Mathematical & Computational Applications, Vol. 2, No. 1, pp. 21-29, 1997.
©Association for Scientific Research

MİNİMAX OPTIMAL CONTROL FOR ONE CLASS OF UNCERTAIN SYSTEMS

V. Dzhafarov*, A. Karamancıoğlu*, S. Çetintaş**

*Dept. of Electrical and Electronic Eng. Osmangazi University Bademlik 26030 Eskişehir, TURKEY.
**Dept. of Mathematics, Dumlupınar University 43100 Kütahya, TURKEY.

Abstact

A controllable system described by linear differential equation with uncertainties in the initial condition and forcing function is considered. We aim to find a control which minimizes a cost function having terminal and integral parts. Using game theory and convex analysis, under some sufficient conditions, the optimal control is obtained.

We consider a linear system whose model is given by the differential equation

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t) \\ x(t_0) = x_0 \in S; \quad t \in [t_0, t_f], \end{cases} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in P \subset \Re^p$, and $v(t) \in Q \subset \Re^q$ such that P and Q are known compact sets. The continuous matrices $A(t), B(t),$ and $C(t)$ have compatible dimensions. The control input $u(t)$ and disturbance input $v(t)$ are not known. Even though the initial condition x_0 is not known the compact set S which it belongs to is known.

Let us associate the system given by Eqn. (1) with the cost function

$$J[u(.), x_0, v(.)] = h\,[x(t_f)] + \int_{t_0}^{t_f} g[t, u(t)]dt \tag{2}$$

where $x(t_f)$ is the value of solution of Eqn. (1) at $t = t_f$, $h: \Re^n \to \Re$ is convex, and $g: [t_0, t_f] \times P \to \Re$ is continuous with respect to t and convex with respect to x. It is desired to select control function u minimizing the cost J. Also it is given that the state vector x in the system given by Eqn.(1) is not observable, therefore, $u(t)$ will be designed without having feedback from it.

If $u(t) \in P$ is measurable with respect to t in $[t_0, t_f]$ then the function u is called an admissible control. Let us denote the set of admissible control functions by U. Assuming that the unknown inputs are measurable and taking values from the set Q we denote such set of unknown inputs by V.

Now we can restate the problem in the terms defined above: It is desired to find $u^0 \in U$ such that

$$\min_{u(.)\in U} \sup_{x_0\in S, v(.)\in V} J[u(.),x_0,v(.)] = \sup_{x_0\in S, v(.)\in V} J[u^0(.),x_0,v(.)] =: J^0 \tag{3}$$

The function u^0 satisfying Eqn. (3) is called guaranteed optimal control.

In this paper we present sufficient conditions for the solution of problem given by (3), and we present illustrative examples.

If the initial position is known, i.e. $S=\{x_0\}$, then the problem (1)-(3) is a programmed minimax problem and can be investigated by methods of $[1-3]$.

Schmitendorf $[4]$ gives sufficient conditions for nonlinear systems with unknown initial vector x_0 and unknown parameters, but the conditions obtained there are very complicated.

In $[5]$ the problem (1)-(3) without integral term in the cost functional was investigated.

Let us present the following notation to use in the sequel. Let $\Phi(t,\tau)$ denote the fundamental matrix for the homogeneous system $\dot{x}(t)=A(t)x(t)$, and let $h^*(x)$ denote the complex conjugate of $h(x)$ defined by $h^*(l):=\sup_{x\in\Re^n}[<l,x>-h(x)]$, where $<\cdot,\cdot>$ denotes the scalar product. Let $L:=\{l\in\Re^n : h^*(l)<+\infty\}$. It can be shown that if $h(x)$ satisfies the Lipschitz conditions in $\Re^n$ then the set L is bounded. In order to express the results compactly also define

$$\varphi_1(l) := \max_{x_0\in S} <l,\Phi(t_f,\tau)x_0>,$$

$$\varphi_2(l) := \int_{t_0}^{t_f} \max_{v\in Q} <l,\Phi(t_f,\tau)C(\tau)v> d\tau,$$

$$\varphi(l) := \varphi_1(l)+\varphi_2(l)-h^*(l),$$

and

$$\mathcal{J}(x,u(.)) = \sup_{\ell\in L}\left[\varphi(\ell)+\langle \ell,x\rangle+\int_{t_0}^{t_f} g(t,u(t))dt\right].$$

Let $x(t,u(\cdot))$ denote the solution of

$$\begin{cases}\dot{x}(t)=\Phi(t_f,t)B(t)u(t)\\ x(t_0)=0\end{cases} \tag{4}$$

Utilizing Cauchy formula to obtain the solution of Eqn. (1), and from the property of interchangability of inner and outer maximums we can express the problem given by (3) in terms of the notations introduced above:
For the system (4) find $u^0\in U$ satisfying

$$\min_{u(.)\in U} \tilde{J}\left[x\left(t_f, u(.)\right), u(.)\right] = \tilde{J}\left[x\left(t_f, u^0(.)\right), u^0(.)\right] = J^0 \tag{5}$$

The initial condition and uncertainty in the problem given by (1)-(3) have been transferred into the supremum. Due to the definition of functional $\tilde{J}$, compared to (1)-(3) the solution of (4)-(5) is sometimes easier. We, however, analyze the problem (1)-(3) next.

Lemma 1: $[1-2]$ The following are true:

$$\min_{u(.)\in U}\left[\int_{t_0}^{t_1}\left\langle \ell, \Phi(t_f, t)B(t)U(t)\right\rangle dt + \int_{t_0}^{t_1} g(t, u(t))dt\right] = \int_{t_0}^{t_1}\min_{u\in P}\left[\left\langle \ell, \Phi(t_f, t)B(t)U\right\rangle + g(t,u)\right]dt, \tag{6}$$

$$\max_{v(.)\in V}\left[\int_{t_0}^{t_1}\left\langle \ell, \Phi(t_f, t)C(t, v(t))\right\rangle dt\right] = \int_{t_0}^{t_1}\max_{v\in Q}\left\langle \ell, \Phi(t_f, t)C(t, v)\right\rangle dt, \tag{7}$$

and $u^*(t,\ell)$ computed from

$$\min_{u\in P}\left\langle \ell, \Phi(t_f, t)B(t)u\right\rangle = \left\langle \ell, \Phi(t_f, t)B(t)u^*(t,\ell)\right\rangle$$

qualifies as a minimum for the lefthand side of (6). □

Let X and Y be nonempty metric compact sets, and let $f: X\times Y \to \Re$ be a continuous function. Then, the pair $(x_0, y_0) \in X\times Y$ satisfying

$$f(x_0, y) \le f(x_0, y_0) \le f(x, y_0) \tag{8}$$

for all $x\in X$ and $y \in Y$ is called a saddle point. If there exists a saddle point, then

$$\min_{x\in X}\ \max_{y\in Y} f(x,y) = \max_{y\in Y}\ \min_{x\in X} f(x,y) \tag{9}$$

holds. In Eqn. (9) the outer minimum and outer maximum pair is a saddle point. The vector x_0 which minimizes the lefthand side of (9) and the vector y_0 which maximizes righthand side of (9) are called minimax and maximin respectively.

Lemma 2: Let (x_0, y_0) be a saddle point. Then the vector x_0 minimizes the function $m(x) = f(x, y_0)$. □

Lemma 2 is a consequence of Eqn.(9).

If y_0 is a maximin vector qualifying as a maximum for the righthand side of (9) then the vector minimizing $m(x) = f(x, y_0)$ is not necessarily the minimax vector. Consider the example below.

Example: Let $X = Y = [0,1]$ and f be defined by

$$f(x,y) = \begin{cases} 0 & ; \quad 0 \le x \le y \le 1 \\ x - y & ; \quad 0 \le y \le x \le 1 \end{cases}$$

Then, clearly,

$$\min_{x \in X} \max_{y \in Y} f(x,y) = \min_{x \in X} x = 0$$

and x_0 is the unique minimax element. Similarly

$$\max_{y \in Y} \min_{x \in X} f(x,y) = \max_{y \in Y} 0 = 0$$

and each $y_0 \in [0,1]$ is a maximin element. If $y_0 = 0.5$ is selected then

$$g(x) = \begin{cases} 0 & ; \quad 0 \le x \le 0.5 \\ x - 0.5 & ; \quad 0.5 \le x \le 1 \end{cases}$$

and the set of minimizing x is computed as $\{x | \ 0 \le x \le 0.5\}$. Only x=0 is the minimax element in this set. □

Let us reconsider the problem given by (1)-(3):

Theorem 1: Let the function $\varphi(.)$ be concave. Then the guaranteed optimal control can be computed from

$$J^0 = \max_{\ell \in L} \left[\varphi(\ell) + \int_{t_0}^{t_f} \min_{u \in P} \left\{ \left[\ell, \Phi(t_f, t) B(t) u \right] + g(t,u) \right\} dt \right]. \tag{10}$$

The guaranteed optimal control $u^0(.)$ solving (3) satisfies:

$$\min_{u \in P} \left[\langle \ell^0, \Phi(t_f, t) B(t) u \rangle + g(t,u) \right] = \left[\langle \ell^0, \Phi(t_f, t) B(t) u^0(t) \rangle + g(t, u^0(t)) \right] \tag{11}$$

where ℓ^0 is the vector maximizing expression (10).

Proof: It is known that [6]

$$h(x) = \sup_{\ell \in L} \left[\langle \ell, x \rangle - h^*(\ell) \right].$$

Considering Lemma 1, this can be written as

$$J^0 = \min_{u(.)\in U} \max_{\ell\in L} J(u(.),\ell)$$

with

$$J(u(.),\ell) = \varphi(\ell) + \left\langle \ell, \int_{t_0}^{t_f} \Phi(t_f,t)B(t)u(t)dt \right\rangle + \int_{t_0}^{t_f} g(t,u(t))dt$$

Since the function $J(u(.),\ell)$ is linear with respect to $u(.)$ and, due to the hypothesis, concave with respect to ℓ, then according to the minimax theorem [7] there exists a saddle point $\left(u^0(.),\ell^0\right)$ and

$$J^0 = \max_{\ell\in L} \min_{u\in U} J(u(.),\ell).$$

Considering Lemma 1 we can write

$$\begin{aligned} J^0 &= \max_{\ell\in L}\left\{\varphi(\ell) + \min_{u(.)\in U}\left[\left\langle \ell, \int_{t_0}^{t_f} \Phi(t_f,t)B(t)u(t) \right\rangle dt + \int_{t_0}^{t_f} g(t,u(t))dt\right]\right\} \\ &= \max_{\ell\in L}\left\{\varphi(\ell) + \int_{t_0}^{t_f} \min_{u\in P}\left[\langle \ell, B(t)u\rangle + g(t,u)\right]dt\right\} \end{aligned}$$

According to Lemma 2, the function $u^0(.)$

$$\varphi(\ell) + \int_{t_0}^{t_f} \left\langle \ell^0, \Phi\left(t_f,t\right)B(t)u(t)\right\rangle dt + \int_{t_0}^{t_f} g\left(t,u(t)\right)dt$$

minimizes the following expression:

$$\begin{aligned} \min_{u(.)\in U}\left[\left\langle \ell^0, \int_{t_0}^{t_f} \Phi\left(t_f,t\right)B(t)u(t)dt \right\rangle + \int_{t_0}^{t_f} g\left(t,u(t)\right)dt\right] = \left\langle \ell^0, \int_{t_0}^{t_f} \Phi\left(t_f,t\right)B(t)u^0(t)dt \right\rangle \\ + \int_{t_0}^{t_f} g\left(t,u^0(t)\right)dt \qquad (12) \end{aligned}$$

This equation with Lemma 2 imply

$$\int_{t_0}^{t_f}\left[\min_{u\in P}\left\{\left\langle \ell^0, \Phi\left(t_f,t\right)B(t)u\right\rangle + g(t,u)\right\} - \left\{\left\langle \ell^0, \Phi\left(t_f,t\right)B(t)u^0(t)\right\rangle + g(t,u^0(t))\right\}\right]dt = 0. (13)$$

Since the integrand is nonnegative we obtain (11) from (13). □

Corollary 1: Let the function $g(t,u)$ be strictly convex with respect to u. Then $u^0(t)$ satisfying the condition (11) is the optimal minimax control.

Proof: Since the function $g(t,u)$ is strictly convex with respect to u, the function $u^0(t)$ minimizing the expression (11) is unique. According to Lemma 2 it is the optimal control. □

Let us define the set P by

$$P = \left\{ p = (p_1, p_2, \dots, p_k) : \sum_{i=1}^{k} p_i = 1, \ \ p_i \geq 0 \right\}.$$

Theorem 2: Let the function $\varphi(\ell)$ be convex, L be the convex hull of $\{\ell_1, \dots, \ell_k\}$. In the sequel it will be denoted by $co\{\ell_1, \dots, \ell_k\}$. Then

$$J^0 = \max_{p\in P} \left[\sum_{i=1}^{k} p_i \varphi(\ell_i) + \int_{t_0}^{t_f} \min_{u\in P} \left[\left\langle \widetilde{\ell}, \Phi(t_f, t) B(t) u \right\rangle + g(t,u) \right] dt \right]. \tag{14}$$

If $u^0(t)$ is the guaranteeing optimal control then it satifies

$$\min_{u\in P} \left[\left\langle \widetilde{\ell}^0, \Phi(t_f,t) B(t) u \right\rangle + g(t,u) \right] = \left\langle \widetilde{\ell}^0, \Phi(t_f,t) B(t) u^0(t) \right\rangle + g(t, u^0(t)) \tag{15}$$

where $\widetilde{\ell} = \sum_{i=1}^{k} p_i \ell_i$, $\widetilde{\ell}^0 = \sum_{i=1}^{k} p_i^0 \ell_i$ and the vector $p^0 = (p_1^0, p_2^0, \dots, p_k^0)$ maximizes the righthand side of expression (12).

Proof: Note that by the hypothesis of the theorem, the function $J(u(.), \ell)$ is convex with respect to ℓ. The following equalities can be shown easily:

$$\max_{\ell \in L = co\{\ell_1, \dots, \ell_k\}} J(u(.), \ell) = \max_{i \in \{1, \dots, k\}} J(u(.), \ell_i) \tag{16}$$

and

$$\max_{i \in \{1, \dots, k\}} J(u(.), \ell_i) = \max_{p\in P} \sum_{i=1}^{k} p_i J(u(.), \ell_i). \tag{17}$$

Thus we obtain

$$J^0 = \min_{u(.)\in U} \max_{p\in P} \sum_{i=1}^{k} p_i J(u(.), \ell_i).$$

The expression under minimax is convex with respect to u and linear with respect to p. Using the minimax theorem we obtain

$$J^0 = \max_{p\in P} \min_{u(.)\in U} \sum_{i=1}^{k} p_i J(u(.), \ell_i). \tag{18}$$

Using Lemma 1, the expression (12) can be obtained from (16). The remaining part of the proof uses substitution of $\tilde{\ell}^0$ for ℓ^0 in that of Theorem 1.

Corollary 2: If L is the convex hull of $\{\ell_1, \ell_2, \ldots, \ell_k\}$, the function $\varphi(\ell)$ is convex, and the function $g(t,u)$ is strictly convex with respect to u then the control $u^0(t)$ obtained from (13) is optimal minimax control. □

Example 1: Consider

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = u_2 + v \end{cases}, \quad \|u\| = \sqrt{u_1^{\,2} + u_2^{\,2}} \le 2, \quad |v| \le 1, \quad x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \quad \begin{cases} x_1^0 \in [0,1] \\ x_2^0 \in [0,1] \end{cases}$$

and

$$J(x(1)) = |x_1(1)| + x_2(1).$$

Here $P = \{u: \|u\| \le 1\}$, $Q = \{v \; : |v| \le 1\}$, $S = \{x = (x_1, x_2): \; 0 \le x_1 \le 1, \; 0 \le x_2 \le 1\}$, $h(x) = h(x_1, x_2) = |x_1| + x_2$, and $g(t,x) = 0$ are given.

Direct calculation shows that

$$h^*(\ell_1, \ell_2) = \begin{cases} 0 \; ; |\ell_1| \le 1, \ell_2 = 1 \\ +\infty; \; otherwise \end{cases},$$

$$\Phi(t,\tau) = \begin{pmatrix} 1 & t-\tau \\ 0 & 1 \end{pmatrix},$$

$$\varphi_1(\ell) = \max_{x_1^0 \in [0,1]} \ell_1 x_1 + \max_{x_2^0 \in [0,1]} (\ell_1 + 1) x_2 ,$$

$$\varphi_2(\ell) = \frac{\ell_1}{2} + 1 ,$$
$$\varphi(\ell) = \varphi_1(\ell) + \varphi_2(\ell),$$

and

$$L = \{(\ell_1, \ell_2): |\ell_1| \le 1, \ell_2 = 1\} = co\{(1 \;\; 1)^T, (1 \;\; -1)^T\}.$$

From the expressions above $\varphi(\ell)$ is convex and L is polygon set. Therefore

$$J^0 = \min_{|u(t)|\le 1} \max_{|\ell_1|\le 1} \left\{ \varphi_1(\ell_1) + \frac{\ell_1}{2} + 1 + \left\langle \begin{pmatrix} \ell_1 \\ 1 \end{pmatrix}, \int_0^1 \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} dt \right\rangle \right\}$$

$$= 1 + \min_{u(\cdot)\in U} \max_{\ell_1 \in [-1,1]} \left\{ \varphi(\ell_1) + \frac{\ell_1}{2} + \int_0^1 \left(\ell_1 u_1(t) + [(1-t)\ell_1 + 1] u_2(t) \right) dt \right\}$$

$$= 1 + \max_{p\in P} \min_{u(\cdot)\in U} \left\{ p_1 \left(\varphi_1(-1) - \frac{1}{2} + \int_0^1 [-u_1(t) + tu_2(t)] dt \right) + \right.$$

$$\left. + p_2 \left(\varphi_1(1) + \frac{1}{2} + \int_0^1 [u_1(t) + (2-t)u_2(t)] dt \right) \right\}$$

The control $u^0(t)$ satisfying (13) is unique, therefore by Theorem 2 the control

$$u^0(t) = \begin{cases} \dfrac{-2}{\sqrt{1+(2-t)^2}} \\ \dfrac{-2(2-t)}{\sqrt{1+2(1-t)}} \end{cases}$$

is minimax control. □

Example 2: Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}, \quad t \in [0,1], \quad |u| \le 1, \quad x_1(0) \in [-1,1], \quad x_2(0) \in [-1,1]$$

with cost functional

$$J = |x_1(1)| + \int_0^1 \left(u^2(t) + u(t) \right) dt .$$

Here $P = [-1,1]$, $Q = \{0\}$, $h(x) = h(x_1, x_2) = |x_1|$, and $g(t,u) = u^2 + u$ are given. Therefore

$$h^*(\ell) = h^*(\ell_1, \ell_2) = \begin{cases} 0 \ ; |\ell_1| \le 1, \ell_2 = 0 \\ +\infty ; otherwise \end{cases},$$

$$L = dom \ h^* = \{(\ell_1, 0) : |\ell_1| \le 1\} = co\{(-1,0),(1,0)\},$$

and

$$\varphi(\ell)=\varphi_1(\ell)-h^*(\ell)=\varphi_1(\ell)=2|\ell_1|.$$

After direct calculation we use Theorem 2 to obtain

$$J^0=\max_{\substack{(p_1,p_2)\\ p_1+p_2=1\\ p_1,p_2\geq 0}}\left\{p_1\varphi(-1)+p_2\varphi(1)-\frac{1}{4}\int_0^1\left[(p_1-p_2)(\tau-1)+1\right]^2 d\tau\right\}$$

$$=\max\left[-\frac{1}{3}p_1^{\,2}+\frac{5}{6}p_1+\frac{17}{12}\right] \tag{18}$$

and

$$\min_{|u|\leq 1}\left\{\left[\left(p_1^0-p_2^0\right)(\tau-1)+1\right]u+u^2\right\}=\left[\left(p_1^0-p_2^0\right)(\tau-1)+1\right]u^0(\tau)+\left(u^0(\tau)\right)^2. \tag{19}$$

Maximum value in (18) is attained at $p_1^0=1$ and $J^0=\frac{23}{12}$; hence $p_1^0=1,\ p_2^0=0.$

Minimum value in (19) is attained at

$$u^0(\tau)=-\frac{\left(p_1^0-p_2^0\right)(\tau-1)+1}{2}=-\frac{\tau}{2}.$$

Thus $u^0(\tau)=-\frac{\tau}{2}$ is minimax control, and optimal value of functional is equal to $\frac{23}{12}$.

□

REFERENCE

[1] Krasovskii.N.N. and Subbotin.A.I., Game-Theoretical problems, Springer-Verlag, 1986.

[2] Subbotin.A.I. and Chentsov.A.G., Optimization of guaranteed result (in Russian), Moscow, 1981.

[3] Dzhafarov.V., Regular programmed maximin in differential games, J. of Appl. Math. and Mech. ,vol. 46, no:5, 1983.

[4] Schmitendorf, W.E., Minmax Control of Systems with Uncertainly in the İnitail State and in the State Equations, IEEE Transactions on Automatic Control, p.439-443, June 1977.

[5] Dzhafarov,V. and Karamancıoğlu,A., Başlangıç Belirsizlikli ve Bilinmeyen Girişli Sistemlerde Minmaks Kontrol, 6th National Congress of Electrical Engineering, Bursa, s.981-984, Sept. 1995.

[6] Rochafellar.R.T., Convex Analysis, Princeton, NJ, Princeton Univ. Press, 1970.

[7] Fan Tszi, Minimax Theorems, In:Infinite Antagomistic Games, Moscow, 1963.