on the results obtained in Section 3 and the Arzelà–Ascoli theorem [11].

Theorem 4. Suppose that $Q_k > 0$, $0 \leq k \leq j$ are given real numbers, and define the nonzero constant M to be the maximum of $|\phi(z, g_0, g_1, \dots, g_j)|$ on the set $A_q^* \times E$, where:

$$E = \{(g_0, g_1, \dots, g_j), |g_k| \leq 2Q_k, 0 \leq k \leq j\}.$$

Furthermore, suppose that:

$$MC_{2n+1, k} \leq Q_k, \quad \max_{z \in A_q^*} \left| D_{q^{-1}}^{(k)} P_n(z; q) \right| = \tilde{p}_k \leq Q_k. \quad (24)$$

Then, the boundary value problem (3) and (4) has a solution in E .

Proof. First, we define the set:

$$J(A_q^*) := \left\{ g \in C_{q^{-1}}^j(A_q^*) : \|D_{q^{-1}}^k g\| = \max_{z \in A_q^*} |D_{q^{-1}}^j g| \leq 2Q_k, 0 \leq k \leq j \right\}.$$

Notice, we can verify that $J(A_q^*)$ is a closed convex subset of the space $C_{q-1}^j(A_q^*)$. Consider an operator $T : C_{q-1}^j(A_q^*) \rightarrow C_{q-1}^{2n}(A_q^*)$ as follows:

$$(Tg)(z) = P_n(z; q) + \int_0^1 |H_n(z, qs)| \phi(s, g(s)) d_qs. \quad (25)$$

In view of Theorem 3, any fixed point of (25) is a solution of the complementary boundary value problem (3) and (4).

Next, we prove that T maps $J(A_q^*)$ into itself. Let $g(z) \in J(A_q^*)$. Then, from (24), (25), and Lemma 5, we get:

$$\begin{aligned} \left| D_{q-1}^{(k)}(Tg)(z) \right| &\leq \left| D_{q-1}^{(k)} P_n(z; q) \right| + M \int_0^1 \left| D_{q-1}^{(k)} H_n(z, qs) \right| d_qs \\ &\leq Q_k + M C_{2n+1,k} \leq 2Q_k, \quad 0 \leq k \leq j. \end{aligned} \quad (26)$$

Thus, $T(J(A_q^*)) \subseteq J(A_q^*)$. Furthermore, since $J(A_q^*)$ is a compact set, Inequality (26) implies that the sets:

$$\mathcal{F}_k := \{D_{q-1}^{(k)}(Tg)(z) : g(z) \in J(A_q^*), \quad 0 \leq k \leq j\},$$

are bounded and then uniformly equi-continuous on $J(A_q^*)$. Therefore, from the Arzelà-Ascoli theorem, the closure of $T(J(A_q^*))$ is compact. Thus, by the Schauder fixed point theorem, we can find a fixed point of T in E that satisfies the boundary value problem (3) and (4). \square

Corollary 1. Assume that the function $\phi(z, g_0, g_1, \dots, g_j)$ satisfies the following condition on $A_q^* \times \mathbb{R}^{j+1}$:

$$|\phi(z, g_0, g_1, \dots, g_j)| \leq L + \sum_{k=0}^j L_k |g_k|^{\alpha_k}, \quad (27)$$

where L, L_k are nonnegative constants and $0 \leq \alpha_k < 1$. Then, the boundary value problem (3) and (4) has a solution.

Proof. By using (27), for $g(x) \in J(A_q^*)$, we get:

$$\left| \phi(z, g(z), D_{q-1}g(z), D_{q-1}^2g(z), \dots, D_{q-1}^jg(z)) \right| \leq N,$$

where $N := L + \sum_{k=0}^j L_k (2Q_k)^{\alpha_k}$. Hence, the result follows by observing that the hypothesis of Theorem 4 is satisfied and replacing M by N such that Q_k , $(0 \leq k \leq j)$ are sufficiently large. \square

Theorem 5. Suppose that the function $\phi(z, g_0, g_1, \dots, g_j)$ on the compact set $A_q^* \times E_1$ satisfies the following conditions:

$$|\phi(z, g_0, g_1, \dots, g_j)| \leq L + \sum_{k=0}^j L_k |g_k|, \quad (28)$$

where

$$\begin{aligned} E_1 &= \{(g_0, g_1, \dots, g_j), |g_k| \leq \tilde{p}_k + C_{2n+1,k} \frac{L+c}{1-\theta}, 0 \leq k \leq j\}, \\ \max_{z \in A_q^*} \left| D_{q-1}^{(k)} P_n(z; q) \right| &= \tilde{p}_k, \quad c = \sum_{k=0}^j L_k \tilde{p}_k, \quad \theta = \sum_{k=0}^j C_{2n+1,k} L_k < 1. \end{aligned} \quad (29)$$

Then, the boundary value problem (3) and (4) has a solution in E_1 .

Proof. Let $y(z) = g(z) - P_n(z; q)$. Then, the boundary value problem (3) and (4) is equivalent to the following problem:

$$\begin{aligned} (-1)^n D_{q^{-1}}^{2n+1} y(z) &= \phi\left(z, (y + P_n)(z), D_{q^{-1}}(y + P_n), \dots, D_{q^{-1}}^j(y + P_n)\right), \\ y(0) &= (D_{q^{-1}}^{2k-1} y)(0) = (D_{q^{-1}}^{2k-1} y)(1) = 0, \quad k = 0, 1, \dots, n. \end{aligned}$$

For $y \in C_{q^{-1}}^j(A_q^*)$, we define:

$$\|y\|_q = \max\left\{\sup_{z \in A_q^*} |D_{q^{-1}}^k y(z)|, \quad 0 \leq k \leq j\right\},$$

and we consider the operator $T_1 : C_{q^{-1}}^j(A_q^*) \rightarrow C_{q^{-1}}^j(A_q^*)$, which is defined by:

$$\begin{aligned} (T_1 y)(z) &= \int_0^1 |H_n(z, qs)| \times \\ &\phi\left(s, y(s) + P_n(s), D_{q^{-1}}(y + P_n)(s), \dots, D_{q^{-1}}^k(y + P_n)(s)\right) d_qs. \end{aligned} \quad (30)$$

We will use the same technique of the proof in Theorem 4. Therefore, it is sufficient to prove that T_1 maps the set:

$$J_1(A_q^*) := \left\{y(z) \in C_{q^{-1}}^j(A_q^*) : \|y\|_q \leq C_{2n+1,j} \frac{L+c}{1-\theta}\right\}$$

into itself. For this, let $y(z) \in J_1(A_q^*)$. It immediately follows that:

$$\left((y + P_n)(z), D_{q^{-1}}(y + P_n)(z), \dots, D_{q^{-1}}^j(y + P_n)(z)\right) \in E_1,$$

and then:

$$\left|D_{q^{-1}}^k(y + P_n)(z)\right| \leq \tilde{p}_k + C_{2n+1,k} \frac{L+c}{1-\theta}, \quad 0 \leq k \leq j. \quad (31)$$

Thus, from (28), (30), (31), and Lemma 5, we get:

$$\begin{aligned} \left|D_{q^{-1}}^j(T_1 y)(z)\right| &\leq \int_0^1 \left|D_{q^{-1}}^j H_n(z, qs)\right| \left(L + \sum_{k=0}^j L_k \left|D_{q^{-1}}^k(y + P_n)(s)\right|\right) d_qs \\ &\leq C_{2n+1,j} \left[L + \sum_{k=0}^j L_k C_{2n+1,k} \frac{L+c}{1-\theta} + \sum_{k=0}^j L_k \tilde{p}_k\right] \\ &= C_{2n+1,j} \left[L + \theta \frac{L+c}{1-\theta} + c\right] \\ &= C_{2n+1,j} \left[\theta \frac{L+c}{1-\theta} + \frac{(L+c)}{1-\theta} (1-\theta)\right] \\ &= C_{2n+1,j} \frac{(L+c)}{1-\theta}. \end{aligned}$$

□

Theorem 6. Suppose that the function $\phi(z, g_0, g_1, \dots, g_j)$ on the compact set $A_q^* \times E_2$ satisfies the Lipschitz condition:

$$|\phi(z, g_0, g_1, \dots, g_j) - \phi(z, f_0, f_1, \dots, f_j)| \leq \sum_{k=0}^j L_k |g_k - f_k|, \quad (32)$$

where E_2 is the same as E_1 in (29), with:

$$L = \max_{z \in A_q^*} |\phi(z, 0, 0, \dots, 0)|.$$

Then, the boundary value problem (3) and (4) has a unique solution in E_2 .

Proof. Since the Lipschitz condition (32) implies (28), the existence of a solution follows from Theorem 5. To prove the uniqueness, let $g(z)$ and $y(z)$ be two solutions of the boundary value problem (3) and (4) in E_2 . Then, as in Theorem 5, it follows that $\|g - y\| \leq \theta \|g - y\|$, and since $\theta < 1$, we get $g(z) = y(z)$. \square

Remark 3. If $\phi = 0$, then the complementary q -Lidstone boundary value problem (3) and (4) has a unique solution $g(z) = P_n(z; q)$.

We illustrate Theorem 4 by the following example:

Example 1. Consider the complementary q -Lidstone boundary value problem:

$$(-1)^n D_{q^{-1}}^3 g(z) = \phi(z, g, D_{q^{-1}} g, \dots, D_{q^{-1}}^j g), \quad z \in A_q^*, \quad (33)$$

$$g(0) = q^3, \quad D_{q^{-1}} g(0) = -q^3, \quad D_{q^{-1}} g(1) = q^3, \quad (34)$$

where $0 \leq j \leq 2$ is fixed and $n = 1$. By using Equation (14), we can compute the q^{-1} -derivative of $P_n(z; q)$ to get:

$$(D_{q^{-1}} P_n)(z; q) = q^2(2z - 1).$$

Therefore,

$$\begin{aligned} P_n(z; q) &= \int_0^{qz} (D_{q^{-1}} P_n)(t; q) d_q t + g(0) \\ &= \frac{2q^4}{1+q} z^2 - q^3 z + q^3, \\ \tilde{p}_0 &= \max_{z \in A_q^*} |P_n(z)| = P_n(0) = q^2, \quad \tilde{p}_1 = \max_{z \in A_q^*} |D_{q^{-1}} P_n(z)| = q^2, \\ \tilde{p}_2 &= \max_{z \in A_q^*} |D_{q^{-1}}^{(2)} P_n(z)| = 2q^2. \end{aligned} \quad (35)$$

We consider two cases:

Case 1. Suppose $j = 0$ and $\phi(z, g) = \frac{\sqrt{q}}{12(1-q)^2} z g^2$. Then, the q -differential equation:

$$(-1)^n D_{q^{-1}}^{(3)} g(z) = \frac{\sqrt{q}}{12(1-q)^2} z g^2, \quad z \in A_q^*,$$

with the boundary condition (34) has a solution in the set $E = \{g(z), |g| \leq 2Q_0\}$, provided $MC_{3,0} \leq Q_0$ and $Q_0 \geq \tilde{p}_0 = q^2$ where:

$$M = \max_{(z,g) \in A_q^* \times E} |\phi(z, g)| = \frac{\sqrt{q}}{3(1-q)^2} Q_0^2, \quad C_{3,0} = q^{\frac{3}{2}} (1-q)^2,$$

and then, $q^2 \leq Q_0 \leq \frac{3}{q^2}$.

Case 2. Suppose that $j = 2$ and:

$$\phi(z, g, D_{q^{-1}} g, D_{q^{-1}}^2 g) = z^2 g + \sqrt{z} D_{q^{-1}} g - D_{q^{-1}}^2 g,$$

and define the set:

$$E = \{(g, D_{q^{-1}}g, D_{q^{-1}}^2g) : Q_j \leq |D_{q^{-1}}^jg| \leq 2Q_j, 0 \leq j \leq 2\}.$$

Assume that $Q_j \geq \tilde{p}_j$ and $MC_{3,j} \leq Q_j$, $j = 0, 1, 2$, where:

$$M = \max_{(z,g) \in A_q^* \times E} |\phi(z, g(z))| = 2Q_0 + 2Q_1 - Q_2.$$

By using Lemma 5, we obtain $C_{3,0} = q^{\frac{3}{2}}(1-q)^2$, $C_{3,1} = q^{\frac{1}{2}}(1-q)^2$, and $C_{3,2} = q + \frac{q^2}{1+q}$. Thus, the condition $MC_{3,j} \leq Q_j$ implies that:

$$\begin{aligned} \left[\frac{1}{q^{\frac{3}{2}}(1-q)^2} - 2 \right] Q_0 - 2Q_1 + Q_2 &\geq 0 \\ \left[\frac{1}{q^{\frac{1}{2}}(1-q)^2} - 2 \right] Q_1 - 2Q_0 + Q_2 &\geq 0 \\ \left[\frac{1+q}{q+2q^2} + 1 \right] Q_2 - 2Q_0 - Q_1 &\geq 0. \end{aligned} \quad (36)$$

Pick $Q_0 = Q_1 = q^2$ and $Q_2 = 4q^2$, which satisfy (36), and the condition $Q_j \geq \tilde{p}_j$ for $j = 0, 1, 2$. Thus, by Theorem 4, the q -differential equation:

$$(-1)^n D_{q^{-1}}^{(3)}g(z) = \phi(z, g, D_{q^{-1}}g, D_{q^{-1}}^{(2)}g), \quad z \in A_q^*$$

has a solution in E .

The following example illustrates Theorem 5:

Example 2. Consider the complementary q -Lidstone boundary value problem:

$$\begin{aligned} (-1)^n D_{q^{-1}}^{(3)}g(z) &= z + \frac{(1-q)z}{4}g(z) + \left(\frac{z}{2}\right)^2 D_{q^{-1}}g(z) + \\ &\frac{(q-1)^2}{8} D_{q^{-1}}^{(2)}g(z), \quad z \in A_q^*, \end{aligned} \quad (37)$$

with the boundary condition (34). The interpolating polynomial $P_q(z)$ is given in Example 1. Note that:

$$|\phi(z, g_0, g_1, g_2)| \leq 1 + \frac{1}{4}|g(z)| + \frac{1}{4}|D_{q^{-1}}g(z)| + \frac{1}{8}|D_{q^{-1}}^{(2)}g(z)|, \quad z \in A_q^*.$$

Then, the conditions in Theorem 5 are satisfied with:

$$\begin{aligned} L &= 1, \quad L_0 = \frac{1}{4}, \quad L_1 = \frac{1}{4}, \quad L_2 = \frac{1}{8}, \\ c &= \sum_{k=0}^2 L_k \tilde{p}_k = \frac{3}{4}q^2, \quad \theta = \sum_{k=0}^2 C_{3,k} L_k < \frac{3}{4} < 1. \end{aligned}$$

Thus, the complementary q -Lidstone boundary value problems (34)–(37) have a solution in:

$$\begin{aligned} E_1 = \left\{ (g_0, g_1, g_2), \quad |g_0| = |g(z)| \leq q^2 + q^{\frac{3}{4}}(1-q)^2 \frac{1 + \frac{3}{4}q^2}{1-\theta}, \right. \\ |g_1| \leq q^2 + q^{\frac{1}{2}}(1-q)^2 \frac{1 + \frac{3}{4}q^2}{1-\theta}, \quad |g_2| \leq \left(q + \frac{q^2}{1+q}\right) \frac{1 + \frac{3}{4}q^2}{1-\theta}, \\ \left. \theta = \frac{1}{4}q^{\frac{3}{4}}(1-q)^2 + \frac{1}{4}q^{\frac{1}{2}}(1-q)^2 + \frac{1}{8}\left(q + \frac{q^2}{1+q}\right) \right\}. \end{aligned}$$

5. Concluding Remarks

The q -Lidstone polynomials are defined in analogy with the well known Lidstone polynomials through the q -translation operator and the q -analogue of the Bernoulli polynomials. These polynomials of degree $2n + 1$ satisfy analogue conditions of the Lidstone polynomials with respect to the q -differential operator D_q^{-1} . It was recently proven, that under certain conditions, an entire function f can be expanded with respect to the points 0 and 1 in terms of the q -Lidstone polynomials.

In [6], we studied the boundary value problems, which consist of an even order q -differential equation and the q -Lidstone boundary conditions. This paper extended this technique to solve some problems. We introduced the complementary q -Lidstone interpolating polynomial of degree $2n$, which involves interpolating data at the odd-order q -derivatives in zero and one, and provided a q -Peano representation of the error function.

This work provided the basis for several applications that we can search in the future. Firstly, we are interested in studying the possibility of extending q -Lidstone and complementary q -Lidstone interpolation polynomials to triangular domains. The analogous problem for the classical case was posed by Agarwal and Wong [12] and studied in [13,14]. Secondly, we are interested in applying such expansions to the construction of the boundary-type quadrature formula on triangles (see [15]) or to a solution of Hermite–Birkhoff interpolation problems on scattered data (see [16,17]).

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