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ON THE PENTANACCI NUMBERS

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Abstract- In this paper, we give recurrence relation obtained by using from [1] for Pentanacci sequence. Furthermore, we construct generating matrix for P_{6n}, S_{6n}. Finally, we represent relationships between Pentanacci sequence and permanents of certain matrices.

Key Words- Pentancci numbers, generating matrix, sums, permanents

1.INTRODUCTION

Interestingly enough, the amazing Fibonacci numbers occur in quite unexpected place in nature. The number of petals in many flowers is often a Fibonacci number. For instance, count the number of petals in the flowers. Enchanter's nightshade has two petals, iris and trillium three, wild rose five and delphinium and cosmos eight. Fibonacci numbers also found in some spiral arrangements of leaves on the twigs of plants and trees. From any leaf on a branch, count up the number of leaves until you reach the leaf directly above it, the number of leaves is often a Fibonacci number. On basswood and elm trees, this number is 2; on beach and hazel trees, it is 3; on and willow trees, it is 13.

There is application of Fibonacci numbers in biology, physics, chemistry. The atomic number Z of an atom is number of protons in it. The periodic table shows an interesting relationship between the atomic numbers of inert gas and Fibonacci numbers. There are six inert gases-helium, neon, argon, krypton, xenon and radon- and they are exceptionally stable chemically. With the exception of helium, their atomic numbers are approximately the same as the Fibonacci numbersF_7 through F_{11}. Optics, the branch of physics that deal with light and vision, has found yet another appearance of Fibonacci numbers in the real world.

Fibonacci numbers occur in relation to music. Fibonacci numbers have found their way into the art of also. In [7], author mentioned in detail from this like innumerable application.

Pentanacci numbers is a generalization of Fibonacci numbers. Author is mention in [8] a application of Pentanacci numbers related to ancestor of humanity. Pentanacci constant is equal to 1.9659. This constant is indicates that rates of growth for fifth cousin pedigrees is Pentanacci constant.

Feng [3] studied about derivation different reccurence relation on the Tribonacci numbers and their sums. Kilic [4] gave generating matrices for the sumsof Tribonacci numbers and $4n$ subscripted Tribonacci sequence, T_{4n}, and their sums. In addition on, Kilic [4] touch on relationships between these sequences and permanents of some matrices. Kilic and Tasci [5] obtained some interesting relationships between the permanents of some tridiagonal matrices with applications to the negatively and positively subscripted usual Fibonacci and Lucas numbers. Kilic and Tasci [2] gave a generalizationof the Pell numbers in matrix representation. Parpar and Nalli [1] used a simple method to derive different recurrence relations on the recursive sequence order-k and their sums.

Parpar and Nalli [1] defined the recursive sequence order-k as following

$$L_n = \sum_{i=1}^{k} L_{n-i}$$

with $L_0 = 0$, $L_1 = 1$, $L_2 = 2^0 = 1$, $L_3 = 2^1 = 2$,…, $L_k = 2^{k-3}$ which are more general than Fibonacci, Tribonacci,…etc. sequences. And authors obtained identities for this definition.

If we take $k = 5$ in upper definition, $\{L_n\}$ is be Pentanacci sequence. Thus the Pentanacci sequence, $\{P_n\}$, is defined by the recurrence relation for $n \geq 4$

$$P_{n+1} = P_n + P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4}$$

with $P_0 = 0$, $P_1 = P_2 = 1$, $P_3 = 2$, $P_4 = 4$. The few first Pentanacci numbers are $0,1,1,2,4,8,16,31,61,120,236, \ldots$

Furthermore, negative subscript Pentanacci numbers are calculated by for $n < 0$

$$P_n = P_{n+5} - P_{n+4} - P_{n+3} - P_{n+2} - P_{n+1}$$

where $P_0 = 0$, $P_1 = P_2 = 1$, $P_3 = 2$, $P_4 = 4$. The few first negative subscript Pentanacci numbers are $0,0,0,1,-1,0,0,0,2,-3,1,0,0,4,-8, \ldots$

Let for $n \geq 0$

$$S_n = \sum_{k=0}^{n} P_k$$

for $n < 0$

$$S_n = \sum_{k=-1}^{n} P_k$$

2. ON THE PENTANACCI SEQUENCE $\{P_{6n}\}$

In this section, we consider the $6n$ subscripted Pentanacci numbers. First third-order linear recurrence relation for the $6n$ subscripted Pentanacci numbersis obtained by method which is told by Parpar and Nalli [1]. And we give a new generating matrix for $\{P_{6n}\}$ and obtain new formulas for the sequence $\{P_{6n}\}$ by using from Parpar and Nalli [1].

Lemma 1: For $n \geq 4$,

$$P_{6(n+1)} = 57P_{6n} + 42P_{6(n-1)} + 22P_{6(n-2)} + 7P_{6(n-3)} - 38156249P_{6(n-4)}$$

where $P_0 = 0$, $P_6 = 16$, $P_{12} = 912$, $P_{18} = 52656$, $P_{24} = 3040048$.

Let for $n \geq 0$

$$S_{6n} = \sum_{k=0}^{n} P_{6k} \tag{1}$$

for $n < 0$

$$S_{6n} = \sum_{k=-1}^{n} P_{6k} \qquad (2)$$

Define the 6×6 matrices F and G_n defined by

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 57 & 42 & 22 & 7 & -38156249 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$G_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ S_{6n} & P_{6(n+1)} & D_n & E_n & H_n & -38156249P_{6n} \\ S_{6(n-1)} & P_{6n} & D_{n-1} & E_{n-1} & H_{n-1} & -38156249P_{6(n-1)} \\ S_{6(n-2)} & P_{6(n-1)} & D_{n-2} & E_{n-2} & H_{n-2} & -38156249P_{6(n-2)} \\ S_{6(n-3)} & P_{6(n-2)} & D_{n-3} & E_{n-3} & H_{n-3} & -38156249P_{6(n-3)} \\ S_{6(n-4)} & P_{6(n-3)} & D_{n-4} & E_{n-4} & H_{n-4} & -38156249P_{6(n-4)} \end{bmatrix}$$

where

$$D_n = 42P_{6n} + 22P_{6(n-1)} + 7P_{6(n-2)} - 38156249P_{6(n-3)}$$
$$E_n = 22P_{6n} + 7P_{6(n-1)} - 38156249P_{6(n-2)}$$
$$H_n = 7P_{6n} - 38156249P_{6(n-1)}$$

and S_{6n} is given by (1) and (2).

Lemma2: If $n \geq 5$, then $S_n = 1 + S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$

Proof: Induction on n.

Since $S_{6n} = P_{6n} + S_{6(n-1)}$ and considering Lemma 2, we have the following corollary without proof.

Corollary 1: If $n > 4$, then $F^n = G_n$.

Define the 6×6 matrices L and T_n as shown:

$$L = \begin{bmatrix} 58 & -15 & -20 & -15 & -38156256 & 38156249 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$T_n = \frac{1}{P_6}\begin{bmatrix} S_{6(n+1)} & -X_n & -Y_n & -Z_n & -V_n & 38156249S_{6n} \\ S_{6n} & -X_{n-1} & -Y_{n-1} & -Z_{n-1} & -V_{n-1} & 38156249S_{6(n-1)} \\ S_{6(n-1)} & -X_{n-2} & -Y_{n-2} & -Z_{n-2} & -V_{n-2} & 38156249S_{6(n-2)} \\ S_{6(n-2)} & -X_{n-3} & -Y_{n-3} & -Z_{n-3} & -V_{n-3} & 38156249S_{6(n-3)} \\ S_{6(n-3)} & -X_{n-4} & Y_{n-4} & -Z_{n-4} & -V_{n-4} & 38156249S_{6(n-4)} \\ S_{6(n-4)} & -X_{n-5} & Y_{n-5} & -Z_{n-5} & -V_{n-5} & 38156249S_{6(n-5)} \end{bmatrix}$$

where

$$X_n = 15S_{6n} + 20S_{6(n-1)} + 15S_{6(n-2)} + 38156256S_{6(n-3)} - 38156249S_{6(n-4)}$$
$$Y_n = 20S_{6n} + 15S_{6(n-1)} + 38156256S_{6(n-2)} - 38156249S_{6(n-3)}$$
$$Z_n = 15S_{6n} + 38156256S_{6(n-1)} - 38156249S_{6(n-2)}$$
$$V_n = 38156256S_{6n} - 38156249S_{6(n-1)}$$

and S_{6n} given (1) and (2).

Theorem 1: If $n > 0$, then $L^n = T_n$.

Proof: The proof follows from the induction method.

Corollary 2: For $n > 5$, the sequence $\{S_{6n}\}$ satisfies the following recursion

$$S_{6n} = 58S_{6(n-1)} - 15S_{6(n-2)} - 20S_{6(n-3)} - 15S_{6(n-4)} - 38156256S_{6(n-5)} + 38156249S_{6(n-6)}$$

where $S_0 = 0$, $S_6 = 16$, $S_{12} = 928$, $S_{18} = 53584$, $S_{24} = 3093632$ and $S_{30} = 178608096$.

3. DETERMINANTAL REPRESENTATIONS

In this section, we give relationships between the sequence $\{P_{6n}\}$with its sums and the permanents of certain matrices.

Define the $n \times n$ matrix $F(n)$ as shown:

$$F(n) = \begin{bmatrix} 1 & 1 & & & & & & & 0 \\ 1 & 1 & 1 & & & & & & \\ 1 & 1 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & & & 1 \\ & 1 & 1 & & & & \ddots & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & 1 \\ 0 & & & & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then $perF(n) = P_{n+1}$ where P_n is the nth Pentanacci number.

For $n > 1$, define the $n \times n$ matrix $M_n = [m_{i,j}]$ with $m_{6,j} = m_{i,i} = 1$ for all i, $m_{i+1,i} = 1$ for $1 \le i \le n-1$, $m_{i,i+1}$=1 for $1 \le i \le 5$, $m_{i+2,i} = 1$ for $1 \le i \le n-2$, $m_{i+3,i} = 1$ for $1 \le i \le n-3$, $m_{i+4,i} = 1$ for $1 \le i \le n-4$, $m_{i+5,i} = 1$ for $1 \le i \le n-5$ and 0 otherwise.

Theorem 2: If $n > 1$, then $perM_n = \sum_{i=0}^{n} P_i$.

Proof: Induction on n.

Define $n \times n$ matrix $U_n = [u_{i,j}]$ with $u_{i,i} = 2$ for $1 \leq i \leq n$, $u_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $u_{i,i+5} = -1$ for $1 \leq i \leq n-5$ and then 0 otherwise.

Theorem 3: Then for $n > 6$

$$perU_n = S_{n+1}$$

where S_n is as before and $perU_1 = 2$, $perU_2 = 4$, $perU_3 = 8$, $perU_4 = 16$, $perU_5 = 32$, $perU_6 = 63$.

Proof: Expanding the $perU_n$ according to the last column, we obtain

$$perU_n = 2perU_{n-1} - perU_{n-6} \tag{3}$$

Since $perU_1 = S_2 = \sum_{i=0}^{1} P_i$, $perU_2 = S_3 = \sum_{i=0}^{2} P_i$, $perU_3 = S_4 = \sum_{i=0}^{3} P_i$, $perU_4 = S_5 = \sum_{i=0}^{4} P_i$, $perU_5 = S_6 = \sum_{i=0}^{5} P_i$, $perU_6 = S_7 = \sum_{i=0}^{6} P_i$, then by Corollary 2, the recurrence relation in (3) generate the sums of Pentanacci numbers. Thus we have the conclusion.

Now we derive a similar relation for terms of sequence $\{P_n\}$. Define $n \times n$ matrix $H_n = [h_{i,j}]$ with $h_{i,i} = 57$ for $1 \leq i \leq n$, $h_{i,i+1} = 42$ for $1 \leq i \leq n-1$, $h_{i,i+2} = 22$ for $1 \leq i \leq n-2$, $h_{i,i+3} = 7$ for $1 \leq i \leq n-3$, $h_{i,i+4} = -38156249$ for $1 \leq i \leq n-4$, $h_{i+1,i} = 1$ for $1 \leq i \leq n-1$ and 0 otherwise.

Theorem 4: Then for $n > 1$

$$perH_n = \frac{P_{6(n+1)}}{P_6}$$

where $perH_1 = \frac{P_{12}}{P_6}$.

Proof: Expanding the $perT_{n+1}$ according to the last column, by our assumption and the definition of H_n, we obtain

$$H_{6(n+1)} = 57H_{6n} + 42H_{6(n-1)} + 22H_{6(n-2)} + 7H_{6(n-3)} + 38156249H_{6(n-4)} \tag{4}$$

Since $perH_1 = \frac{P_{12}}{P_6}$, $perH_{21} = \frac{P_{18}}{P_6}$, $perH_3 = \frac{P_{24}}{P_6}$, $perH_4 = \frac{P_{30}}{P_6}$, $perH_5 = \frac{P_{36}}{P_6}$, by Lemma 2, the recurrence relation (4) generates the $\frac{P_{6(n+1)}}{P_6}$. The theorem is proven.

For $n > 1$, we define the $n \times n$ matrix Z_n as in the compact form, by the definition of H_n

$$Z_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ 0 & & H_{n-1} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

Theorem 5: If $n > 1$, then

$$perZ_n = \frac{\sum_{i=1}^{n} P_{6i}}{P_6}$$

Proof: (Induction on n) If $n = 2$, then $perZ_2 = \frac{\sum_{i=1}^{2} P_{6i}}{P_6} = 58$. Suppose that the equation holds for n. We show that the equation holds for $n+1$. Thus, by the definition of H_n and Z_n, expanding$perZ_{n+1}$, according to the first column gives us $perZ_{n+1} = perZ_n + perH_n$. By our assumption and Theorem 4, we have the conclusion.

In this paper, we derive reccurence relations for $\{P_{6n}\}$ and $\{S_{6n}\}$. But if desired, can be applied to different sequence as Fibonacci, Tribonacci,… etc.

4. APPLICATIONS

In this section we give applications of theorems in section 3.

Example1: For $n = 8$, $F(8)$ matrix is as follow

$$F(8) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $per\ F(8) = 120 = F_9$.

Example2: For $n = 10$, M_{10} matrix is as follow

$$M_{10} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $per\ M_{10} = 480 = \sum_{i=0}^{10} P_i$.

Example3: For $n = 10$, U_{10} matrix is as follow

$$U_{10}=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

and $per\ U_{10} = 944 = S_{11}$.

Example4: For $n = 6$, H_6 matrix is as follow

$$H_6=\begin{bmatrix} 57 & 42 & 22 & 7 & -38156249 & 0 \\ 1 & 57 & 42 & 22 & 7 & -38156249 \\ 0 & 1 & 57 & 42 & 22 & 7 \\ 0 & 0 & 1 & 57 & 42 & 22 \\ 0 & 0 & 0 & 1 & 57 & 42 \\ 0 & 0 & 0 & 0 & 1 & 57 \end{bmatrix}$$

and $per\ H_6 = 32214492796 = \frac{P_{42}}{P_6}$.

Example5: For $n = 7$, Z_7 matrix is as follow

$$Z_7=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 57 & 42 & 22 & 7 & -38156249 & 0 \\ 0 & 1 & 57 & 42 & 22 & 7 & -38156249 \\ 0 & 0 & 1 & 57 & 42 & 22 & 7 \\ 0 & 0 & 0 & 1 & 57 & 42 & 22 \\ 0 & 0 & 0 & 0 & 1 & 57 & 42 \\ 0 & 0 & 0 & 0 & 0 & 1 & 57 \end{bmatrix}$$

and $per\ Z_7 = 32214538870 = \frac{S_{42}}{P_6}$.

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