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The Regularization-Homotopy Method for the Two-Dimensional Fredholm Integral Equations of the First Kind

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Abstract: In this work, we consider two-dimensional linear and nonlinear Fredholm integral equations of the first kind. The combination of the regularization method and the homotopy perturbation method, or shortly, the regularization-homotopy method is used to find a solution to the equation. The application of this method is based upon converting the first kind of equation to the second kind by applying the regularization method. Then the homotopy perturbation method is employed to the resulting second kind of equation to obtain a solution. A few examples including linear and nonlinear equations are provided to show the validity and applicability of this approach.

Keywords: Fredholm integral equations; regularization method; homotopy perturbation method

MSC: 45A05; 65R20; 65J20

1. Introduction

Integral equations appear in many scientific applications with a very wide range from physical sciences to engineering. An immense amount of work has been done on solving them. The literature is very dense on the subject. Many analytical and numerical techniques have been constructed so far and it is still expanding [1–4].

In particular, Fredholm integral equations of the first kind appear in many physical and engineering applications. There are numerous articles and books on the investigation of analytical and numerical solutions of one dimensional Fredholm integral equations of the first kind [1–3].

In general, integral equations are classified as either first or second kind depending on where the unknown function $u(x)$ appears. If it appears only inside the integral sign, it is called an integral equation of the first kind, otherwise, it is called an integral equation of the second kind. The appearance of the unknown function only inside the integral sign introduces some difficulties. These, for instance, include applying known useful methods introduced for solving the second kind of equations to the first kind. To overcome this, one either has to modify the existing techniques, transform the integral equation, or construct a new method if it is possible. The regularization method is a method that transforms the integral equation of the first kind into the second kind. We will make use of this technique in the subsequent sections.

First kind Fredholm integral equations are usually considered to be ill-posed problems. That means, solutions may not exist and if it exists, it may not be unique [3,5,6].

The one dimensional linear and nonlinear Fredholm integral equations are of the form

$$f(x) = \lambda \int_a^b K(x,t)u(t)\,dt, \tag{1}$$

and

$$f(x) = \lambda \int_a^b K(x,t)F(u(t))\,dt\,, \tag{2}$$

respectively. In these equations $f(x)$, the kernel $K(x,t)$, and a constant parameter λ are given. The independent variable x is taken from a closed and bounded region. $F(u(x))$ is a nonlinear function of $u(x)$ and the desired function is $u(x)$. There are some analytical and numerical approaches to find exact or approximate solutions for Equations (1) and (2) in the literature. The one that we particularly focus on in this article is the regularization-homotopy method introduced by A. Wazwaz in [7]. We investigate this method further and show that it is applicable to the two-dimensional Fredholm integral equations of the first kind (see the next section).

The two-dimensional linear Fredholm integral equations has the following form:

$$f(x,t) = \lambda \int_a^b \int_c^d K(x,t,y,z)u(y,z)\,dy\,dz, \tag{3}$$

and the nonlinear equation has the form:

$$f(x,t) = \lambda \int_a^b \int_c^d K(x,t,y,z)F(u(y,z))\,dy\,dz. \tag{4}$$

In these equations $f(x,t)$, the kernel $K(x,t,y,z)$, and a constant parameter λ are given. $F(u(x,t))$ is a nonlinear function of $u(x,t)$ and the desired function is $u(x,t)$.

Research on the two-dimensional case has been getting more attention recently [8–15]. The main goal in this work is to extend the regularization-homotopy method introduced in [7] for one dimensional Fredholm integral equations of the first kind to two-dimensional Fredholm integral equations of the first kind. This method can also applied for obtaining numerical solutions of the Fredholm integral equations of the first kind. Motivated by [14–16], one possible application area could be image restoration and denoising.

2. The Regularization Method

The regularization method was first introduced by A. N. Tikhonov [17,18], and D. L. Phillips [2]. The application of the regularization method transforms the first kind integral equations into the second. The details for one dimensional case can be found in [2,17–19]. We instead focus on the two-dimensional case. The regularization method for the two-dimensional Fredholm integral equations of the first kind was introduced in [13]. We now briefly explain the method. Like in one dimensional case, the regularization method transforms the first kind of equation:

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)u(y,z)\,dy\,dz \tag{5}$$

and the nonlinear equation:

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)F(u(y,z))\,dy\,dz \tag{6}$$

to the second kind of equation:

$$\alpha u_\alpha(x,t) = f(x,t) - \int_a^b \int_c^d K(x,t,y,z)u_\alpha(y,z)\,dy\,dz \tag{7}$$

and

$$\alpha u_\alpha(x,t) = f(x,t) - \int_a^b \int_c^d K(x,t,y,z)F(u_\alpha(y,z))\,dy\,dz, \tag{8}$$

respectively, where α is a small positive parameter. Notice that one could express Equations (7) and (8) as

$$u_\alpha(x,t) = \frac{1}{\alpha} f(x,t) - \frac{1}{\alpha} \int_a^b \int_c^d K(x,t,y,z) u_\alpha(y,z)\, dy\, dz \tag{9}$$

and

$$u_\alpha(x,t) = \frac{1}{\alpha} f(x,t) - \frac{1}{\alpha} \int_a^b \int_c^d K(x,t,y,z) F(u_\alpha(y,z))\, dy\, dz, \tag{10}$$

respectively. It was shown in [20] that the solution of Equation (9) or (10) as $\alpha \to 0$ approaches $u(x,t)$ which is the solution of Equation (5) or (19). In other words,

$$u(x,t) = \lim_{\alpha \to 0} u_\alpha(x,t).$$

We now state some existence and uniqueness results from the operator theory [1,21]. Let

$$A : C([a,b] \times [c,d]) \to C([a,b] \times [c,d])$$

and the integral operator

$$Au(x,t) = \int_c^d \int_a^b K(x,t,y,z) u(y,z)\, dy\, dz \quad x \in [a,b], t \in [c,d]. \tag{11}$$

Theorem 1. *Let $K : C([a,b] \times [c,d] \times [a,b] \times [c,d]) \to \mathbb{R}$ be continuous, then the operator (11) is bounded with the norm:*

$$||A||_\infty = \max_{x \in [a,b], t \in [c,d]} \int_c^d \int_a^b |K(x,t,y,z)|\, dy\, dz\,. \tag{12}$$

Proof. See [21]. □

Theorem 2. *Let A be a bounded operator on $C([a,b] \times [c,d])$ with $||A|| < 1$ and I denotes the identity operator. Then $I - A$ has a bounded inverse on $C([a,b] \times [c,d])$, which is given by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \tag{13}$$

and satisfies

$$||(I - A)^{-1}|| \leq \frac{1}{1 - ||A||}. \tag{14}$$

Proof. See [1]. □

We also want to note that for any $\alpha > 0$, Equation (10) can be written in operator form as

$$u - Au = f \tag{15}$$

With this notation, theorem 2 ensures that $||A||_\infty < 1$ is a sufficient condition for existence and uniqueness of the solution of Equation (15) [21].

3. The Homotopy Perturbation Method

In this section, we investigate the application of the homotopy perturbation (HPM) to the two-dimensional Fredholm integral equations of the first kind. The HPM is a coupling of perturbation method and homotopy in topology. To see the basic idea behind the HPM, let us consider an equation of the form:

$$L(u) = 0, \tag{16}$$

where L is any integral operator. Then a convex homotopy with an embedding parameter $p \in [0,1]$ can be defined by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{17}$$

where *F(u)* is a functional operator with known solutions. It is then easy to see that

$$H(u,p) = 0 \tag{18}$$

implies

$$H(u,0) = F(u) \quad \text{and} \quad H(u,1) = L(u)$$

One can infer from Equations (17) and (18), as the the embedding parameter monotonically increases from 0 to 1, the trival problem (*F(u)* = 0) deforms the original problem (*L(u)* = 0) [22]. For more detailed information on the HPM, we refer the reader to [23,24].

4. The Regularization-Homotopy Method

We investigate the first kind linear equation:

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)u(y,z)\,dy\,dz \tag{19}$$

and the nonlinear equation:

$$f(x,t) = \int_a^b \int_c^d K(x,t,y,z)F(u(y,z))\,dy\,dz.$$

In what follows we focus on explaining the regularization-homotopy method for the linear case. We just make a note about the nonlinear case since it will be treated similarly. We finally give an algorithm about how to apply the method.

We recall from Section 2 that the regularization method transform Equation (19) to the following equation:

$$u_\alpha(x,t) = \frac{1}{\alpha}f(x,t) - \frac{1}{\alpha}\int_a^b \int_c^d K(x,t,y,z)u_\alpha(y,z)\,dy\,dz \tag{20}$$

Since the aim of this article is to extend the homotopy-regularization method introduced in [7], we construct the homotopy as follows:

$$H(u_\alpha,p) = (1-p)F(u_\alpha) + pL(u_\alpha) = 0, \tag{21}$$

where

$$F(u_\alpha) = u_\alpha(x,t),$$
$$L(u_\alpha) = u_\alpha(x,t) - \frac{1}{\alpha}f(x,t) + \frac{1}{\alpha}\int_a^b \int_c^d K(x,t,y,z)u_\alpha(y,z)\,dy\,dz$$

and $p \in [0,1]$ is an embedding parameter monotonically increases from 0 to 1 [7]. The homotopy perturbation method allows writing

$$u_\alpha = u_{\alpha,0} + pu_{\alpha,1} + p^2u_{\alpha,2} + \dots \tag{22}$$

as a power series in p and setting $p = 1$, *i.e.*,

$$u_\alpha = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_{\alpha,n} \tag{23}$$

Now, if we expand Equation (21) , we obtain

$$(1-p)u_\alpha(x,t) + p\Big[u_\alpha(x,t) - \frac{1}{\alpha}f(x,t) + \frac{1}{\alpha}\int_a^b\int_c^d K(x,t,y,z)u_\alpha(y,z)\,dy\,dz\Big] = 0$$

or

$$u_\alpha(x,t) + p\Big[-\frac{1}{\alpha}f(x,t) + \frac{1}{\alpha}\int_a^b\int_c^d K(x,t,y,z)u_\alpha(y,z)\,dy\,dz\Big] = 0 \tag{24}$$

Substituting Equation (22) into (24) and combining like terms, we get

$$\begin{aligned}
p^0 &: u_{\alpha,0}(x,t) = 0,\\
p^1 &: u_{\alpha,1}(x,t) = \frac{1}{\alpha}f(x,t),\\
p^2 &: u_{\alpha,2}(x,t) = -\frac{1}{\alpha}\int_a^b\int_c^d K(x,t,y,z)u_{\alpha,1}(y,z)\,dy\,dz,\\
&\vdots\\
p^{n+1} &: u_{\alpha,n+1}(x,t) = -\frac{1}{\alpha}\int_a^b\int_c^d K(x,t,y,z)u_{\alpha,n}(y,z)\,dy\,dz, \quad n \ge 1
\end{aligned} \tag{25}$$

Thus, we obtain a formula for the components of the solution. If we substitute these components into Equation (23), we obtain a solution if it exists. That is, Equation (23) holds if a solution exists.

We note that the nonlinear equations will be treated similarly. We first make a change of variables and then transform the nonlinear equation into a linear equation so that the above algorithm can be applied. At the end, we reintroduce the original variable and as a result we obtain the solution for the nonlinear equation. Although there are some modifications of HPM (MHPM) which were introduced and applied for solving two-dimensional Fredhom integral equations in [25–27], we will not investigate the MHPM further. We will instead limit our focus to extend the regularization-homotopy method.

We now summarize how to apply the regularization-homotopy method to Equation (5). For Equation (19), we first make a change of variables to transform the nonlinear equation into a linear one. We then apply the following steps:

- Apply the regularization method to transform the linear Fredholm integral equations of the first kind into second kind,
- Apply the homotopy perturbation method to find an approximate solution,
- Let the regularization parameter $\alpha \to 0$ to obtain a solution.

5. Illustrative Examples

We note that we assume the kernel $k(x,t,y,z)$ is separable, *i.e.*, $k(x,t,y,z) = g(x,t)h(y,z)$. We also require the function $f(x,t)$ involve components matched by $g(x,t)$. This is a necessary condition for a solution to exist [19].

Example 1:

Consider the following linear Fredholm integral equation of the first kind [28]:

$$xt = \int_0^1\int_0^1 xte^{y+z}u(y,z)\,dy\,dz. \tag{26}$$

The regularization method transforms Equation (26) to

$$u_\alpha(x,t) = \frac{1}{\alpha}xt - \frac{1}{\alpha}\int_0^1\int_0^1 xte^{y+z}u_\alpha(y,z)\,dy\,dz. \tag{27}$$

From Equation (24) we have

$$u_\alpha(x,t) = p\Big(\frac{1}{\alpha}xt - \frac{1}{\alpha}\int_0^1\int_0^1 xte^{y+z}u_\alpha(y,z)\,dy\,dz\Big). \tag{28}$$

Following the steps in Equation (25), we get

$$\begin{aligned}
p^0 &: u_{\alpha,0}(x,t) = 0,\\
p^1 &: u_{\alpha,1}(x,t) = \frac{1}{\alpha}xt,\\
p^2 &: u_{\alpha,2}(x,t) = -\frac{1}{\alpha}\int_0^1\int_0^1 xte^{y+z}u_{\alpha,1}(y,z)\,dy\,dz,\\
&\qquad\qquad\quad = -\frac{xt}{\alpha^2},\\
p^3 &: u_{\alpha,3}(x,t) = -\frac{1}{\alpha}\int_0^1\int_0^1 xte^{y+z}u_{\alpha,2}(y,z)\,dy\,dz,\\
&\qquad\qquad\quad = \frac{xt}{\alpha^3},\\
p^4 &: u_{\alpha,4}(x,t) = -\frac{1}{\alpha}\int_0^1\int_0^1 xte^{y+z}u_{\alpha,3}(y,z)\,dy\,dz,\\
&\qquad\qquad\quad = -\frac{xt}{\alpha^4},\\
&\ \vdots
\end{aligned} \tag{29}$$

Thus, the approximate solution becomes

$$\begin{aligned}
u_\alpha(x,t) &= \frac{1}{\alpha}xt\Big(1 - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^3} + \dots\Big)\\
&= \frac{xt}{\alpha+1}.
\end{aligned}$$

Letting $\alpha \to 0$, we obtain the exact solution as

$$u(x,t) = xt.$$

There are other solutions to this equation. For instance,

$$u(x,t) = \frac{x^2t^2}{(e-2)^2}, \frac{x^3t^3}{(6-2e)^2}, \frac{x^4t^4}{(9e-24)^2}\cdots$$

This is expected because Fredholm integral equations of the first kind are often ill-posed problems. That means solutions may not exist and if it exists it may not be unique.

Example 2:

Consider the following linear Fredholm integral equation of the first kind [10]:

$$\frac{1}{2}(e^2-1)e^{x+y} = \int_0^1\int_0^1 e^{x+y+s+t}u(s,t)\,ds\,dt. \tag{30}$$

The regularization method transforms Equation (30) to

$$u_\alpha(x,y) = \frac{1}{2\alpha}(e^2-1)e^{x+y} - \frac{1}{\alpha}\int_0^1\int_0^1 e^{x+y+s+t}u(s,t)\,ds\,dt. \tag{31}$$

Now, to construct homotopy let

$$u_\alpha(x,y) = p\Big(\frac{1}{2\alpha}(e^2-1)e^{x+y} - \frac{1}{\alpha}\int_0^1\int_0^1 e^{x+y+s+t}u_\alpha(s,t)\,ds\,dt\Big) \tag{32}$$

$$\begin{aligned}
p^0 : u_{\alpha,0}(x,y) &= 0,\\
p^1 : u_{\alpha,1}(x,y) &= \frac{1}{2\alpha}(e^2-1)e^{x+y},\\
p^2 : u_{\alpha,2}(x,y) &= -\frac{1}{\alpha}\int_0^1\int_0^1 e^{x+y+s+t}u_{\alpha,1}(s,t)\,ds\,dt,\\
&= -\frac{1}{8\alpha^2}(e^2-1)^3e^{x+y},\\
p^3 : u_{\alpha,3}(x,y) &= -\frac{1}{\alpha}\int_0^1\int_0^1 e^{x+y+s+t}u_{\alpha,2}(s,t)\,ds\,dt,\\
&= \frac{1}{32\alpha^3}(e^2-1)^5e^{x+y},\\
p^4 : u_{\alpha,4}(x,y) &= -\frac{1}{\alpha}\int_0^1\int_0^1 e^{x+y+s+t}u_{\alpha,3}(s,t)\,ds\,dt,\\
&= -\frac{1}{128\alpha^4}(e^2-1)^7e^{x+y},\\
&\ \vdots
\end{aligned} \tag{33}$$

Thus, the approximate solution becomes

$$\begin{aligned}
u_\alpha(x,y) &= \frac{1}{2\alpha}(e^2-1)e^{x+y}\Big(1 - \frac{1}{2^2\alpha}(e^2-1)^2 + \frac{1}{2^4\alpha^2}(e^2-1)^4 - \frac{1}{2^6\alpha^3}(e^2-1)^6 + \dots\Big)\\
&= \frac{2(e^2-1)e^{x+y}}{2^2\alpha + (e^2-1)^2}.
\end{aligned}$$

Letting $\alpha \to 0$, we obtain the exact solution as

$$u(x,y) = \frac{2e^{x+y}}{e^2-1}.$$

There are other solutions to this equation. For instance,

$$u(x,y) = \frac{9(e^2-1)e^{2x+2y}}{2(e^3-1)^2}, \frac{8e^{3x+3y}}{(e^2-1)(e^2+1)^2}, \dots$$

Having infinitely many solutions for this equation is quite normal because it is an ill-posed problem.

Example 3:

Consider the following nonlinear Fredholm integral equation of the first kind [10]:

$$\frac{x}{6(1+y)} = \int_0^1\int_0^1 \frac{x}{1+y}(1+s+t)u^2(s,t)\,ds\,dt. \tag{34}$$

We first transform the nonlinear Equation (34) to a linear equation by using the change of variable

$$v(s,t) = u^2(s,t) \tag{35}$$

so that Equation (34) becomes

$$\frac{x}{6(1+y)} = \int_0^1 \int_0^1 \frac{x}{1+y}(1+s+t)v(s,t)\, ds\, dt. \tag{36}$$

Once we obtain a solution to Equation (36), then reversing Equation (35), *i.e.*, $u(s,t) = \pm\sqrt{v(s,t)}$, we obtain the desired solutions. The regularization method transform Equation (36) to

$$v_\alpha(x,y) = \frac{x}{6\alpha(1+y)} - \frac{1}{\alpha}\int_0^1 \int_0^1 \frac{x}{1+y}(1+s+t)v_\alpha(s,t)\, ds\, dt. \tag{37}$$

Now, to construct homotopy let

$$v_\alpha(x,y) = p\Big(\frac{x}{6\alpha(1+y)} - \frac{1}{\alpha}\int_0^1 \int_0^1 \frac{x}{1+y}(1+s+t)v_\alpha(s,t)\, ds\, dt\Big), \tag{38}$$

$$\begin{aligned}
p^0 &: v_{\alpha,0}(x,y) = 0,\\
p^1 &: v_{\alpha,1}(x,y) = \frac{x}{6\alpha(1+y)},\\
p^2 &: v_{\alpha,2}(x,y) = -\frac{x}{\alpha(1+y)}\int_0^1 \int_0^1 (1+s+t)v_{\alpha,1}(s,t)\, ds\, dt\\
&\qquad = -\frac{x}{6\alpha^2(1+y)}\Big(\frac{3+2\log(2)}{6}\Big),\\
p^3 &: v_{\alpha,3}(x,y) = -\frac{x}{\alpha(1+y)}\int_0^1 \int_0^1 (1+s+t)v_{\alpha,2}(s,t)\, ds\, dt\\
&\qquad = \frac{x}{6\alpha^3(1+y)}\Big(\frac{3+2\log(2)}{6}\Big)^2,\\
p^4 &: v_{\alpha,4}(x,y) = -\frac{x}{\alpha(1+y)}\int_0^1 \int_0^1 (1+s+t)v_{\alpha,3}(s,t)\, ds\, dt\\
&\qquad = -\frac{x}{6\alpha^4(1+y)}\Big(\frac{3+2\log(2)}{6}\Big)^3,\\
&\qquad \vdots
\end{aligned} \tag{39}$$

Thus, the approximate solution becomes

$$\begin{aligned}
v_\alpha(x,y) &= \frac{x}{6\alpha(1+y)}\left(1 - \frac{1}{\alpha}\Big(\frac{3+2\log(2)}{6}\Big) + \frac{1}{\alpha^2}\Big(\frac{3+2\log(2)}{6}\Big)^2 - \frac{1}{\alpha^3}\Big(\frac{3+2\log(2)}{6}\Big)^3 + \dots\right)\\
&= \frac{x}{(1+y)(6\alpha+3+2\log(2))}.
\end{aligned}$$

Letting $\alpha \to 0$, we obtain the exact solution as

$$v(x,y) = \frac{x}{(1+y)(3+2\log(2))}$$

Since

$$\begin{aligned}
u(x,y) &= \pm\sqrt{v(x,y)},\\
&= \pm\sqrt{\frac{x}{(1+y)(3+2\log(2))}}.
\end{aligned} \tag{40}$$

These are exactly the same solutions obtained in [10]. There are other solutions as well.

6. Conclusions

In this article, we extend the application of the regularization-homotopy method to the two-dimensional linear and nonlinear Fredholm integral equations of the first kind. The method is a combination of two powerful methods. Three examples are considered and exact solutions are obtained by using the regularization-homotopy method.

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