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SYMMETRY REDUCTIONS AND EXACT SOLUTIONS OF A VARIABLE COEFFICIENT (2+1)-ZAKHAROV-KUZNETSOV EQUATION

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Abstract - We study the generalized (2+1)-Zakharov-Kuznetsov (ZK) equation of time dependent variable coefficients from the Lie group-theoretic point of view. The Lie point symmetry generators of a special form of the class of equations are derived. We classify the Lie point symmetry generators to obtain the optimal system of one-dimensional subalgebras of the Lie symmetry algebras. These subalgebras are then used to construct a number of symmetry reductions and exact group-invariant solutions to the underlying equation.

Key Words - Generalized ZK equation; solitons; Lie symmetries; optimal system; symmetry reduction; group-invariant solutions

1. INTRODUCTION

The study of the exact solutions of nonlinear evolution equations plays an important role to understand the nonlinear physical phenomena which are described by these equations. The importance of deriving such exact solutions to these nonlinear equations facilitate the verification of numerical methods and helps in the stability analysis of solutions.

In this paper, we study the exact solutions of one such nonlinear evolution equation, the generalized (2+1)-Zakharov-Kuznetsov equation of the form

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_{xyy} = 0 \tag{1}$$

of time dependent variable coefficients. Here *f(t), g(t)* and *h(t)* are arbitrary smooth functions of the variable t and $fgh \neq 0$. The equation (1) models the nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields [1]. The equation (1) also appears in different forms in many areas of Physics, Applied Mathematics and Engineering (see for example [2, 3]).

The transformation

$$\tilde{t} = \int f(t)dt, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{u} = u \tag{2}$$

maps equation (1) to

$$\tilde{u}_t + \tilde{f}(t)\tilde{u}\tilde{u}_x + \tilde{g}(t)\tilde{u}_{xxx} + \tilde{h}(t)\tilde{u}_{xyy} = 0, \quad (3)$$

where $\tilde{f}(t) = 1$, $\tilde{g}(t) = g(t)/f(t)$ and $\tilde{h}(t) = h(t)/f(t)$. Therefore, with out loss of generality we can consider the equations of the general form

$$u_t + uu_x + a(t)u_{xxx} + b(t)u_{xyy} = 0 \quad (4)$$

in our analysis as all the results of the class (4) can be extended to the class (1) by the transformation (2).

In [4], travelling wave-like solutions for the equation (1) were obtained. In [5] and [6], similarity reductions and some exact solutions were obtained for the special cases of the class of equations (4) using symmetry group method. For the theory and application of the Lie symmetry methods, see e.g., the Refs. [7, 8, 9, 10]. Recently, in [11] the method of Lie groups is utilized to derive solutions to an integrable equation governing short waves in a long-wave model.
The outline of the paper is as follows. In Section 2, we present the Lie point symmetries of a special case of the equation (4). In Section 3, we construct the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the special form of the equation. Moreover, using the optimal system of subalgebras symmetry reductions and exact group-invariant solutions of the underlying equation are obtained. Finally, in Section 4 concluding remarks are made.

2. LIE POINT SYMMETRIES

In this section, we consider a special case of the class of equations (4). That is, for the time dependent coefficients $a(t) = a_0/t$ and $b(t) = b_0/t$, where a_0 and b_0 are arbitrary constants, we utilize the Lie symmetry group method to obtain symmetry reductions and group-invariant solutions of the underlying equation. Therefore, the equation that is going to be studied in this paper takes the form

$$u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} = 0. \quad (5)$$

A vector field

$$X = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \psi(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \quad (6)$$

is a generator of point symmetry of the equation (5) if

$$X^{[3]}\left(u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} \right)\Big|_{(5)} = 0, \quad (7)$$

where the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{xyy}\frac{\partial}{\partial u_{xyy}},$$

the coefficients $\zeta_t, \zeta_x, \zeta_{xxx}$ and ζ_{xyy} are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\psi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi),\\
\zeta_{xy} &= D_y(\zeta_x) - u_{xt} D_y(\tau) - u_{xx} D_y(\xi) - u_{xy} D_y(\psi),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi) - u_{xxy} D_x(\psi),\\
\zeta_{xyy} &= D_y(\zeta_{xy}) - u_{xyt} D_y(\tau) - u_{xxy} D_y(\xi) - u_{xyy} D_y(\psi).
\end{aligned}$$

Here D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots, i = 1,2,3,$$

and $(x^1, x^2, x^3) = (t, x, y)$.

The coefficient functions τ, ξ, ψ and η are calculated by solving the determining equation (7). Since τ, ξ, ψ and η are independent of the derivatives of u, the coefficients of like derivatives of u in (7) can be equated to yield an over determined system of linear partial differential equations (PDEs). Therefore, the determining equation for symmetries after lengthy calculations yield

$$\tau = \tau(t), \xi = \xi(t,x), \psi = \psi(y), \xi_{xx} = 0, \eta_{xu} = 0, \eta_{uu} = 0, \tag{8}$$

$$(-1/t^2)\tau + (1/t)\tau_t - (3/t)\xi_x = 0, \tag{9}$$

$$(-1/t^2)\tau + (1/t)\tau_t - (1/t)\xi_x - (2/t)\psi_y = 0, \tag{10}$$

$$2\eta_{yu} - \psi_{yy} = 0, \tag{11}$$

$$\eta + \tau_t u - \xi_t - \xi_x u + (b_0/t)\eta_{yyu} = 0, \tag{12}$$

$$\eta_t + \eta_x u + (a_0/t)\eta_{xxx} + (b_0/t)\eta_{xyy} = 0. \tag{13}$$

Solving the determining equations (8)-(13) for τ, ξ, ψ and η, we obtain the following symmetry group generators given by

$$X_1 = \partial_x, X_2 = \partial_y, X_3 = t\partial_x + \partial_u, X_4 = t\partial_t - u\partial_u.$$

3. SYMMETRY REDUCTIONS AND EXACT GROUP-INVARIANT SOLUTIONS OF THE EQUATION (5)

Here we first construct the optimal system of one-dimensional subalgebras of the Lie algebra admitted by the equation (5). The classification of the one-dimensional subalgebras are then used to reduce the equation (5) into a partial differential equation (PDE) having two independent variables. Then we also study the symmetry properties of the reduced PDE to derive further symmetry reductions and exact group-invariant solutions for the underlying equation.

The results on the classification of the Lie point symmetries of the equation (5) are summarized by the Tables 1, 2 and 3. The commutator table of the Lie point symmetries of the equation (5) and the adjoint representations of the symmetry group of (5) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (5) which is given in Table 3 (for more details of the approach see [8] and the references therein).

Table 1. Commutator table of the Lie algebra of equation (5)

	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	0
X_3	0	0	0	$-X_3$
X_4	0	0	X_3	0

Table 2. Adjoint table of the Lie algebra of equation (5)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	X_4
X_3	X_1	X_2	X_3	$X_4+\varepsilon X_3$
X_4	X_1	X_2	$e^{-\varepsilon}X_3$	X_4

Table 3. Subalgebra, group invariants, group-invariant solutions of (5)

N	X	α	β	Group – invariant solution
1	$X_4+\lambda X_1+\mu X_2$	$x-\lambda\ln t$	$y-\mu\ln t$	$u=\frac{1}{t}h(\alpha,\beta)$
2	$X_2+\upsilon X_1$	t	$x-\upsilon y$	$u=h(\alpha,\beta)$
3	$X_3+\varepsilon X_1$	t	y	$u=\frac{x}{(t+\varepsilon)}+h(\alpha,\beta)$
4	$X_3+\delta X_2+\varepsilon X_1$	t	$\delta x-(t+\varepsilon)y$	$u=\frac{x}{(t+\varepsilon)}+h(\alpha,\beta)$
5	X_1	t	y	$u=h(\alpha,\beta)$

Here $\varepsilon=0,\pm1,\delta=\pm1$ and λ,μ and υ are arbitrary constants.

Case 1. In this case, the group-invariant solution corresponding to the symmetry generator $X_4+\lambda X_1+\mu X_2$ reduces the equation (5) to the PDE

$$\lambda h_{\alpha} + \mu h_{\beta} - hh_{\alpha} + h - a_0 h_{\alpha\alpha\alpha} - b_0 h_{\alpha\beta\beta} = 0. \tag{14}$$

Now the equation (14) admits the following symmetry generators given by

$$X_1 = \partial_{\alpha}, \quad X_2 = \partial_{\beta}.$$

(a) X_1

The group-invariant solution corresponding to X_1 is $h = H(\gamma)$, where $\gamma = \beta$ is the group invariant of X_1, the substitution of this solution into the equation (14) and solving we obtain a solution $u(t,x,y) = Ce^{-y/\mu}$ for (5), here C is a constant.

(b) $X_1 + \rho X_2$, where ρ is a constant.

$X_1 + \rho X_2$ leads to the group-invariant solution $h = H(\gamma)$, where $\gamma = \beta - \rho\alpha$ is the group invariant. Substitution of this solution into the equation (14) gives rise to the ordinary differential equation (ODE)

$$(\rho^3 a_0 + \rho b_0)H'' + \rho HH' + (\mu - \lambda\rho)H' + H = 0, \tag{15}$$

here `prime' denotes differentiation with respect to γ.

Case 2. The group-invariant solution arising from $X_2 + \upsilon X_1$ reduces the equation (5) to the PDE

$$h_{\alpha} + hh_{\beta} + \frac{(a_0 + b_0\upsilon^2)}{\alpha} h_{\beta\beta\beta} = 0. \tag{16}$$

The equation (16) admits the following three Lie point symmetry generators

$$X_1 = \partial_{\beta}, \quad X_2 = \alpha\partial_{\beta} + \partial_h, \quad X_3 = -\alpha\partial_{\alpha} + h\partial_h.$$

The optimal system of one-dimensional subalgebras are $X_3 + cX_1, X_2 + dX_1, X_1$, where c is an arbitrary real constant and $d = 0, \pm 1$.

(a) $X_3 + cX_1$

The group-invariant solution corresponding to $X_3 + cX_1$ is $h = \dfrac{1}{\alpha} H(\gamma)$, where $\gamma = \beta + c\ln\alpha$ is the group invariant of $X_3 + cX_1$, the substitution of this solution into the equation (16) results in the following ODE

$$(a_0 + b_0\upsilon^2)H'' + HH' + a_0 H' - H = 0, \tag{17}$$

here `prime' denotes differentiation with respect to γ.

(b) $X_2 + dX_1$.

$X_2 + dX_1$ leads to the group-invariant solution $h = \dfrac{\beta}{(\alpha + d)} + H(\gamma)$, where $\gamma = \alpha$ is the group invariant. Substitution of this solution into the equation (16) gives the solution

$$u(t,x,y)=\frac{x-\upsilon y+C}{(t+d)},$$

where C is a constant.

(c) X_1

The symmetry generator X_1 gives the trivial solution $u(t,x,y)=C$, where C is a constant.

Case 3. The group-invariant solution that corresponds to $X_3+\varepsilon X_1$ reduces the equation (5) to the PDE

$$h_\alpha+\frac{h}{\alpha+\varepsilon}=0. \qquad (18)$$

Hence the solution of the equation (5) is given by

$$u(t,x,y)=\frac{x+H(y)}{(t+\varepsilon)},$$

where $H(y)$ is an arbitrary function of its argument.

Case 4. The $X_3+\delta X_2+\varepsilon X_1$-invariant solution reduces the equation (1) to the PDE

$$h_\alpha+\frac{\beta}{(\alpha+\varepsilon)}h_\beta+\delta h h_\beta+\frac{h}{(\alpha+\varepsilon)}+\left[\frac{a_0\delta^3}{\alpha}+\frac{b_0\delta(\alpha+\varepsilon)^2}{\alpha}\right]h_{\beta\beta\beta}=0. \qquad (19)$$

(a) $\varepsilon=0$

In this case, the PDE (19) becomes

$$h_\alpha+\frac{\beta}{\alpha}h_\beta+\delta h h_\beta+\frac{h}{\alpha}+\left[\frac{a_0\delta^3}{\alpha}+b_0\delta\alpha\right]h_{\beta\beta\beta}=0. \qquad (20)$$

The equation (20) admits the Lie algebra spanned by the following symmetry generators

$$X_1=\alpha\partial_\beta,\quad X_2=\delta\partial_\beta-1/\alpha\partial_h.$$

(i) X_1

The group-invariant solution corresponding to X_1 is $h=H(\gamma)$, where $\gamma=\alpha$ is the group invariant of X_1, the substitution of this solution into the equation (20) and solving we obtain the solution $u(t,x,y)=(x+C)/t$, where C is a constant.

(ii) $X_1+\omega X_2$, where ω is a constant.

The grou-invariant solution corresponding to $X_1+\omega X_2$ is $h=-\beta/\alpha(\omega\alpha+\delta)+H(\gamma)$, where $\gamma=\alpha$ is the group invariant of $X_1+\omega X_2$, the substitution of this solution into the equation (20) and solving we obtain the solution

$$u(t,x,y)=\frac{\omega x+y+C}{\omega t+\delta},$$

where C is a constant.

(b) $\varepsilon\neq 0$.

In this instance, the PDE (19) admits the following symmetry generators

$$X_1=\delta\partial_\beta-1/(\alpha+\varepsilon)\partial_h,\quad X_2=\delta\alpha\partial_\beta+\varepsilon/(\alpha+\varepsilon)\partial_h.$$

(i) X_1

The group-invariant solution corresponding to X_1 is $h=-\beta/\delta(\alpha+\varepsilon)+H(\gamma)$, where $\gamma=\alpha$ is the group invariant of X_1, the substitution of this solution into the equation (19) and solving we obtain the solution $u(t,x,y)=\frac{y}{\delta}+C$, where C is a constant.

(ii) $X_2+\omega X_1$, where ω is a constant.

The $X_2+\omega X_1$-invariant solution is given by $h=(\varepsilon-\omega)\beta/\delta(\alpha+\omega)(\alpha+\varepsilon)+H(\gamma)$, where $\gamma=\alpha$ is the group invariant of $X_2+\omega X_1$, the substitution of this solution into the equation (19) and solving we obtain the solution

$$u(t,x,y)=\frac{\delta x-(\varepsilon-\omega)y+C}{\delta(t+\omega)},$$

where C is a constant.

Case 5. The X_1-invariant solution reduces the equation (5) to $h_\alpha=0$. Hence the solution of the equation (5) is given by $u(t,x,y)=H(y)$, where $H(y)$ is an arbitrary function of its argument.

4. CONCLUDING REMARKS

In this paper we have studied the generalized (2+1)-ZK equation with time dependent variable coefficients using the Lie symmetry group method. We derived the Lie point symmetry generators of a special form of the underlying class of equations. The Lie symmetry classification with respect to the special form of the time dependent variable coefficients equation was presented. We used this classification of optimal system of one-dimensional subalgebras of the Lie symmetry algebras to construct symmetry reductions and exact group-invariant solutions for the special form of the equation.

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