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Adjustable Bézier Curves with Simple Geometric Continuity Conditions

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Abstract: This paper aims to simplify the continuity conditions of Bézier curves. For this purpose, a special family of Bézier curves with three parameters, to be called adjustable Bézier curves, is constructed. They have the same structure as the quartic Bézier curves. The newly constructed curves possess some of the basic properties of Bézier curves, such as the convex hull property, symmetry, geometric invariance, etc., and they have shape adjustability. Moreover, under the geometric continuity of order 1 (G^1) conditions of the usual Bézier curves, the adjustable Bézier curves can reach geometric continuity of order k (G^k); here, k is one of the parameters of the newly constructed curves. The recursive evaluation algorithm of the new curves is provided. We also discuss how to construct the adjustable Bézier curves with a given tangent polygon. Numerical examples illustrate the correctness and validity of the proposed method.

Keywords: Bézier curve; shape parameter; recursive algorithm; geometric continuity; tangent polygon

1. Introduction

In a computer-aided geometric design, the Bézier model is a powerful tool for constructing free-form curves and surfaces [1]. It has a number of desirable properties that account for its popularity. It should be noted, however, that this method, like many others, might not be capable of meeting all the requirements that might appear during the design process.

A drawback of the Bézier method is that the position of the curves and surfaces is fixed relative to the given control points. To overcome this shortcoming, many attempts have been made to incorporate shape parameters into the Bézier model [2–13]. These models are constructed on different spaces, such as polynomial function space [2–8], trigonometric function space [9–12], and hyperbolical function space [13]. All the curves constructed in [2–13] are extensions of the classical Bézier curves. They inherit most of the properties of the Bézier method and enjoy other properties ideal for modeling.

These methods have improved the shape adjustment ability of Bézier curves. However, when using the Bézier method to describe complex shapes, the problem of smooth joining still needs to be solved. The problem of continuity has also been studied in the literature when considering the degree reduction of Bézier curves with boundary constraints [14,15]. In general, the higher the requirement for smoothness is, the more complex the smooth joining condition is. Most of the new methods [2–6,8–13] in the literature have not reduced the difficulty of this problem. Yan and Liang present a special family of Bézier curves that have the same structure as the quadratic Bézier curves [7]. The geometric continuity conditions of these newly constructed curves are relatively simple. When using them to construct composite curves, the adjacent segments can reach continuity of an arbitrary order as long as we select the appropriate parameter.

In this paper, we extend the curves in [7] and define the special family of Bézier curves with the same structure as the quartic Bézier curves. The newly constructed curves have an adjustable shape as well as simple geometric continuity conditions. An important advantage of these newly constructed curves is the presence of the free parameters α, k, and s, which can be used to adjust the shape of the curve and to construct composite curves that are geometric-continuous up to a given order at the joins. More specifically, the parameter k relates to boundary continuity, s relates to the total degree of the newly constructed curves, and parameter α controls the proximity of the resulting curve and the control polygon.

2. Blending Functions

2.1. Construction of the Blending Functions

Definition 1. *Let k and s be integers, and $k \geq 1$, $1 \leq s \leq k$, $m = 2k+s+1$, $\alpha \in (0,1]$. For $t \in [0,1]$, we can use the Bernstein basis functions $B_i^m(t) = C_m^i t^i (1-t)^{m-i}$, $i = 0, \ldots, m$, to define the following functions:*

$$\begin{cases} f_0(t;k,s,\alpha) = B_0^m(t) + (1-\alpha)\sum\limits_{i=1}^{k} B_i^m(t), \\ f_1(t;k,s,\alpha) = \alpha \sum\limits_{i=1}^{k} B_i^m(t), \\ f_2(t;k,s,\alpha) = \sum\limits_{i=k+1}^{k+s} B_i^m(t), \\ f_3(t;k,s,\alpha) = \alpha \sum\limits_{i=k+s+1}^{2k+s} B_i^m(t), \\ f_4(t;k,s,\alpha) = (1-\alpha) \sum\limits_{i=k+s+1}^{2k+s} B_i^m(t) + B_m^m(t). \end{cases} \tag{1}$$

We call $f_i(t;k,s,\alpha)$, $i = 0, \ldots, 4$, Bernstein-like blending functions (BL) associated with parameters k s, and α.

For simplicity, we will abbreviate $f_i(t;k,s,\alpha)$ and $B_i^m(t)$ by f_i and B_i^m whenever there is no confusion. Sometimes we will write $f_i(t;k,s,\alpha)$ as $f_i(t)$.

2.2. Properties of the Blending Functions

Proposition 1. *The BL functions have the following properties:*

(a) *Degeneracy: When $k = s = \alpha = 1$, the BL functions are the quartic Bernstein basis functions.*

(b) *Non-negativity: When $\alpha \in (0,1]$, for any k $(k \geq 1)$ and s $(1 \leq s \leq k)$, we have $f_i \geq 0$, $i = 0, \ldots, 4$.*

(c) *Normalization: For any k $(k \geq 1)$, s $(1 \leq s \leq k)$, and $\alpha \in (0,1]$, we have $\sum\limits_{i=0}^{4} f_i = 1$.*

(d) *Symmetry: For any k $(k \geq 1)$, s $(1 \leq s \leq k)$, $\alpha \in (0,1]$, and $t \in [0,1]$, we have $f_i(t) = f_{4-i}(1-t)$, and $i = 0, \ldots, 4$.*

(e) *Endpoint property: For any k $(k \geq 1)$, s $(1 \leq s \leq k)$, and $\alpha \in (0,1]$ we have*

$$\begin{cases} f_0(0) = 1, \quad f_i(0) = 0 \quad (i = 1,2,3,4), \\ f_i(1) = 0 \quad (i = 0,1,2,3), \quad f_4(1) = 1, \end{cases} \tag{2}$$

And, for any L, when $1 \leq L \leq k$, we have

$$\begin{cases} f_0^{(L)}(0) = (-1)^L L! C_m^L \alpha, f_1^{(L)}(0) = (-1)^{L-1} L! C_m^L \alpha, f_i^{(L)}(0) = 0\ (i = 2,3,4), & (3a) \\ f_i^{(L)}(1) = 0\ (i = 0,1,2), f_3^{(L)}(1) = -L! C_m^L \alpha, f_4^{(L)}(1) = L! C_m^L \alpha. & (3b) \end{cases}$$

(f) *Linear independence: For any k $(k \geq 1)$, s $(1 \leq s \leq k)$, and $\alpha \in (0,1]$, the BL functions $f_i(t;k,s,\alpha)$, $i = 0, \ldots, 4$ are linearly independent.*

Proof. We only prove (e) and (f), since the others are easy to obtain.

(e) The conclusions in (2) are obvious. We only prove (3a) and (3b). If

$$f(t) = a_{n-1} + a_n t^n + a_{n+1} t^{n+1} + \ldots + a_m t^m,$$

where $n \in N^+$, $n \leq m$, then

$$f^{(L)}(0) = \begin{cases} 0, & 1 \leq L < n, \\ L! a_L, & n \leq L \leq m. \end{cases} \tag{4}$$

Note that the Bernstein basis functions can be written as

$$B_i^m(t) = \sum_{j=i}^{m} C_m^i C_{m-i}^{j-i} (-1)^{j-i} t^j.$$

Therefore, from (4), we obtain

$$\begin{cases} \frac{d^L B_i^m(t)}{dt^L}|_{t=0} = 0, & 1 \leq L < i, \\ \frac{d^L B_i^m(t)}{dt^L}|_{t=0} = (-1)^{L-i} L! C_m^i C_{m-i}^{L-i}, & i \leq L \leq m. \end{cases} \tag{5}$$

From (1) and (5), it follows that, when $1 \leq L \leq k$,

$$f_1^{(L)}(0) = \alpha \sum_{i=1}^{L} (-1)^{L-i} L! C_m^i C_{m-i}^{L-i} = \alpha L! C_m^L \sum_{i=1}^{L} C_L^i (-1)^{L-i} = (-1)^{L-1} L! C_m^L \alpha, \tag{6}$$

and

$$f_i^{(L)}(0) = 0, \quad i = 2, 3, 4. \tag{7}$$

Besides, from the relation $\sum_{i=0}^{4} f_i = 1$, we have $\sum_{i=0}^{4} f_i^{(L)} = 0$, so

$$f_0^{(L)}(0) = -\sum_{i=1}^{4} f_i^{(L)}(0). \tag{8}$$

Substituting (6) and (7) into (8) shows that, when $1 \leq L \leq k$,

$$f_0^{(L)}(0) = (-1)^L L! C_m^L \alpha. \tag{9}$$

From (6), (7) and (9), we see that (3a) holds. Then, from the symmetry of the BL functions we obtain (3b) immediately.

(f) Let us consider the linear combination

$$\sum_{i=0}^{4} a_i f_i = 0, \tag{10}$$

where $a_i \in R$, $i = 0, \ldots, 4$. Substituting (1) into (10) and rearranging the terms, we obtain

$$a_0 B_0^m + [(1-\alpha)a_0 + \alpha a_1] \sum_{i=1}^{k} B_i^m + a_2 \sum_{i=k+1}^{k+s} B_i^m + [\alpha a_3 + (1-\alpha)a_4] \sum_{i=k+s+1}^{2k+s} B_i^m + a_4 B_m^m = 0.$$

From the linear independence of the Bernstein basis functions, it follows that

$$\begin{cases} a_0 = 0, \\ (1-\alpha)a_0 + \alpha a_1 = 0, \\ a_2 = 0, \\ \alpha a_3 + (1-\alpha)a_4 = 0, \\ a_4 = 0. \end{cases} \tag{11}$$

Obviously, when $\alpha \neq 0$, the solutions of (11) are $a_i = 0, \quad i = 0, \ldots, 4$. This shows that the BL functions are linearly independent. □

3. Adjustable Bézier Curves

3.1. Construction of the Adjustable Bézier Curves

Definition 2. *Given five control points $\boldsymbol{V}_i \in \mathbb{R}^d$, $d = 2, 3$, and $i = 0, \ldots, 4$, we can use the BL functions to define an adjustable Bézier curve by*

$$\boldsymbol{f}(t) = \sum_{i=0}^{4} f_i(t; k, s, \alpha)\boldsymbol{V}_i, \quad t \in [0, 1]. \tag{12}$$

Proposition 2. *The adjustable Bézier curves can be expressed as Bézier curves of degree m, that is,*

$$\boldsymbol{f}(t) = \sum_{i=0}^{4} f_i \boldsymbol{V}_i = \sum_{i=0}^{m} B_i^m \boldsymbol{R}_i, \quad t \in [0, 1],$$

where the control points $\boldsymbol{R}_i$, $i = 0, \ldots, m$, are given by

$$\boldsymbol{R}_i = \begin{cases} \boldsymbol{V}_0, & i = 0, \\ (1-\alpha)\boldsymbol{V}_0 + \alpha \boldsymbol{V}_1, & i = 1, \ldots, k, \\ \boldsymbol{V}_2, & i = k+1, \ldots, k+s, \\ \alpha \boldsymbol{V}_3 + (1-\alpha)\boldsymbol{V}_4, & i = k+s+1, \ldots, 2k+s, \\ \boldsymbol{V}_4, & i = m. \end{cases} \tag{13}$$

Proof. Substituting (1) into (12) yields

$$\boldsymbol{f}(t) = B_0^m \boldsymbol{V}_0 + \sum_{i=1}^{k} B_i^m [(1-\alpha)\boldsymbol{V}_0 + \alpha \boldsymbol{V}_1] + \sum_{i=k+1}^{k+s} B_i^m \boldsymbol{V}_2 + \sum_{i=k+s+1}^{2k+s} B_i^m [\alpha \boldsymbol{V}_3 + (1-\alpha)\boldsymbol{V}_4] + B_m^m \boldsymbol{V}_4.$$

Now let

$$\boldsymbol{f}(t) = \sum_{i=0}^{m} B_i^m \boldsymbol{R}_i.$$

The above two relations imply (13) immediately. □

Remark 1. *For an adjustable Bézier curve with control polygon $\boldsymbol{V}_0\boldsymbol{V}_1\boldsymbol{V}_2\boldsymbol{V}_3\boldsymbol{V}_4$, considered a Bézier curve, the control polygon is $\boldsymbol{R}_0\boldsymbol{R}_1 \ldots \boldsymbol{R}_m$. From (13), we see that the polygon $\boldsymbol{R}_0\boldsymbol{R}_1 \ldots \boldsymbol{R}_m$ has at most five different control points $\boldsymbol{V}_0\boldsymbol{R}_1\boldsymbol{V}_2\boldsymbol{R}_{m-1}\boldsymbol{V}_4$. When we change the value of α, the polygon $\boldsymbol{V}_0\boldsymbol{R}_1\boldsymbol{V}_2\boldsymbol{R}_{m-1}\boldsymbol{V}_4$ will be changed accordingly. The greater the value of α is, the smaller the difference between the polygons $\boldsymbol{V}_0\boldsymbol{R}_1\boldsymbol{V}_2\boldsymbol{R}_{m-1}\boldsymbol{V}_4$ and $\boldsymbol{V}_0\boldsymbol{V}_1 \ldots \boldsymbol{V}_4$. In particular, when $\alpha = 1$, the two polygons coincide. By changing the value of s, the weight of the vertex $\boldsymbol{V}_2$ will be changed. The greater the value is, the greater the weight is. The weight of $\boldsymbol{R}_1$ and $\boldsymbol{R}_{m-1}$ depend on k. Moreover, the greater the value of k, the greater is the weight of $\boldsymbol{R}_1$ and $\boldsymbol{R}_{m-1}$.*

For any $t \in [0,1]$, in order to obtain the corresponding point $\boldsymbol{f}(t)$ on an adjustable Bézier curve, we can first convert the curve to a Bézier curve and then use the de Casteljau algorithm [1] for Bézier curves. That is,

$$\boldsymbol{f}(t) = \sum_{i=0}^{4} f_i \boldsymbol{V}_i = \sum_{i=0}^{m} B_i^m \boldsymbol{R}_i = \sum_{i=0}^{m-1} B_i^{m-1} \boldsymbol{R}_i^1 = \ldots = \sum_{i=0}^{1} B_i^1 \boldsymbol{R}_i^{m-1} = \boldsymbol{R}_0^m.$$

The points $\boldsymbol{R}_i$, $i = 0, \ldots, m$, are defined by (13), and the intermediate points $\boldsymbol{R}_i^l$ are defined recursively by

$$\boldsymbol{R}_i^l = (1-u)\boldsymbol{R}_i^{l-1} + u\boldsymbol{R}_{i+1}^{l-1},$$

where $l = 1, \ldots, m$, $i = 0, \ldots, m-l$, and $\boldsymbol{R}_i^0 = \boldsymbol{R}_i$, $i = 0, \ldots, m$.

Figure 1 shows the process of recursive evaluation of an adjustable Bézier curve with $k = s = 2$ and $\alpha = \frac{1}{2}$ at $t = \frac{1}{2}$. In this figure, the points obtained in different recursive steps are shown in different colors and marks.

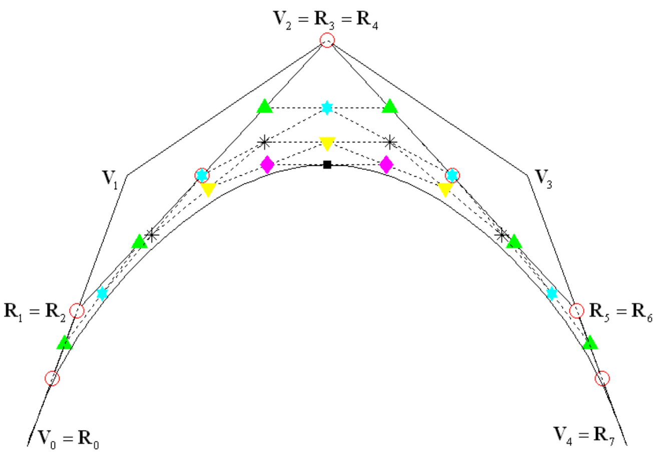


Figure 1. Recursive evaluation process of an adjustable Bézier curve. The red circles, green triangles, blue stars, black stars, yellow triangles, purple rhombuses represent the points received in step 1 to step 6 of the recurrence, respectively. The black square represents the point received in the last step of recurrence.

3.2. Properties of the Adjustable Bézier Curves

From the properties of the BL functions, we obtain the following properties of the adjustable Bézier curves.

(1) *Convex hull property*: The adjustable Bézier curves lie inside the convex hull of the control points. This is true, since the BL functions are nonnegative on $[0,1]$ and sum to 1.

(2) *Geometric invariance*: From (13), we know that the adjustable Bézier curves are affine combinations of their control points. Thus, their shape is independent of the choice of the coordinate system.

(3) *Symmetry*: The points $\boldsymbol{V}_i$, $i = 0, \ldots, 4$, and $\boldsymbol{V}_i$, $i = 4, \ldots, 0$, define two adjustable Bézier curves with the same shape but different parameterization.

(4) *Geometric property at the endpoints*: From (2), (3) and (12) we get

$$\begin{cases} \boldsymbol{f}(0) = \boldsymbol{V}_0, \\ \boldsymbol{f}(1) = \boldsymbol{V}_4. \end{cases} \tag{14}$$

$$\begin{cases} \boldsymbol{f}^{(L)}(0) = (-1)^{L-1} L! C_m^L \alpha (\boldsymbol{V}_1 - \boldsymbol{V}_0), \\ \boldsymbol{f}^{(L)}(1) = L! C_m^L \alpha (\boldsymbol{V}_4 - \boldsymbol{V}_3), \quad 1 \le L \le k. \end{cases} \tag{15}$$

Remark 2. *From (14), we see that the adjustable Bézier curves interpolate the first and last vertices of the control polygon. From (15), we see that, at each endpoint of the adjustable Bézier curves, the derivative vectors of orders 1 to k are collinear.*

(5) *Shape adjustability property*: Even if the control points of an adjustable Bézier curve are fixed, its shape can still be adjusted by changing the values of the three parameters k, s, and α.

Figures 2–4 illustrate the influence of the parameters k, s, and α on the shape of the adjustable Bézier curves. In Figure 2A, the curves are generated by fixing $k = 2$ and $s = 1$. In Figure 2B, the curves are generated by fixing $k = s = 3$. In Figure 2, Curves 1 to 4 take $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, respectively. In Figure 3A, Curves 1 to 4 are generated by fixing $k = 4$, $\alpha = \frac{1}{2}$, taking $s = 1, 2, 3, 4$, respectively. In Figure 3B, Curves 1 to 3 are generated by fixing $k = 5$, $\alpha = 1$, taking $s = 1, 3, 5$, respectively. In Figure 4A, the curves are generated by fixing $s = 1$ and $\alpha = \frac{1}{3}$. In Figure 4B, the curves are generated by fixing $s = 1$, $\alpha = \frac{2}{3}$. In Figure 4 Curves 1 to 3 take $k = 1, 3, 5$, respectively.

From Figure 2, we see that, when k and s are fixed at a time and α varies, the greater the value of α, the better keeps the adjustable Bézier curve the characteristics of its control polygon. From Figure 3, we see that, when k and α are fixed at a time and s varies, the greater the value of s, the more approximates the middle of the adjustable Bézier curve to the control point $\mathbf{V}_2$. From Figure 4, we see that, when s and α are fixed at a time and k varies, the greater the value of k, the more each end of the adjustable Bézier curve approximates the first and last edges of its control polygon.

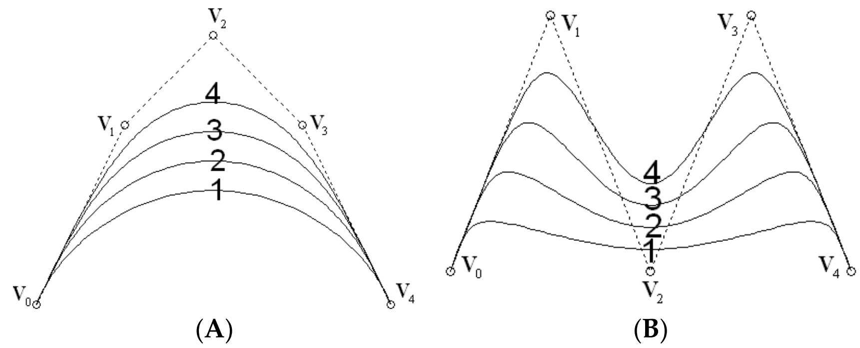


Figure 2. The adjustable Bézier curves with the same parameters k and s but different α. (**A**) $k = 2$, $s = 1$; (**B**) $k = s = 3$.

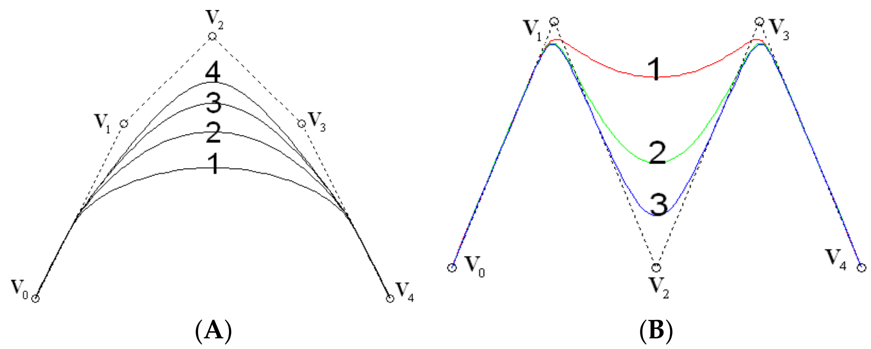


Figure 3. The adjustable Bézier curves with the same parameters k and α but different s. (**A**) $k = 4$, $\alpha = \frac{1}{2}$; (**B**) $k = 5$, $\alpha = 1$.

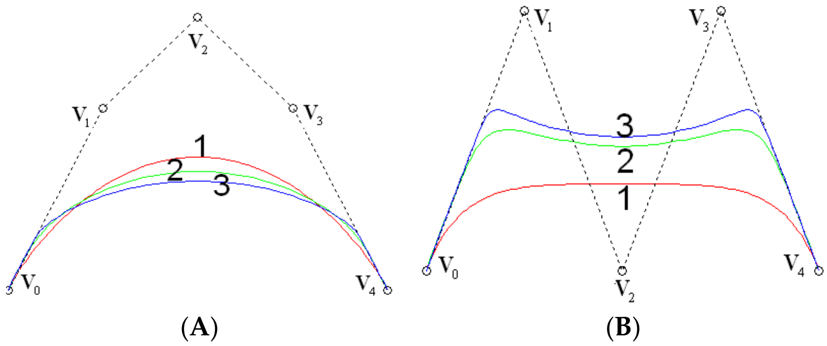


Figure 4. The adjustable Bézier curves with the same parameters s and α but different k. (**A**) $s = 1$, $\alpha = \frac{1}{3}$; (**B**) $s = 1$, $\alpha = \frac{2}{3}$.

Remark 3. *The adjustable Bézier curves can be seen as Bézier curves, and the relationship between Bézier curves and their control polygons is well known. Therefore, from the effect of k, s, and α on the control polygons of Bézier curves (see Remark 1), it is easy to understand the effect of the parameters on the shape of the adjustable Bézier curves described in the previous paragraph.*

4. Composite Adjustable Bézier Curves

When describing complex shapes, a single curve segment often cannot meet the requirements, so it is necessary to use composite curves. When we tackle the problem of composite Bézier curves, we immediately encounter the problem of continuity constraints between the pieces. Continuity constraint problems will also lead to discussing degree reduction of Bézier curves with boundary constraints. Next, we will discuss how to construct composite curves and how to achieve a smooth connection when using the adjustable Bézier curves.

There are two kinds of smoothness in computer-aided geometric design; the parametric continuity and the geometric continuity. The use of parametric continuity disallows many parameterizations that generate geometrically smooth curves. The geometric continuity is a relaxed form of the parametric continuity.

Here, we discuss the k^{th} order geometric continuity (G^k continuity) conditions of the adjustable Bézier curves. In [1,16,17], the definition of G^k continuity is given. Further, the practical Beta-constraints for the geometric continuity of curves are provided in [16,17]. According to the Beta-constraints, we have the following conclusion.

Lemma 1. *Let $t \in [0,1]$, and assume that the curves $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ are joined at $\boldsymbol{f}(1) = \boldsymbol{g}(0)$. If for $1 \le L \le k$,*

$$\begin{cases} \boldsymbol{f}^{(L)}(1) = S_L(\boldsymbol{V}_1 - \boldsymbol{V}_0), \\ \boldsymbol{g}^{(L)}(0) = (-1)^{L-1} S_L(\boldsymbol{V}_1 - \boldsymbol{V}_0), \end{cases} \tag{16}$$

where S_L is a constant related to L, then the two curves are G^k–continuous at the join.

Proof. To make the two curves G^k–continuous, it is necessary that (taken from the Beta-constraints in [16])

$$\begin{pmatrix} \boldsymbol{g}'(0) \\ \boldsymbol{g}''(0) \\ \boldsymbol{g}'''(0) \\ \boldsymbol{g}^{(4)}(0) \\ \vdots \\ \boldsymbol{g}^{(k)}(0) \end{pmatrix} = \begin{pmatrix} \beta_1 & & & & & \\ \beta_2 & \beta_1^2 & & & & \\ \beta_3 & 3\beta_1\beta_2 & \beta_1^3 & & & \\ \beta_4 & 4\beta_1\beta_3 + 3\beta_2^2 & 6\beta_1^2\beta_2 & \beta_1^4 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \beta_k & \cdots & & & & \beta_1^k \end{pmatrix} \begin{pmatrix} \boldsymbol{f}'(1) \\ \boldsymbol{f}''(1) \\ \boldsymbol{f}'''(1) \\ \boldsymbol{f}^{(4)}(1) \\ \vdots \\ \boldsymbol{f}^{(k)}(1) \end{pmatrix}, \tag{17}$$

where $\beta_1 > 0$. Substituting (16) into (17), we obtain

$$\begin{pmatrix} S_1 \\ -S_2 \\ S_3 \\ -S_4 \\ \vdots \\ (-1)^{k-1}S_k \end{pmatrix} = \begin{pmatrix} \beta_1 & & & & & \\ \beta_2 & \beta_1^2 & & & & \\ \beta_3 & 3\beta_1\beta_2 & \beta_1^3 & & & \\ \beta_4 & 4\beta_1\beta_3 + 3\beta_2^2 & 6\beta_1^2\beta_2 & \beta_1^4 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \beta_k & \cdots & & & & \beta_1^k \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ \vdots \\ S_k \end{pmatrix}. \tag{18}$$

It is clear that (18) has a unique solution for β_i, $i = 1, \ldots, k$, with $\beta_1 = 1$. Thus, the two curves are G^k–continuous. □

From (15) and Lemma 1, we obtain the following result.

Proposition 3. *Assume that two adjustable Bézier curves,*

$$\boldsymbol{f}_1(t) = \sum_{i=0}^{4} f_i(t; k_1, s_1, \alpha_1)\boldsymbol{V}_i, \quad t \in [0, 1],$$

and

$$\boldsymbol{f}_2(t) = \sum_{i=0}^{4} f_i(t; k_2, s_2, \alpha_2)\boldsymbol{V}_i^*, \quad t \in [0, 1],$$

are joined at $\boldsymbol{f}_1(1) = \boldsymbol{f}_2(0)$, i.e., $\boldsymbol{V}_4 = \boldsymbol{V}_0^*$. When

$$\boldsymbol{V}_1^* - \boldsymbol{V}_0^* = C(\boldsymbol{V}_4 - \boldsymbol{V}_3) \quad (C > 0)\ , \tag{19}$$

for any $s_i \in [1, k_i]$ and $\alpha_i \in (0, 1]$, $i = 1, 2$, the two curves are G^k–continuous, where $k = min\{k_1, k_2\}$.

Remark 4. *From Proposition 3 we see that, when the control points of two position continuous adjustable Bézier curves meet the condition (19), the continuity of the composite curve depends only on $k = min\{k_1, k_2\}$, and has nothing to do with s_i and α_i, $i = 1, 2$. Thus, if we want to change the continuity of the composite curve, we can alter k by altering k_1 or (and) k_2. Of course, this way, both the continuity and the shape of the composite curve will change. If we want to adjust the shape of the composite curves without disrupting the continuity, we can keep k_1 and k_2 unchanged, but change s_1, α_1 and/or s_2, α_2. In short, there are many ways to adjust the continuity and the shape of the composite adjustable Bézier curve.*

Figures 5 and 6 show some different composite adjustable Bézier curves defined by the same control polygon. In Figure 5, Curve Segments 1 and 5 take $k = s = 1$, $\alpha = \frac{1}{2}$; 2 and 6 take $k = s = 2$, $\alpha = \frac{1}{2}$; 3 and 7 take $k = s = 3$, $\alpha = \frac{2}{3}$; 4 and 8 take $k = s = 4$, $\alpha = 1$. In these curve segments, 1 and any one of 5–8, and 5 and any one of 1–4, are G^1 continuous; 2 and any one of 6–8, and 6 and any one of 2–4, are G^2 continuous; 3 and 7, 3 and 8, and 4 and 7 are G^3 continuous; 4 and 8 are G^4 continuous. In Figure 6A, all curve segments take $k = s = 3$. Curve Segments 1 and 4 take $\alpha = \frac{1}{4}$; 2 and 5 take $\alpha = \frac{1}{2}$; 3 and 6 take $\alpha = \frac{3}{4}$. In these curve segments, any one of 1–3 and any one of 4–6 are G^3continuous. In Figure 6B, the parameter settings of Curves 1 to 5 are as follows: $k = s = 2$, $\alpha = 1$; $k = 2$; $s = 1$, $\alpha = \frac{1}{3}$; $k = s = 3$, $\alpha = 1$; $k = 3$, $s = 2$, $\alpha = \frac{2}{3}$; $k = 3$, $\alpha = \frac{1}{3}$. In these curve segments, any one of 1–2 and any one of 3–5 are G^2 continuous.

Remark 5. *The two adjustable Bézier curves can be connected in a G^k fashion under the condition of relatively simple; however, all the continuities up to order k have to be zero. By construction, these curves have k vanishing derivatives at both endpoints. This is a strong restriction, and we hope it can be improved in future research.*

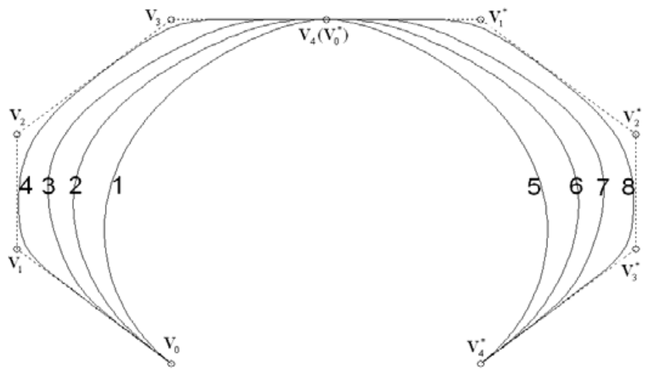
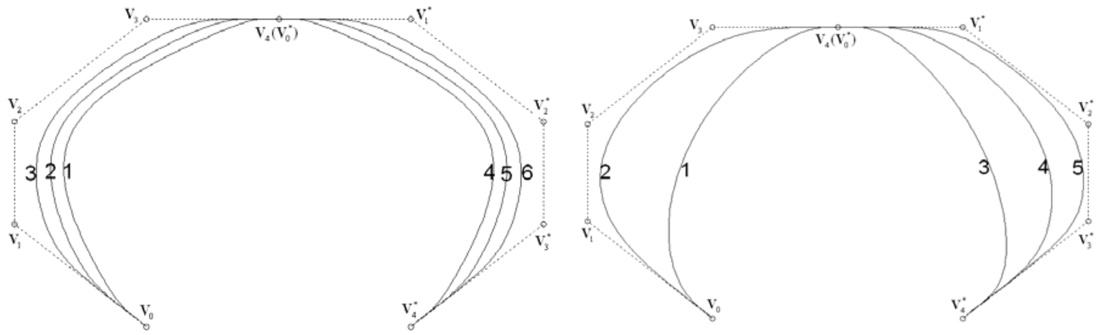


Figure 5. Composite adjustable Bézier curves with different continuity and shape.



(**A**) (**B**)

Figure 6. Composite adjustable Bézier curves with the same continuity but different shape. (**A**) $k = s = 3$; (**B**) the parameters are different from each other.

5. Adjustable Bézier Curves with Tangent Polygon

In geometric design, we often need to solve the following problem: when the tangent lines of the outline are known, how is a curve approaching them found? E.g., how is a curve tangent to the given polygon defined? This problem has been discussed previously (for instance, [18–20]).

As an application, we will next discuss how to construct an adjustable Bézier curve that is tangential to a given polygon at the specified points of tangency.

Assume that there is a closed polygon with vertices $\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, where $\boldsymbol{P}_0 = \boldsymbol{P}_n$. The purpose of this section is to construct a closed composite adjustable Bézier curve tangent to each edge of the polygon. Let the tangent point at the i^{th} segment of the polygon be

$$\boldsymbol{T}_i = (1 - \lambda_i)\boldsymbol{P}_{i-1} + \lambda_i \boldsymbol{P}_i, \tag{20}$$

where $\lambda_i \in (0, 1)$, $i = 1, \ldots, \mathrm{n}$, are the adjustable parameters of the tangent points.

To this aim, we add a virtual point $\boldsymbol{T}_{n+1} = \boldsymbol{T}_1$, and we construct one adjustable Bézier curve between two adjacent tangent points $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$, $i = 1, 2, \ldots, n$. Hence, the whole composite adjustable Bézier curve is composed of n curve segments. The control points of the i^{th} segment

$$\boldsymbol{f}_i(t) = \sum_{j=0}^{4} f_j(t; k_i, s_i, \alpha_i)\boldsymbol{V}_{ij}, \quad t \in [0, 1],$$

are chosen as follows:

$$\begin{cases} \boldsymbol{V}_{i0} = \boldsymbol{T}_i, \\ \boldsymbol{V}_{i1} = \boldsymbol{V}_{i2} = \boldsymbol{V}_{i3} = \boldsymbol{P}_i, \\ \boldsymbol{V}_{i4} = \boldsymbol{T}_{i+1}, \end{cases} \tag{21}$$

where $i = 1, \ldots, n$.

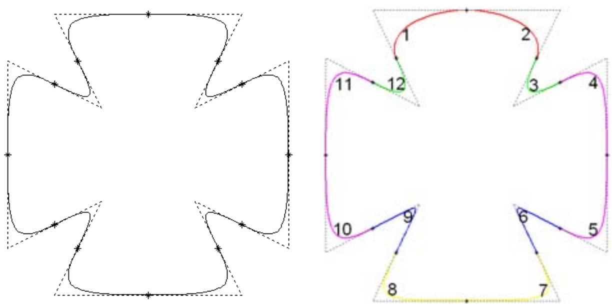
According to (14), (15), (20) and (21), we obtain

$$\begin{cases} \boldsymbol{f}_i(1) = \boldsymbol{f}_{i+1}(0) = \boldsymbol{T}_{i+1}, \\ \boldsymbol{f}_i^{(L)}(1) = L!C_{m_i}^L \alpha_i (\boldsymbol{T}_{i+1} - \boldsymbol{P}_i) = L!C_{m_i}^L \alpha_i \lambda_{i+1} (\boldsymbol{P}_{i+1} - \boldsymbol{P}_i), \quad 1 \le L \le k_i, \\ \boldsymbol{f}_{i+1}^{(L)}(0) = (-1)^{L-1} L!C_{m_{i+1}}^L \alpha_{i+1} (\boldsymbol{P}_{i+1} - \boldsymbol{T}_{i+1}) = (-1)^{L-1} L!C_{m_{i+1}}^L \alpha_{i+1} (1 - \lambda_{i+1})(\boldsymbol{P}_{i+1} - \boldsymbol{P}_i), \quad 1 \le L \le k_{i+1}, \end{cases} \tag{22}$$

where $m_i = 2k_i + s_i + 1$, $i = 1, \ldots, n-1$. According to (22), we know that the i^{th} segment of the composite curve is tangent to the polygon at $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$. Moreover, the i^{th} and $(i+1)^{\text{th}}$ curve segments are G^k continuous, where $k = \min\{k_i, k_{i+1}\}$.

Remark 6. *According to (21), we know that the vertices and tangent points of the given polygon can directly determine all the control points of the composite adjustable Bézier curve; hence, the method is very simple. Figure 7 intuitively shows its flexibility and effectiveness.*

Figure 7 shows two composite adjustable Bézier curves with the same tangent polygon and tangent points. In this figure, the dotted line is the given tangent polygon; the points marked with an asterisk are the tangent points. All of the adjustable parameters of the tangent points take the same value $\frac{1}{2}$. In Figure 7A, all curve segments take $k = s = 3$ and $\alpha = 0$. Hence, all the adjacent curve segments are G^3 continuous. In Figure 7B, Curve Segments 1 and 2 take $k = s = 1$ and $\alpha = 0$; 3 and 12 take $k = s = 2$ and $\alpha = 0$; 4, 5, 10, and 11 take $k = s = 3$ and $\alpha = 0$; 6 and 9 take $k = s = \alpha = 1$; 7 and 8 take $k = s = 4$ and $\alpha = 0$. Hence, Curve Segments 1 and 2, 2 and 3, 5 and 6, 6 and 7, 8 and 9, 9 and 10, and 12 and 1 are G^1 continuous; 3 and 4, and 11 and 12, are G^2 continuous; 4 and 5, and 10 and 11, are G^3 continuous; 7 and 8 are G^4 continuous.



(A) **(B)**

Figure 7. Adjustable Bézier curves tangential to the given polygon. (**A**) $k = s = 3$, $\alpha = 0$; (**B**) each segment with different parameters.

6. Conclusions

The adjustable Bézier curves defined in this paper inherit the endpoint characteristics and most of the good properties of the classical Bézier curves. Compared with ordinary Bézier curves, the adjustable Bézier curves have two main advantages; a flexible shape and simple continuity conditions. These two merits simplify the use of adjustable Bézier curves in curve design. The tensor product surfaces defined by the BL functions can be expected to have similarly beneficial properties and thus deserve further study.

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