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Existence of Solutions for Fractional Integro-Differential Equations with Non-Local Boundary Conditions

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Abstract: In this paper, we study the existence of solutions for a new class of boundary value problems of non-linear fractional integro-differential equations. The existence result is obtained with the aid of Schauder type fixed point theorem while the uniqueness of solution is established by means of contraction mapping principle. Then, we present some examples to illustrate our results.

Keywords: fractional differential equations; Caputo fractional derivative; fixed point theorem

MSC: 34A08, 34B15

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1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines such as physics, aerodynamics, polymer rheology, regular variations in thermodynamics, biophysics, blood flow phenomena, electrical circuits, biology, etc. In fact, the tools of fractional calculus have considerably improved the mathematical modeling of many real world problems. For theoretical development and applications of the subject, we refer the reader to [1–11] and the references cited therein.

The nonlocal boundary conditions are important in describing some peculiarities happening inside the domain of physical, chemical or other processes [12], while the integral boundary conditions provide the means to assume an arbitrary shaped cross-section of blood vessels in computational fluid dynamics (CFD) studies of blood flow problems [13,14].

Non-local boundary value problems of nonlinear fractional order differential equations have recently been investigated by several researchers. The domain of study ranges from the theoretical aspects to the analytic and numerical methods for fractional differential equations.

Agarwal et al. [4] discussed the existence of solutions for a boundary value problem of integro-differential equations of fractional order

$$-D^{\alpha}x(t) = Af(t, x(t)) + BI^{\beta}g(t, x(t)), 2 < \alpha \leq 3, t \in [0, 1]$$

with non-local three-point boundary conditions $D^{\delta}x(t) = 0$, $D^{\delta+1}x(t) = 0$, $D^{\delta}x(1) - D^{\delta}x(\eta) = a$, where $0 < \delta \leq 1$, $\alpha - \delta > 3$, $0 < \beta < 1, 0 < \eta < 1$.

Motivated by the works mentioned, in this paper, we investigate the existence and uniqueness of solutions for the non-linear fractional integro-differential equation

$$^{c}D^{p}x(t) = f(t, x(t), ^{c}D^{q}x(t), I^{r}x(t)) \tag{1}$$

with non-local boundary conditions

$$\begin{cases} {}^cD^q x(\alpha) + {}^c D^q x(\beta) = bx'(1) \\ x(0) + x(1) = aI^r x(\gamma), \end{cases} \tag{2}$$

where ${}^cD^p, {}^c D^q$ denote the Caputo fractional derivative of order p and q respectively, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, a, b are real constants, $1 < p \leq 2$, $0 < q < 1$, $p > q + 1$, $r > 0$, $t \in [0,1]$ and $0 < \alpha, \beta, \gamma < 1$.

These boundary conditions are interesting and important from a physical point.

The rest of paper is arranged as follows: In Section 2 we recall some basic definitions of fractional calculus and present an auxiliary lemma. Section 3 contains main existence and uniqueness results. We also present some examples to to illustrate our results.

2. Preliminaries

First of all, we recall some basic definitions of fractional calculus [7].

Definition 1. *For a function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order p is defined as*

$${}^cD^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{n-p-1} f^{(n)}(s) ds, \qquad n-1 < p < n, \quad n = [p] + 1,$$

where $[p]$ denotes the integer part of real number p and $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2. *The Riemann-Liouvill fractional integral of order p is defined as*

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds, \qquad p > 0,$$

provided the integral exists.

Now, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 1. *Let $1 < p \leq 2, 0 < q < 1, r > 0, p > q + 1, \alpha, \beta, \gamma > 0, a, b \in \mathbb{R}, t \in I := [0,1]$, $\frac{a\gamma^r}{\Gamma(r+1)} \neq 2$ and $y(t) \in C([0,1], \mathbb{R})$. Then the unique solution of the boundary value problem ${}^cD^p x(t) = y(t)$ with boundary conditions $x(0) + x(1) = aI^r x(\gamma)$ and ${}^cD^q x(\alpha) + {}^c D^q x(\beta) = bx'(1)$ is given by*

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds + \frac{1}{\varphi_2} \left[\frac{a}{\Gamma(r+p)} \int_0^\gamma (\gamma - s)^{p+r-1} y(s) ds - \frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds \right] \\ & - \frac{1}{\varphi_1} \left(t - \frac{\varphi_3}{\varphi_2} \right) \left[\frac{b}{\Gamma(p-1)} \int_0^1 (1-s)^{p-2} y(s) ds - \frac{1}{\Gamma(p-q)} \int_0^\alpha (\alpha - s)^{p-q-1} y(s) ds \right. \\ & \left. - \frac{1}{\Gamma(p-q)} \int_0^\beta (\beta - s)^{p-q-1} y(s) ds \right]. \end{aligned}$$

where $\varphi_1 = \frac{\alpha^{-q+1} + \beta^{-q+1}}{\Gamma(-q+2)} - b$, $\varphi_2 = 2 - \frac{a\gamma^r}{\Gamma(r+1)}$ and $\varphi_3 = 1 - \frac{a\gamma^{r+1}}{\Gamma(r+2)}$.

Proof. It is well known [7] that the general solution of the equation ${}^cD^p x(t) = y(t)$ is given by

$$x(t) = I^p y(t) + c_0 + c_1 t, \tag{3}$$

where c_0, c_1 are arbitrary constants. By using the boundary conditions, we get

$$\begin{aligned} c_0 = \frac{1}{\varphi_2}\Bigg[& \frac{a}{\Gamma(r+p)}\int_0^{\gamma}(\gamma-s)^{p+r-1}y(s)ds - \frac{1}{\Gamma(p)}\int_0^1(1-s)^{p-1}y(s)ds \\ & - \frac{\varphi_3}{\varphi_1}\Bigg(\frac{b}{\Gamma(p-1)}\int_0^1(1-s)^{p-2}y(s)ds - \frac{1}{\Gamma(p-q)}\Big(\int_0^{\alpha}(\alpha-s)^{p-q-1}y(s)ds \\ & + \int_0^{\beta}(\beta-s)^{p-q-1}y(s)ds\Big)\Bigg)\Bigg] \end{aligned}$$

and

$$c_1 = \frac{1}{\varphi_1}\left[\frac{b}{\Gamma(p-1)}\int_0^1(1-s)^{p-2}y(s)ds - \frac{1}{\Gamma(p-q)}\left(\int_0^{\alpha}(\alpha-s)^{p-q-1}y(s)ds + \int_0^{\beta}(\beta-s)^{p-q-1}y(s)ds\right)\right]$$

Substituting the values of c_0, c_1 in (3), completes the proof. □

3. The Main Results

For $0 < r < 1$, let $\mathcal{W} = \{x : x, ^cD^q x(t) \in C([0,1],\mathbb{R})\}$ denote the Banach space of all continuous functions defined on $[0,1]$ into $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)| + |^cD^q x(t)|, t \in [0,1]\}$.

Using Lemma 1, we define an operator $T : \mathcal{W} \to \mathcal{W}$ associated with the problem (1)–(2) as

$$\begin{aligned} (Tx)(t) = & \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, x(s), ^cD^q x(s), I^r x(s))ds \\ & + \frac{1}{\varphi_2}\Bigg[\frac{a}{\Gamma(r+p)}\int_0^{\gamma}(\gamma-s)^{p+r-1} f(s, x(s), ^cD^q x(s), I^r x(s))ds \\ & - \frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1} f(s, x(s), ^cD^q x(s), I^r x(s))ds\Bigg] \\ & - \frac{1}{\varphi_1}\left(t - \frac{\varphi_3}{\varphi_2}\right)\Bigg[\frac{b}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2} f(s, x(s), ^cD^q x(s), I^r x(s))ds \\ & - \frac{1}{\Gamma(p-q)}\Bigg(\int_0^{\alpha}(\alpha-s)^{p-q-1} f(s, x(s), ^cD^q x(s), I^r x(s))ds \\ & + \int_0^{\beta}(\beta-s)^{p-q-1} f(s, x(s), ^cD^q x(s), I^r x(s))ds\Bigg)\Bigg]. \end{aligned} \tag{4}$$

Observe that the problem (1)–(2) has solutions if and only if the operator T has fixed points.

Theorem 1 ([15]). *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \varepsilon Tu, 0 < \epsilon < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 2. *Assume that there exists $h \in C([0,1],\mathbb{R}^+)$ such that $|f(t, x(t), ^cD^q x(t), I^r x(t))| \leq h(t)$ for $t \in [0,1]$ with $\max_{t\in[0,1]} |h(t)| = \|h\|$. Then the problem (1)–(2) has at least one solution on $[0,1]$.*

Proof. As a first step, we show that the operator T is completely continuous. Let $\mathcal{D} \subseteq \mathcal{W}$ be a bounded set. Then for each $x \in \mathcal{D}$, we get

$$\begin{aligned}
|(Tx)(t)| \leq & \frac{1}{\Gamma(p)} \int_0^t |t-s|^{p-1} |h(s)| ds \\
& + \frac{1}{|\varphi_2|} \left[\frac{|a|}{\Gamma(r+p)} \int_0^\gamma |\gamma - s|^{p+r-1} |h(s)| ds \right. \\
& \left. + \frac{1}{\Gamma(p)} \int_0^1 |1-s|^{p-1} |h(s)| ds \right] \\
& + \frac{1}{|\varphi_1|} \left| t - \frac{\varphi_3}{\varphi_2} \right| \left[\frac{|b|}{\Gamma(p-1)} \int_0^1 |1-s|^{p-2} |h(s)| ds \right. \\
& + \frac{1}{\Gamma(p-q)} \left(\int_0^\alpha |\alpha - s|^{p-q-1} |h(s)| ds \right. \\
& \left. \left. + \int_0^\beta |\beta - s|^{p-q-1} |h(s)| ds \right) \right],
\end{aligned}$$

on the taking the norm for $t \in [0,1]$, we obtain

$$\begin{aligned}
\|Tx\| \leq \|h\| & \left(\frac{1}{\Gamma(p+1)} + \frac{1}{|\varphi_2|} \left[\frac{|a|}{\Gamma(r+p+1)} \gamma^{r+p} + \frac{1}{\Gamma(p+1)} \right] \right. \\
& \left. + \frac{1}{|\varphi_1|} \left(1 + \frac{|\varphi_3|}{|\varphi_2|} \right) \left[\frac{|b|}{\Gamma(p)} + \frac{|\alpha|^{p-q} + |\beta|^{p-q}}{\Gamma(p-q+1)} \right] \right)
\end{aligned}$$

We set $h_1 = \frac{1}{\Gamma(p+1)} + \frac{1}{|\varphi_2|} \left(\frac{|a|}{\Gamma(r+p+1)} \gamma^{r+p} + \frac{1}{\Gamma(p+1)} \right) + \frac{1}{|\varphi_1|} \left(1 + \frac{|\varphi_3|}{|\varphi_2|} \right) \left(\frac{|b|}{\Gamma(p)} + \frac{|\alpha|^{p-q} + |\beta|^{p-q}}{\Gamma(p-q+1)} \right)$. In a similar manner, we find that

$$\begin{aligned}
\|{}^c D^q Tx\| \leq \|h\| & \left(\frac{1}{\Gamma(p-q+1)} + \frac{1}{|\varphi_2|} \left[\frac{|a|}{\Gamma(r+p-q+1)} \gamma^{r+p-q} + \frac{1}{\Gamma(p-q+1)} \right] \right. \\
& \left. + \left(1 + \frac{|\varphi_3|}{|\varphi_2|} \right) \left[\frac{|b|}{|\varphi_1| \Gamma(p-q)} + \frac{|\alpha|^{p-2q} + |\beta|^{p-2q}}{|\varphi_1| \Gamma(p-2q+1)} \right] \right)
\end{aligned}$$

Put $h_2 = \frac{1}{\Gamma(p-q+1)} + \frac{1}{|\varphi_2|} \left(\frac{|a|}{\Gamma(r+p-q+1)} \gamma^{r+p-q} + \frac{1}{\Gamma(p-q+1)} \right) + \frac{1}{|\varphi_1|} \left(1 + \frac{|\varphi_3|}{|\varphi_2|} \right) \left(\frac{|b|}{\Gamma(p-q)} + \frac{|\alpha|^{p-2q} + |\beta|^{p-2q}}{\Gamma(p-2q+1)} \right)$.

Next, for each $x \in \mathcal{D}$ and $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| = & \Bigg| \frac{1}{\Gamma(p)} \int_0^{t_2} (t_2 - s)^{p-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& - \frac{t_2}{\varphi_1} \Bigg[\frac{b}{\Gamma(p-1)} \int_0^1 (1-s)^{p-2} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& - \frac{1}{\Gamma(p-q)} \Bigg(\int_0^\alpha (\alpha - s)^{p-q-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& + \int_0^\beta (\beta - s)^{p-q-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \Bigg) \Bigg] \\
& - \frac{1}{\Gamma(p)} \int_0^{t_1} (t_1 - s)^{p-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& + \frac{t_1}{\varphi_1} \Bigg[\frac{b}{\Gamma(p-1)} \int_0^1 (1-s)^{p-2} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& - \frac{1}{\Gamma(p-q)} \Bigg(\int_0^\alpha (\alpha - s)^{p-q-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \\
& + \int_0^\beta (\beta - s)^{p-q-1} f(s, x(s), {}^c D^q x(s), I^r x(s)) ds \Bigg) \Bigg] \Bigg| \\
& \leq \|h\| \Bigg[\frac{|t_2^p - t_1^p| + 2(t_2 - t_1)^p}{\Gamma(p+1)} \\
& + \frac{1}{|\varphi_1|} \Big| t_2 - t_1 \Big| \Bigg(\frac{|b|}{\Gamma(p)} + \frac{|\alpha|^{p-q} + |\beta|^{p-q}}{\Gamma(p-q+1)} \Bigg) \Bigg].
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| \leq & \Big\| h \Big\| \Bigg[\frac{|t_2^p - t_1^p| + 2(t_2 - t_1)^p}{\Gamma(p+1)} \\
& + \frac{1}{|\varphi_1|} \Big| t_2 - t_1 \Big| \Bigg(\frac{|b|}{\Gamma(p)} + \frac{|\alpha|^{p-q} + |\beta|^{p-q}}{\Gamma(p-q+1)} \Bigg) \Bigg].
\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}
|({}^c D^q Tx)(t_2) - ({}^c D^q Tx)(t_1)| \leq & \Big\| h \Big\| \Bigg[\frac{|t_2^{p-q} - t_1^{p-q}| + 2(t_2 - t_1)^{p-q}}{\Gamma(p-q+1)} \\
& + \frac{1}{|\varphi_1|} \Big| t_2 - t_1 \Big| \Bigg(\frac{|b|}{\Gamma(p-q)} + \frac{|\alpha|^{p-2q} + |\beta|^{p-2q}}{\Gamma(p-2q+1)} \Bigg) \Bigg].
\end{aligned}$$

The functions t, t^p, t^{p-q} are uniformaly continuous on $[0,1]$ since $1 < p \leq 2, p - q > 0$. Therefore, by Arzela-Ascoli Theorem, the sets $\{Tx : x \in \mathcal{D}\}$ and $\{{}^c D^q Tx : x \in \mathcal{D}\}$ are relatively compact in $C([0,1])$. Thus, we deduce that $T(\mathcal{D})$ is a relatively compact subset of $\mathcal{W}$.

Now, we consider the set

$$B = \{x \in \mathcal{W} | x = \chi Tx, 0 < \chi < 1\},$$

and show that it is bounded. Let $x \in B$, then $x = \chi Tx, 0 < \chi < 1$. For any $t \in [0,1]$, it implies from $|x(t)| = \chi|Tx(t)|$ that

$$\|x\| \leq \|h\| \left[\frac{1}{\Gamma(p+1)} + \frac{1}{|\varphi_2|} \left(\frac{|a|}{\Gamma(r+p+1)} \gamma^{r+p} + \frac{1}{\Gamma(p+1)} \right) + \left(1 + \frac{|\varphi_3|}{|\varphi_2|} \right) \left(\frac{|b|}{|\varphi_1|\Gamma(p)} + \frac{|\alpha|^{p-q} + |\beta|^{p-q}}{|\varphi_1|\Gamma(p-q+1)} \right) \right].$$

This shows that the set B is bounded. Thus, by Theorem 1, we conclude that the operator T has at least one fixed point. Consequently , the problem (1)–(2) has at least one solution on $[0,1]$. □

Now, we establish the uniqueness of solutions for problem (1)–(2) by means of classical contraction mapping principle.

Theorem 3. *Assume that $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying the condition*

$$|f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z})| \leq \ell(|x-\overline{x}| + |y-\overline{y}| + |z-\overline{z}|),$$

with $\ell < \frac{1}{4k}$, where $k = \max\{h_1, h_2\}$. Then, the problem (1)–(2) has a unique solution on $[0,1]$.

Proof. First, we show that $TU_\nu \subseteq U_\nu$, where T is the operator defined by (4), $U_\nu = \{x \in \mathcal{W} : \|x\| \leq \nu\}$ with $\frac{1}{2}\nu \geq \frac{k\lambda}{1-4\ell k}$, where $\lambda = \sup_{t\in[0,1]} |f(t,0,0,0)|$. For $x \in U_\nu, t \in [0,1]$, we have

$$\begin{aligned}
|f(t,x(t),{}^c D^q x(t), I^r x(t))| &= |f(t,x(t),{}^c D^q x(t), I^r x(t)) - f(t,0,0,0) + f(t,0,0,0)| \\
&\leq |f(t,x(t),{}^c D^q x(t), I^r x(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq \ell\Big[|x(t)| + |{}^c D^q x(t)| + |I^r x(t)|\Big] + \lambda \\
&\leq \ell\Big[\sup_{t\in[0,1]}\{|x(t)| + |{}^c D^q x(t)|\} + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}|x(s)|ds\Big] + \lambda \\
&\leq \ell\Big[\|x\| + \frac{1}{\Gamma(r+1)}\sup_{t\in[0,1]}|x(t)|\Big] + \lambda \\
&\leq \ell\Big[\|x\| + \frac{1}{\Gamma(r+1)}\sup_{t\in[0,1]}\{|x(t)| + |{}^c D^q x(t)|\}\Big] + \lambda \\
&\leq 2\ell\|x\| + \lambda \leq 2\ell v + \lambda.
\end{aligned}$$

Hence,

$$\begin{aligned}
|Tx(t)| \leq (2\ell v + \lambda)\Bigg[&\frac{1}{\Gamma(p+1)} + \frac{1}{|\varphi_2|}\Big(\frac{|a||\gamma|^{r+p}}{\Gamma(r+p+1)} \\
&+ \frac{1}{\Gamma(p+1)}\Big) + \Big(\frac{|\varphi_3|}{|\varphi_2|} + 1\Big)\Big(\frac{|b|}{|\varphi_1|\Gamma(p)} + \frac{1}{|\varphi_1|\Gamma(p-q+1)}\Big(|\alpha|^{p-q} + |\beta|^{p-q}\Big)\Big)\Bigg] \\
&= (2\ell v + \lambda)h_1 \leq (2\ell v + \lambda)k \leq \frac{1}{2}\nu.
\end{aligned}$$

Similarly, we obtain

$$|({}^c D^q Tx)(t)| \leq (2\ell v + \lambda)h_2 \leq (2\ell v + \lambda)k \leq \frac{1}{2}\nu.$$

Therefore, we get that $Tx \in U_\nu$ implies that $TU_\nu \subseteq U_\nu$. Moreover, for $x, y \in \mathcal{W}$ and for any $t \in [0,1]$, we have

$$|Tx(t) - Ty(t)| \leq 2\ell\|x-y\| \sup_{t\in[0,1]} \Big(\frac{1}{\Gamma(p)} \int_0^t |t-s|^{p-1} ds + \frac{1}{|\varphi_2|} \Big[\frac{|a|}{\Gamma(r+p)} \int_0^\gamma |\gamma - s|^{p+r-1} ds$$
$$+ \frac{1}{\Gamma(p)} \int_0^1 |1-s|^{p-1} ds \Big] + \Big|\frac{1}{\varphi_1}\Big| \Big(\Big|t - \frac{\varphi_3}{\varphi_2}\Big| \Big) \Big(\frac{|b|}{\Gamma(p-1)} \int_0^1 |1-s|^{p-2} ds$$
$$+ \frac{1}{\Gamma(p-q)} \Big(\int_0^\alpha |\alpha - s|^{p-q-1} ds + \int_0^\beta |\beta - s|^{p-q-1} ds \Big) \Big) \Big) \leq 2\ell\|x-y\|k.$$

Analogously, we can obtain

$$|^c D^q Tx(t) -^c D^q Ty(t)| \leq 2\ell\|x-y\|h_2 \leq 2\ell\|x-y\|k.$$

Therefore, with the condition $\ell < \frac{1}{4k}$, we deduce that the operator T is a contraction. Hence, it follows Banach's fixed point theorem that the problem (1)–(2) has a unique solution on $[0,1]$. □

Example 1. *Consider the following fractional boundary value problem given by*

$$\begin{cases} ^c D^{\frac{3}{2}} x(t) = 4t^{\frac{1}{3}}(2 + \cos^3 x(t)) + \frac{|^c D^{\frac{1}{3}} x(t)|}{1 + |^c D^{\frac{1}{3}} x(t)|} + \frac{4}{t+5} \big(\frac{I^{\frac{1}{2}} x(t)}{1 + I^{\frac{1}{2}} x(t)} \big) \\ x(0) + x(1) = I^{\frac{1}{2}} x(\frac{1}{2}), \\ ^c D^{\frac{1}{3}} x(\frac{1}{4}) +^c D^{\frac{1}{3}} x(\frac{1}{5}) = 3x'(1). \end{cases} \tag{5}$$

where $p = \frac{3}{2}, q = \frac{1}{3}, r = \frac{1}{2}, a = 1, b = 3, \alpha = \frac{1}{4}, \beta = \frac{1}{5}, \gamma = \frac{1}{2}$, *and*

$$f(t, x(t),^c D^{\frac{1}{3}} x(t), I^{\frac{1}{2}} x(t)) = 4t^{\frac{1}{3}}(2 + \cos^3 x(t)) + \frac{|^c D^{\frac{1}{3}} x(t)|}{1 + |^c D^{\frac{1}{3}} x(t)|} + \frac{4}{t+5} \Big(\frac{I^{\frac{1}{2}} x(t)}{1 + I^{\frac{1}{2}} x(t)} \Big).$$

Clearly $|f(t, x(t),^c D^{\frac{1}{3}} x(t), I^{\frac{1}{2}} x(t))| \leq (12t^{\frac{1}{3}})(1 + \frac{4}{t+5}) = h(t)$ *with* $\|h\| = 21.6$. *Hence, by Theorem 2, the problem (5) has at least one solution on* $[0,1]$.

Example 2. *Consider the following fractional boundary value problem given by*

$$\begin{cases} ^c D^{\frac{7}{4}} x(t) = \frac{2}{t+80} (\sin(x(t)) +^c D^{\frac{3}{5}} x(t) + I^{\frac{1}{3}} x(t) + 3) \\ x(0) + x(1) = I^{\frac{1}{3}} x(\frac{1}{7}), \\ ^c D^{\frac{3}{5}} x(\frac{1}{4}) +^c D^{\frac{3}{5}} x(\frac{1}{6}) = 2x'(1). \end{cases} \tag{6}$$

Here $p = \frac{7}{4}, q = \frac{3}{5}, r = \frac{1}{3}, a = 1, b = 2, \alpha = \frac{1}{4}, \beta = \frac{1}{6}, \gamma = \frac{1}{7}$, *and*

$$f(t, x(t),^c D^{\frac{3}{5}} x(t), I^{\frac{1}{3}} x(t)) = \frac{2}{t+80} (\sin(x(t)) +^c D^{\frac{3}{5}} x(t) + I^{\frac{1}{3}} x(t) + 3).$$

Using the given values of the parameters, we obtain $h_1 \approx 4.5153, h_2 \approx 9.6398$ *and* $k = \max\{h_1, h_2\} \approx 9.64$. *It is easy to see that*

$$|f(t, x(t),^c D^{\frac{3}{5}} x(t), I^{\frac{1}{3}} x(t)) - f(t, y(t),^c D^{\frac{3}{5}} y(t), I^{\frac{1}{3}} y(t))| \leq \frac{1}{40} (|x-y| + |^c D^{\frac{3}{5}} x -^c D^{\frac{3}{5}} y| + |I^{\frac{1}{3}} x - I^{\frac{1}{3}} y|).$$

We have $\ell = \frac{1}{40}$ *and* $4k\ell \approx 0.9640 < 1$. *Therefore, by Theorem 3 we deduce that the problem (6) has a unique solution on* $[0,1]$.

4. Conclusions

In this paper, we have obtained some existence results for a non-linear fractional integro-differential problem with non-local boundary conditions by means of Schauder type fixed point theorem and contraction mapping principle. Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem.

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