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ON F-MONOTONE OPERATORS AND GENERALIZED STRONGLY NONLINEAR VARIATIONAL INEQUALITIES

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Abstract-In this paper we develop the existence theory of generalized strongly nonlinear variational inequality problem involving F-monotone operator in the setting of reflexive Banach spaces and Hausdorff topological vector spaces separately. Our results include Dugundji's and Granas's variational inequality in reflexive Banach spaces and Tan's variational inequality in topological vector spaces, respectively.

1. INTRODUCTION

The concept of strongly nonlinear variational inequality (SNVI) was introduced by Noor [14] and subsequently studied by Nanda [12] . On the other hand the notion of F-monotonicity of operators was introduced by Kato [7,8] and subsequently discussed by Nanda [11]. In fact , the usual concept of monotonicity introduced by Minty [10] is a special case , when F is an identity operator . Furthermore , the concept of generalized variational inequality (GVI) was introduced and studied by Nanda [11] alongwith the concept of F-monotonicity . Another concept called general nonlinear variational inequality was introduced and discussed by Noor [15,16,17]. Recently, Nanda [13] has further introduced the concept of generalized strongly nonlinear variational inequality (GSNVI) which includes the concepts of SNVI and the general variational inequality as special cases , and has studied some existence theorems alongwith F-monotonicity.

The purpose of this paper is to study further the existence theory of GSNVI problem involving F-monotone operators in the reflexive Banach spaces and in topological vector spaces , following the technique of Dugundji and Granas [4] and Tan [18] respectively . Similar studies on GSNVI problem have been undertaken by Siddiqui , Ansari and Kazmi [19]. However, they did not study the problem alongwith F-monotonicity .

2 . PRELIMINARIES

Let X and Y be locally convex spaces . Let F, T and A be nonlinear operators such that $F : D(F) = X \to R(F) \subset Y$ and $T , A: D \subset X \to R \subset Y^*$ where D and R denote the domaim and the range of the operators respectively. Let K be a nonempty closed convex subset of $D \subset X$.

Then

$$\{ x \in K: \langle Tx , F(y-x) \rangle \geq \langle Ax , F(y-x) \rangle \text{ for all } y \in K\} \qquad (1)$$

will be called a GSNVI and any $x \in K$ which satisfies (1) will be called a solution of GSNVI(1). Similarly,

$$\{ x \in K: \langle Ty, F(y-x) \rangle \geq \langle Ay, F(y-x) \rangle, \text{ for all } y \in K. \qquad (2)$$

will be called another GSNVI. Nanda [13] has proved the equivalence of the sets of solution of (1) and (2).

Note that if $X = Y$ and F is the identity map, (1) and (2) reduce to SNVI studied by Nanda [12]. If A is the zero operator, then (1) and (2) reduce to GVI discussed in Nanda [11]. If simultaneously F - Identity map I and $A = O$, then we obtain the usual variational inequalities introduced by Hartman and Stampacchia [6] and studied by many others. We now quote some definitions which will be required in the sequel.

T is said to be F-monotone (see Nanda [11]) if

$\langle Tx - Ty, F(x-y) \rangle \geq 0$ for all $x,y \in D$ and strictly F-monotone if

$\langle Tx - Ty, F(x-y) \rangle > 0$ for all $x,y \in D$, $x \neq y$.

Let $y = x$ and $F = 1$. Then the above concepts are just equivalent to monotonicity and strict monotonicity. If X is reflexive Banach space, $Y = X^*$, X^* is strictly convex and F the duality map, then F- monotonicity of T means that T is aceretive in the sense of Browder [2].

F is said to be symmetric if $F(x) = F(-x)$ and antisymmetric if

$F(x) = -F(-x)$ for all $x \in D(F)$

F is said to be positive homogeneous if

$F(tx) = t F(x)$ for $t > 0$, and additive if

$F(x+y) = F(x) + F(y)$ for $x, y \in D(F)$.

A multivalued mapping $G: K \to 2^x$ is called the KKM mapping if for every finite subset $\{x_1, x_2, \ldots\ldots, x_n\}$ of K, conv. $\{x_1, x_2, \ldots\ldots, x_n\} \subset \bigcup_{i=1} G(x_i)$,

where, conv. (A) denotes convex hull of A, (Fan [5]).

We shall now list some results which will be needed in this paper. The following result was obtained by Dugundji [4].

Theorem 2.1 . Let X be a vector space, K an arbitrary subset of X, $G: K \to 2^x$ a KKM map, such that each G(x) is finitely closed. Then the family $\{ G(x) \mid x \in K \}$ of sets has a finite intersection property.

From this result the following (corollary 1.4) was deduced by Dugundji and Granas [4] which is a slight modification of Ky Fan's theorem [5].

Theorem 2.2 . Let X be a vector space, K an arbitrary subset of X and $G: K \to 2^x$ a KKM map. Assume that there is a set-valued map $\Gamma: K \to 2^x$ such that $G(x) \subset \Gamma(x)$ for each $x \in K$, and for which

$$\cap \{ \Gamma(x) \mid x \in K \} = \cap \{ G(x) \mid x \in K \}.$$

If there is some topology on X such that $\Gamma(x)$ is compact, then $\cap G(x) \neq \varnothing$

The following result was obtained by Lin [9].

Theorem 2.3. Let K be a nonempty , weakly compact convex set in a Hausdorff topological vector space X . Let f and g be two real valued functions on K x K having the following properties :

(i) $g(x,y) \leq f(x,y)$ for all $(x,y) \in K \times K$ and $f(x,y) \leq 0$ for all x K ;

(ii) for each fixed $x \in K$, g (x,y) is a lower semicontinuous function of y on K ;

(iii) for each fixed $y \in K$, the set $\{x \in K \mid f(x, y) > 0\}$ is convex or empty .

Then there exists a point $y_o \in K$ such that $g(x, y_o) \leq 0$ for all $x \in K$.

3. EXISTENCE THEOREM FOR THE SOLUTION OF GSNVI IN THE SETTING OF REFLEXIVE BANACH SPACES

Throughout this section we denote by X a reflexive banach space , X* a dual of X and $<.,.>$ a continuous pairing between X* and X . We prove the following result .

Theorem 3.1. Let K be a nonempty , closed , bounded convex subset of X , F be positive homogeneous , additive , antisymmetric and continuous such that F(K) is dense in X . Let T , -A:K→X* be F-monotone . Suppose further that the maps T , -A are continuous on $L \cap K$ for each one dimensional flat $L \subset X$. Then there is a solution of (1) . Moreover , if T , -A are strictly monotone then the solution of (1) is unique .

Proof . Let the multivalued mappings G , $\Gamma : K \to 2^X$ be defined , for each $x \in K$, as

$$G(x) = \{ y \in K: < Ty , F(x-y) > \geq < Ay , F(x-y) >\} \text{ , and}$$

$$\Gamma(x) = \{ y \in K: < Tx , F(x-y)> \geq < Ax , F(x-y) >\} \text{ , respectively .}$$

To prove the theorem , we are to show that $\cap \{ G(x) : x \in K \} \neq \varnothing$.

First , G is a KKM map . Indeed , let $y_0 \in \text{conv} . \{ x_1 , x_2 , . . . x_n \}$,

$\sum_{i=1} t_i = 1$, $t_i > 0$ and $y_o = \sum_{i=1} t_i x_i$. If $y_o \notin G(x_i)$,

we would have $< T y_o , F(x - y_o) > < < Ay_o , F(x - y_o) >$ for each i = 1 , ... , n and for all $x_i \in K$. This would give

$$< Ty_o, F(x) + F(-y_o) > < < Ay_o,F(x) +F(-y_o) >$$

or $$<Ty_o-Ay_o,F(y_o) > > < Ty_o-Ay_o,F(x) >$$ for all $x \in K$, using the fact that F is additive and antisymmetric .

On account of positive homogenous F, we have

$$< Ty_o - Ay_o , F(y_o) > = < Ty_o - Ay_o , F(\sum_{i=1} t_i x_i) >$$

$$\leq \sum_{i=1} t_i < Ty_o - Ay_o , F(y_o) >$$

$$< <Ty_o-Ay_o,F(y_o) >$$, which is a contradiction

so we have conv. $\{ x_1, x_2, \ldots, x_n \} \subset \bigcup_{i=1} G(x_i)$. Thus G is a KKM map.

Now, we show that $G(x) \subset \Gamma(x)$ for each $x \in K$. For let $y \in G(x)$, so that

$< Ty, F(x-y) > \geq < Ay, F(x-y) >$. By F- monotonicity of T, -A, we have

$$< (Ty - Ay) - (Tx - Ax), F(x - y) > \geq 0$$

or, $$< Ty - Ay, F(x - y) > \geq < Tx - Ax, F(x - y) >$$

or, $< Tx-Ax, F(x - y) > \geq 0$, so that $< Tx, F(x - y) > > < Ax, F(x-y) >$,

i . e. $y \in \Gamma(x)$ and so $G(x) \subset \Gamma(x)$ for each $x \in K$.

Next, we show that $\cap \{ \Gamma(x) \mid x \in K \} \subset \cap \{ G(x) \mid x \in K \}$.

Assume $y_o \in \cap \Gamma(x)$. Choose any $x \in K$ and let $z_t = t x + (1 - t) y_o = y_o - t (y_o - x)$, because K is convex, we have $z_t \in K$ for each $0 < t < 1$. Since, $y_o \in \Gamma(z_t)$ for each $0 < t < 1$, we find that

$$< Tz_t, F(z_t - y_o) > \geq < Az_t, F(z_t - y_o) >,$$

or, $$< Tz_t, F(x - y_o) > \geq < Az_t, F(x - y_o) >.$$

Now, let $t \to 0$, the continuity of T and A on the ray joining y_o and x gives $T(z_t) \to T(y_o)$ and $A(z_t) \to A(y_o)$ weakly in X* (since F(K) is dense in X). Hence,

$< Ty_o, F(x - y_o) > \geq < Ay_o, F(x - y_o) >$. Thus, $y_o \in G(x)$ for each $x \in K$.

Since, K is a closed, bounded, convex set in a reflexive space, it is weakly compact ; therefore, each $\Gamma(x)$, being the intersection of the closed half - space $\{ y \in K \mid < Tx-Ax, F(y) > \geq < Tx-Ax, F(x) > \}$ with K is for the same reason, also weakly compact.

Thus, all the requirements in Theorem 2.2 are satisfied, therefore,

$\cap \{ G(x) \mid x \in K \} \neq \varnothing$. This completes the proof.

Remark 3.2 : For A= 0 and F an identity map simultaneously in Theorem 3.1, we get the main result of Dugundji and Granas [4 , theorem 2.1] which is an extention of work of Hartman 's and Stampacchia 's [6] variational inequality.

4 . EXISTENCE THEOREM FOR THE SOLUTION OF GSNVI IN THE SETTING OF HAUSDORFF TOPOLOGICAL VECTOR SPACES

Let X be a Hausdorff topological vector space, X* the dual space of X (that is, the vector space of all continuous linear functionals on X). We denote the pairing between X* and X by $< w, x >$ for w in X* and x in X. Let K be any nonempty subset of X, a set valued map $f: K \to 2^X$ is called monotone on K if for each x and y in K, each u in f(x), and each w in f(y), Re

$< w - u , y - x > \geq 0$, [3 , p.79] . Let X and Y be topological spaces , and let $f : X \to 2^{y}$ be a set -valued map. We say that f is lower semicontinuous at $x_o \in X$ if for each open set G with $f(x_o) \cap G \neq \emptyset$ there is neighbourhood $N(x_o)$ of x_o such that if $x \in N(x_o)$, then $f(x_o) \cap G \neq \emptyset$; f is lower semicontinuous on X if it is lower semicontinuous at each point of X . Also f is upper semicontinuous at $x_o \in X$ if for each open set G with $f(x_o) \subseteq G$ there exists a neighbourhood $N(x_o)$ of x_o such that if $x \in N(x_o)$, then $f(x) \subseteq G$; f is upper semicontinuous on X if it is upper semicontinuous at each point of X , [1 , 109] . We prove the following .

Theorem 4.1 . Let K be a non- empty ,weakly compact convex set in X and let T , - A: $K \to 2^{x}$ be set valued maps such that for each $x \in K$, T(x) , A(x) are nonempty subsets of X* ; suppose that T , - A are F-monotone respectively and the function F is continuous positive homogeneous , additive and anti-symmetric . Assume that for each one dimensional flat $L \subset X$, $T/L \cap K$ and $-A/L \cap K$ are lower semi-continuous from the topology of X to the weak* topology $\sigma(X^*, X)$ of X* and that for each $y \in K$, there exists a point $x \in K$ and a point $w \in T(x)$ and -A(x) with $\mathrm{Re} < w, F(y-x) > > 0$. Then there exists a point $y_o \in K$ such that

$$\sup_{u \in (T-A)y_o} \mathrm{Re} < u , F(y_o - x) > \leq 0 \text{ for all } x \in K .$$

Proof : By monotonicity of T and -A , for each $x , y \in K$,$w \in (T - A)(x)$ and $u \in (T-A)(y)$, we have

$$\mathrm{Re} < T(x), F(x-y) > \leq \mathrm{Re} < A(x), F(x-y) >,$$

and

$$\mathrm{Re} < T(y), F(x-y) > \leq \mathrm{Re} < A(y), F(x-y) > 0 .$$

Then

$$\mathrm{Re} < T(y) - A(y) - (T(x) - A(x)), F(y-x) > \geq 0 , \text{ so that}$$

$$\mathrm{Re} < T(y) - A(y), F(y-x) > \geq \mathrm{Re} < T(x) - A(x), F(y-x) > ,$$

or

$$\mathrm{Re} < T(x) - A(x), F(y-x) > \leq \mathrm{Re} < T(y) - A(y), F(y-x) > .$$

Then

$$\sup_{w \in (T-A)x} \mathrm{Re} < w, F(y-x) > \leq \inf_{u \in (T-A)y} \mathrm{Re} < u, F(x-y) > \text{ for all } x,y \in K$$

For each $x , y \in K$ define

$$g(x, y) = \sup_{w \in (T-A)x} \mathrm{Re} < w , F(y-x) > ,$$

$$f(x, y) = \inf_{u \in (T-A)y} \mathrm{Re} < u , F(y-x) > .$$

(i) We have $g(x, y) \leq f(x, y)$ for all $x , y \in X$, and $f(x, x) = 0$ for all $x \in K$, because $F(0) = 0$.

(ii) It is easy to check that for each $x \in K$, $g(x, y)$ is a weakly lower semi-continuous function of y on K , because F is continuous .

(iii) For each fixed $y \in K$, the set $G(x) = \{ x \in K : F(x, y) > 0 \}$ is convex . To see this , let $x_1, x_2 \in G(x)$ and $\alpha \in [0.1]$; then $f(x_1, y) > 0$, $f(x_2, y) > 0$. Let $o < s < \min f(x_i, y)$, $i = 1, 2$.

Then $\inf_{u \in (T-A)y} \operatorname{Re} < u , F(y - x_i) > = f(x_i, y) > s$.

It follows that $\operatorname{Re} < u, F(x_i) > < \operatorname{Re} < u, F(y) > -s$ for all $u \in (T\text{-}A)y$ and F additive and antisymmetric , and $i = 1, 2$.

Therefore by the use of positive homogeneousness of F ,

$$\operatorname{Re} < u, F(\alpha x_1) + F(1-\alpha) x_2 > < \alpha(\operatorname{Re} < u, F(y) > -s) + (1-\alpha)(\operatorname{Re} < u, F(y) > -s) = \operatorname{Re} < u, F(y) > -s .$$

Thus

$$\operatorname{Re} < u , F[y - (\alpha x_1 + (1-\alpha) x_2] > s \text{ for all } u \in (T - A) y .$$

It follows that $f(\alpha x_1 + (1 - \alpha) x_2, y) = \inf_{u \in (T-A)y} \operatorname{Re} < u, F[y - (\alpha x_1) + (1-\alpha) x_2] > \geq s > 0$, that is , $\alpha x_1 + (1 - \alpha) x_2 \in G(x)$, which is the desired result .

Now we equip X with weak topology and we find that all the conditions in Theorem 2.3 are satisfied , therefore , there exists a point $y_o \in K$ such that $g(x, y_o) \leq 0$.

or , $\sup_{w \in (T-A)x} \operatorname{Re} < w, F(y_o - x) > \leq 0$ for all $x \in K$ (*)

Next choose any $x \in K$ and let $z_t = tx + (1 - t)y_o = y_o - t(y_o - x)$ for $t \in [0,1]$. As K is convex , we have $z_t \in K$ for $t \in [0, 1]$. Therefore , by (*) , we have

$\sup_{w \in (T-A)z_t} \operatorname{Re} < w , F(y_o - z_t) > \leq 0$ for all $t \in [0, 1]$, so that

t . $\sup_{w \in (T-A)z_t} < w , F(y_o - x) > \leq 0$ for all $t \in [0, 1]$, and in particular ,

$\sup_{w \in (T-A)z_t} \operatorname{Re} < w , F(y_o - x) > \leq 0$ for all $t \in [0, 1]$. (**)

Let $u_o \in (T - A) y_o$ be arbitrarily fixed . For each $\in > 0$, let

$U_{uo} = \{ u \in X^* : | < u_o - u , F(y_o - x) > | < \in \}$;

then , $\cup_{uo}$ is a $\sigma(X^*, X)$ neighbourhood of u_o . Since T , -A are lower semicontinuous , and $\cup_{uo} \cap (T - A) y_o \neq \varnothing$, there exists a neighbourhood $N(y_o)$ of y_o such that $z \in N(y_o)$ implies that $(T - A)(z) \cap \cup_{uo} \neq \varnothing$. We also observe that $z_t \to y_o$ as $t \to 0$; thus there exists $0 < \delta < 1$ such that for all $t \in (0, \delta)$, we have $z_t \in N(y_o)$. But then $(T - A)(z_t) \cap \cup_{uo} \neq \varnothing$ for $t \in (0,\delta)$.Take any $w \in (T - A)(z_t) \cap \cup_{uo}$ we have $| < u_o - w , F(y_o - x) > | < \in$. This implies that

$$\operatorname{Re} < u_o , F(y_o - x) > < \operatorname{Re} < w , F(y_o - x) > + \in .$$

By (**) , we have $\text{Re} < u_o , F(y_o - x) > < \in$. Since $\in > 0$ is arbitrary ,

$\text{Re} < u_o , F(y_o - x) > \leq 0$. As $u_o \in (T - A) y_o$ is arbitrary ,

$$\sup_{u \in (T-A) y_o} \text{Re} < u , F(y_o - x) \leq 0 \text{ for all } x \in K .$$ This completes the proof .

As another application of Theorem 2.3 , we mention the following result .

Theorem 4.2. Let K be a nonempty , weakly compact, convex set in a Hausdorff topological vector space X and let T , -A : $K \rightarrow 2^{x}$ be F-monotone, where the function F is additive and anti - symmetric . Supose that h: $K \rightarrow R$ is a lower semicontinuous, convex function and that for $y \in K$, there exists a point $x \in X$ with

$$\sup_{w \in (T-A)(x)} \text{Re} < w , F(y - x) > + (h(y) - h(x) \geq 0 .$$

Then there exists a point $y_o \in K$ such that

$$\sup_{w \in (t-A)(x)} \text{Re} < w , F(y_o - x) > \leq h(x) - h(y_o) , \text{ for all } x \in X .$$

Proof : For each x and y in X, each w in (T - A) (x) and each u in (T - A) (y) , define

$$g(x , y) = \sup_{w \in (T-A)(x)} \text{Re} < w , F(y - x) > + h(y) - h(x) ,$$

$$f(x , y) = \inf_{u \in (T-A)(y)} \text{Re} < u , F(y - x) > + h(y) - h(x) .$$

Let X be equipped with the weak topology ; then all the conditions of Theorem 2.3 are satisfied , and by applying Theorem 2.3 , we get the required result .

Remark 4.3 : For F an identity map and A = 0 , we get

(i) Theorem 3 of Tan [18] from our Theorem 4.1 .

(ii) Theorem 5 of Tan [18] , which is a generalization of Theorem 2 of Yen [20] , from our Theorem 4.2 .

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