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GLOBAL NONEXISTENCE OF SOLUTIONS OF THE QUASILINEAR HYPERBOLIC EQUATION OF THE VIBRATIONS OF A RISER

Vural Bayrak
and
Mehmet Can

Istanbul Technical University Mathematics Department
Maslak, 80626 Istanbul TURKIYE

Abstract: In this work, the nonexistence of the global solutions of a quasilinear hyperbolic boundary value problem with dissipative term in the equation is considered. In one space dimension this initial value problem models the behavior of a riser vibrating due to the effects of waves and current. The nonexistence proof is achieved by the use of the so called concavity method. In this method one writes down a functional which represents the norm of the solution in some sense. Then it is proved that this functional satisfies the hypotheses of the concavity lemma. Hence one concludes that one cannot continue the solution for all time by showing that this functional and hence the norm of the solution, would otherwise blow up in finite time.

1. INTRODUCTION

The nonexistence of global solutions of quasilinear hyperbolic equations with dissipative terms in the equations are investigated by J. L. Lions [1], R. T. Glassey [2], H. A. Levine [4, 5, 6], and O. A. Ladyzhenskaya and V. K. Kalantarov [8] and many others. Levine, [7] has a survey article with many relevant references (see also Straughan [9]).

In [4] Levine studied the initial value problem for the following ``abstract" wave equation with dissipation

$$Pu_{tt} + A_1 u_t + Au = f(u)$$

in a Hilbert space where P, A_1 and A are positive linear operators defined on some dense subspace of the Hilbert space and f is a gradient operator with potential F. It is assumed that $(u, f(u)) \geq F(u)$ for all u in the domain of F. The global nonexistence result he proved is the following: If the energy is initially negative then the solution can not be global. This is the same result that he proved in the case that A=0 (see [6].) To our knowledge this was the first global nonexistence theorem for nonlinear wave equations with damping.

However, the functional used for the investigation of initial-boundary value problems with no dissipative terms in the boundary conditions can not be continued directly to the problems with the dissipative terms in the boundary conditions. Whenever damping is present, one must allow for the possibility that the data restrictions could be more severe than without damping.

The tool used in this work is a Lemma to be found in [3,6,8]. From now on we'll call it the Concavity Lemma. The most crucial point in the application of this tool is to find a functional that represents the dissipation on the boundary and satisfies the conditions of the Concavity Lemma.

Let us begin by stating Concavity Lemma [6].

Lemma 1.

If a function

$$\Psi(t) \in C^2, \qquad \Psi(t) \geq 0,$$

satisfies the inequality

$$\Psi''(t)\Psi(t) - (1+\gamma)[\Psi'(t)]^2 \geq 0 \tag{1.1}$$

for some number $\gamma > 0$, then the following hold:

If $\Psi(0) > 0$, $\Psi'(0) > 0$ then for the number

$$t_0 = \frac{\Psi(0)}{\gamma \Psi'(0)} \tag{1.2}$$

There exists a positive number $t_1 \leq t_0$ such that, as $t \to t_1$,

$$\Psi(t) \to +\infty \tag{1.3}$$

The proof of this lemma is quite easy. One observes that from (1.1) we have $\left(\Psi^{-\gamma}\right)'' \leq 0$ as long as $\Psi > 0$. Since the differential inequality tells us that Ψ is convex and $\Psi'(0) > 0, \Psi$ must be increasing and hence cannot change sign. The rest of the lemma follows from the observation that $\Psi^{-\gamma}$ must be below its tangent line at $(0, \Psi^{-\gamma}(0))$ and that the slope of this line is negative. Therefore the line and hence $\Psi^{-\gamma}$ must cross the t axis. The line does so in time t_0 while the function $\Psi^{-\gamma}$ does so in a possibly earlier time t_1.

2. THE INITIAL-BOUNDARY VALUE PROBLEM

Let us assume that the initial boundary value problem

$$u_{tt}+\alpha u_t+2\beta u_{xxxx}-2\left[(ax+b)u_x\right]_x+\frac{\beta}{3}(u_x^3)_{xxx}-\left[(ax+b)u_x^3\right]_x-\beta(u_{xx}^2u_x)_x=f(u), \qquad (2.1)$$
$$(t,x)\in(0,T)\times[0,1]$$

$$u(t,0)=u(t,1)=0, \qquad u_{xx}(t,0)=u_{xx}(t,1)=0, \qquad t\in(0,T), \qquad (2.2)$$

$$u(0,x)=u_0(x), \qquad u_t(0,x)=u_1(x), \qquad x\in[0,1] \qquad (2.3)$$

has a local classical solution.

In the above $T>0$ is an arbitrary number, α,β,a and b are nonnegative numbers. The equation (2.1) models the behavior of a riser vibrating due to effects of waves and current [10].

In this section using the method of concavity ([2], [3]), it will be shown that under certain conditions on f,u_0 and u_1, the IBVP (2.1)-(2.3) has no global solutions.

Let the function $f(u)$, with its primitive $F(u)=\int_0^u f(\xi)d\xi$ satisfies the inequality

$$uf(u)\geq 4(2\gamma+\frac{\alpha}{2}+1)F(u), \qquad \forall u\in R^1 \qquad (2.4)$$

where $\gamma>0$ is a suitable number. Then we can prove the following theorem on the nonexistence of the global solutions of the IBVP (2.1)-(2.3).

Theorem 1.

Let the functions $u_0(x)$, $u_1(x)$ satisfy the inequalities

$$\int_0^1 u_0u_1dx>0,$$

$$\beta\int_0^1 u_0'^2u_0''^2dx+\int_0^1 u_1^2dx+\frac{1}{2}\int_0^1(ax+b)u_0'^4dx$$

$$+2\int_0^1(ax+b)u_0'^2dx+2\beta\int_0^1 u_0''^2dx-2\int_0^1 F(u_0)dx\leq 0 \qquad (2.5)$$

Let $f(u)$ satisfies the condition (2.4) with

$$\gamma > \frac{\alpha}{32\beta\mu_1^2} - \frac{1+\alpha}{4}$$

where μ_1 is the smallest eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$ as to satisfy the Dirichlet condition.

Let

$$t_0 = \frac{\int_0^1 u_0^2 dx}{2\gamma \int_0^1 u_0 u_1 dx} . \tag{2.6}$$

Then there exists a $t_1 \le t_0$ such that

$$\lim_{t \to t_1} \int_0^1 u^2 dx = \infty . \tag{2.7}$$

Proof: Let

$$\Psi(t) = [\Phi(t)]^{-\gamma} \tag{2.8}$$

where

$$\Phi(t) = \int_0^1 u^2(x)\, dx . \tag{2.9}$$

Differentiating (2.8) and (2.9) with respect to t one has

$$\Psi'(t) = -\gamma[\Phi(t)]^{-\gamma-1}\Phi'(t) = -2\gamma[\Phi(t)]^{-\gamma-1}\int_0^1 uu_t dx \tag{2.10}$$

and

$$\Psi''(t) = \gamma(\gamma+1)[\Phi(t)]^{-\gamma-2}\Phi'(t)^2 - \gamma[\Phi(t)]^{-\gamma-1}\Phi''(t) =$$

$$-\gamma[\Phi(t)]^{-\gamma-2}[\Phi''(t)\Phi(t) - (\gamma+1)\Phi'(t)^2] \tag{2.11}$$

Now let us prove the inequalities

$$\Psi'(0) < 0, \text{ and } \Psi''(t) \le 0 \text{ for } t \ge 0 . \tag{2.12}$$

Using (2.10) one finds

$$\Psi'(0) = -2\gamma[\Phi(0)]^{-\gamma-1} \int_0^1 u_0 u_{1t} dx \tag{2.13}$$

and one deduces the resulting inequality $\Psi'(0) < 0$ using the hypothesis of the theorem.

To prove the inequality $\Psi''(0) \le 0$ it is sufficient to show that

$$H \equiv \Phi''(t)\Phi(t) - (\gamma+1)\Phi'(t)^2 \ge 0 \,. \tag{2.14}$$

To prove it let us first compute the derivatives $\Phi'(t)$ and $\Phi''(t)$.

One has

$$\Phi'(t) = 2\int_0^1 u u_t dx \,, \tag{2.15}$$

$$\Phi''(t) = 2\int_0^1 u u_{tt} dx + 2\int_0^1 u_t^2 dx = 4(\gamma+1)\int_0^1 u_t^2 dx + 2\left[\int_0^1 u u_{tt} dx - (2\gamma+1)2\int_0^1 u_t^2 dx\right] \,. \tag{2.16}$$

Using (2.15) and (2.16) in (2.14) one obtains

$$H(t) \equiv 4(\gamma+1)\left\{\int_0^1 u^2 dx \int_0^1 u_t^2 dx - \left[\int_0^1 u u_{tt} dx\right]^2\right\} + 2\Phi(t)\left[\int_0^1 u u_{tt} dx - (2\gamma+1)\int_0^1 u_t^2 dx\right] \,. \tag{2.17}$$

From the Cauchy-Schwarz theorem, the sum of the terms in the bracelet in (2.17) is nonnegative. Hence to prove the inequality (2.14), it suffices to prove that

$$G(t) \equiv \int_0^1 u u_{tt} dx - (2\gamma+1)\int_0^1 u_t^2 dx \ge 0 \,. \tag{2.18}$$

Multiplying both sides of (2.1) by u and integrating over the interval $[0,1]$ after application of integration by parts when necessary, one has

$$\int_0^1 u u_{tt} dx = -\alpha\int_0^1 u u_t dx - 2\beta\int_0^1 u_{xx}^2 dx - 2\beta\int_0^1 u_{xx}^2 u_x^2 dx$$

$$-2\int_0^1 (ax+b) u_x^2 dx - \int_0^1 (ax+b) u_x^4 dx + \int_0^1 u f(u) dx \,. \tag{2.19}$$

On the other hand the fact $\int_0^1 (\frac{u}{2} - u_t)^2 dx \ge 0$ implies

$$-\int_0^1 uu_t dx \geq -\frac{1}{4}\int_0^1 u^2 dx - \int_0^1 u_t^2 dx$$

Hence one obtains

$$G(t) \geq -\frac{\alpha}{4}\int_0^1 u^2 dx - 2(2\gamma + \frac{\alpha}{2} + 1)\int_0^1 u_t^2 - 2\beta\int_0^1 u_{xx}^2 dx - 2\beta\int_0^1 u_{xx}^2 u_x^2 dx -$$

$$-2\int_0^1 (ax+b)u_x^2 dx - \int_0^1 (ax+b)u_x^4 dx + \int_0^1 uf(u)dx \; . \tag{2.20}$$

Next we multiply both sides of (2.1) by u_t and integrate over the region $[0,1]$. Then we integrate with respect to t, over the interval $[0, t]$ to obtain:

$$\int_0^1 u_t^2 dx = -2\alpha\int_0^1\int_0^1 u_t^2 dx\, dt - 2\beta\int_0^1 u_{xx}^2 dx - 2\beta\int_0^1 u_{xx}^2 u_x^2 dx - 2\int_0^1 (ax+b)u_x^2 dx -$$

$$-\int_0^1 (ax+b)u_x^4 dx - 2\int_0^1 F(u)dx + \beta\int_0^1 u_0'^2 u_0''^2 dx + 2\beta\int_0^1 u_0''^2 dx + 2\int_0^1 (ax+b)u_0'^2 dx +$$

$$+\frac{1}{2}\int_0^1 (ax+b)u_0'^4 dx + \int_0^1 u_1^2 dx + 2\int_0^1 F(u_0)dx \; . \tag{2.21}$$

Using this relation in (2.20) and omitting some of the positive terms one obtains

$$G(t) \geq -\frac{\alpha}{4}\int_0^1 u^2 dx + 2\beta(4\gamma + \alpha + 1)\int_0^1 u_{xx}^2 dx + \int_0^1 uf(u)dx - 2(4\gamma + \alpha + 2)\int_0^1 F(u)dx -$$

$$-(4\gamma + \alpha + 2)[\beta\int_0^1 u_0'^2 u_0''^2 dx + 2\beta\int_0^1 u_0''^2 dx + 2\int_0^1 (ax+b)u_0'^2 dx +$$

$$+\frac{1}{2}\int_0^1 (ax+b)u_0'^4 dx + \int_0^1 u_1^2 dx + 2\int_0^1 F(u_0)dx] \; . \tag{2.22}$$

Using the hypotheses of the theorem one has

$$G(t) \geq -\frac{\alpha}{4}\int_0^1 u^2 dx + 2\beta(4\gamma + \alpha + 1)\int_0^1 u_{xx}^2 dx \; . \tag{2.23}$$

Since μ_1 is the smallest eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$, from Courant-Weil theorem one obtains

$$\mu_1^2 \int_0^1 u^2 dx \le \int_0^1 u_{xx}^2 \, dx . \tag{2.24}$$

Hence the inequality (2.23) becomes

$$G(t) \ge [2\beta\mu_1^2(4\gamma + \alpha + 1) - \frac{\alpha}{4}] \int_0^1 u^2 dx \tag{2.25}$$

and for $2\beta\mu_1^2(4\gamma + \alpha + 1) - \frac{\alpha}{4} \ge 0$ one immediately obtains $G(t) \ge 0$.

3. AN EXAMPLE

As an example of the problem above for $\alpha = \frac{1}{2}$, $\beta = a = b = 0.01$, let us consider the initial boundary value problem

$$100u_{tt} + 50u_t + 2u_{xxxx} - 2[(x+1)u_x]_x + \frac{1}{3}(u_x^3)_{xxx} - [(x+1)u_x^3]_x - (u_{xx}^2 u_x)_x = 2500u^5, \quad t \in (0,T), \; x \in [0,1] \tag{3.1}$$

$$u = 0, \qquad u_{xx} = 0, \qquad t \in (0,T), \; x = 0,1 \tag{3.2}$$

$$u(0,x) = u_0(x) = \sin(\pi x), \qquad u_t(0,x) = u_1(x) = 0.01, \qquad x \in [0,1] \tag{3.3}$$

In the above

$$f(u) = 25u^5, \qquad F(u) = \frac{25}{6}u^6.$$

Hence for the real number $\gamma = 1/8$ and for all $u \in R^1$, the inequality (2.4) is satisfied.

The hypotheses of Theorem 1 are satisfied by the initial data in (3.3), and the blow up time is estimated as

$$t_0 = \frac{\int_0^1 u_0^2 dx}{2\gamma \int_0^1 u_0 u_1 dx} = 100\pi . \tag{3.4}$$

Using a suitable numerical scheme, the accuracy of this upper bound for the blow up time can be investigated.

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