ON THE COUPLING OF AUXILIARY PARAMETER, ADOMIAN'S POLYNOMIALS AND CORRECTION FUNCTIONAL

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Abstract-In this paper, we apply He's variational iteration method (VIM) coupled with an auxiliary parameter and Adomian's polynomials which proves very effective to control the convergence region of approximate solution. The proposed algorithm is tested on generalized Hirota–Satsuma coupled KdV equation and numerical results explicitly reveal the complete reliability, efficiency and accuracy of the suggested technique. It is observed that the approach may be implemented on other nonlinear models of physical nature.
Key words-Variational iteration method; Auxiliary parameter; Adomian's polynomials, Hirota–Satsuma coupled KdV equation.

1. INTRODUCTION

The rapid development of nonlinear sciences, (see Abbasbandy [1-3], Bildik [4], Chun [5], Demirbağ [6], Geng [7], Ganji [8], Hirota [9], Herişanu [10], He [11-21], Kaya [22], Mohyud-Din [23-26], Soltanian [27], Tatari [28], Wu [29], Yu [30], Yong [31]) witnesses number of new analytical and numerical methods. Most of these introduced techniques are coupled with the inbuilt deficiencies including calculation of the so-called Adomian's polynomials, divergent results, limited convergence, lengthy calculations, small parameter assumption and non compatibility with the physical nature of the problems. In order to cope with such deficiencies, He [11-21] developed the variational iteration method (VIM) which was completely independent of the above mentioned inadequacies and has been applied to a wide class of nonlinear problems, (see Abbasbandy [1-3], Bildik [4], Chun [5], Demirbağ [6], Geng [7], Ganji [8], Hirota [9], Herişanu [10], He [11-21], Kaya [22], Mohyud-Din [23-26] and the references therein). The exponential success rate of He's VIM is a true index of its credibility. With the passage of time, several modifications have been introduced in He's variational iteration method (VIM) which has further enhanced the accuracy and efficacy of this algorithm to a tangible level. Abbasbandy [1-3] introduced the insertion of Adomian's polynomials in the correction functional of variational iteration method (VIM) and solved quadratic Riccati differential equation and Klein-Gordon equations. Inspired and motivated by the ongoing research in this area, we propose insertion of an unknown auxiliary parameter h into the correction functional of VIM and the subsequent coupling of Adomian's polynomials in the re-formulated correctional functional. The algorithm has been successfully tested on generalized Hirota–Satsuma coupled KdV equation. It is observed that the suggested insertion of auxiliary parameter h provides a simple way to adjust and control the convergence region of approximate solution for any values of x and t. Numerical results explicitly reveal the complete

reliability, efficiency and accuracy of the proposed algorithm. It is observed that the approach may be implemented on other nonlinear models of physical nature. It is worth mentioning that generalized Hirota–Satsuma coupled KdV equation (Wu [29], Yu [30], Yong [31]):

$$\begin{cases} u_t = \dfrac{1}{2} u_{xxx} - 3uu_x + 3(vw)_x\,, \\[2mm] v_t = -v_{xxx} + 3uv_x\,, \\[2mm] w_t = -w_{xxx} + 3uw_x\,. \end{cases} \tag{1}$$

arises very frequently in applied and engineering sciences.

2. ANALYSIS OF THE PROPOSED ALORITHM

Consider the following functional equation (see Abbasbandy [1-3], Bildik [4], Chun [5], Demirbağ [6], Geng [7], Ganji [8], He [11-21], Mohyud-Din [23-26]):

$$Lu + Ru + Nu = g(x), \tag{2}$$

where L is the highest order derivative that is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. The basic characteristic of He's method is to construct a correction functional for (2), which reads

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left\{ Lu_n(s) + R\tilde{u}_n + N\tilde{u}_n - g(s) \right\}ds, \tag{3}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory (He [11-21], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To solve (2) by He's VIM, we first determine the Lagrange multiplier λ that can be identified optimally via variational theory. Then, the successive approximations $u_n(x)$, $n \geq 0$, of the solution $u(x)$ can be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With λ determined, several approximations $u_n(x)$, $u_n \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{n\to+\infty} u_n(x). \tag{4}$$

In summary, we have the following variational iteration formula for (2):

$$\begin{cases} u_0(x)\,\text{is an arbitrary function}\,, \\[2mm] u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left\{ Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s) \right\}ds, \quad n \geq 0. \end{cases} \tag{5}$$

2.1 Coupling Of Auxiliary Parameter And Correction Functional

An unknown auxiliary parameter h can be inserted into the correction functional (5) of He's VIM. According to this assumption, we construct the following variational iteration formula:

$$\begin{cases} u_0(x) \, is \, an \, arbitrary \, function, \\ u_1(x,h) = u_0(x) + h\int_0^x \lambda(s)\{Lu_0(s) + Ru_0(s) + Nu_0(s) - g(s)\}ds, \quad n \geq 0 \qquad (6) \\ u_{n+1}(x,h) = u_n(x,h) + h\int_0^x \lambda(s)\{Lu_n(s,h) + Ru_n(s,h) + Nu_n(s,h) - g(s)\}ds, \quad n \geq 1. \end{cases}$$

Of course, assuming the Lagrange multiplier λ, has been already identified. It should be emphasized that $u_n(x,h), n \geq 1$ can be computed by symbolic computation software such as Maple or Mathematica. The approximate solutions $u_n(x,h), n \geq 1$ contain the auxiliary parameter h. The validity of the method is based on such an assumption that the approximations $u_n(x,h)$, $u_n \geq 0$ converge to the exact solution $u(x)$. It is the auxiliary parameter h which ensures that the assumption can be satisfied. In general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensures that the approximate solutions are convergent. In fact, the proposed combination is very simple, easier to implement and is capable to approximate the solution more accurately in a bigger interval.

2.2 Coupling Of Auxiliary Parameter, Adomian's Polynomials And Correction Functional

In this algorithm, we will be making the coupling of Adomian's polynomials and the re-formulated correctional functional (6) and obtain the following iterative scheme:

$$\begin{cases} u_0(x) \, is \, an \, arbitrary \, function, \\ u_1(x,h) = u_0(x) + h\int_0^x \lambda(s)\left\{Lu_0(s) + \sum_{n=1}^{\infty} u_n(s,h) + \sum_{n=1}^{\infty} A_n - g(s)\right\}ds, \quad n \geq 0 \qquad (6a) \\ u_{n+1}(x,h) = u_n(x,h) + h\int_0^x \lambda(s)\left\{Lu_n(s,h) + \sum_{n=1}^{\infty} u_n(s,h) + \sum_{n=1}^{\infty} A_n - g(s)\right\}ds, \quad n \geq 1. \end{cases}$$

where A_n are the so-called Adomian's polynomials which can be generated according to the specific algorithms Abbasbandy [1, 2].

3. NUMERICAL EXAMPLES

In this section, we apply the proposed algorithm (6a) to solve two generalized Hirota–Satsuma coupled KdV equations. Numerical results are compared with original variational iteration method (VIM).

Example 3.1 Consider the KdV equation (1) with the initial conditions (Wu [29], Yu [30], Yong [31])

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$$
\begin{cases}
u(x,0) = \dfrac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2(kx), \\[2mm]
v(x,0) = -\dfrac{4k^2 c_0\left(\beta + k^2\right)}{3c_1^2} + \dfrac{4k^2\left(\beta + k^2\right)}{3c_1}\tanh(kx) \\[2mm]
w(x,0) = c_0 + c_1 \tanh(kx),
\end{cases}
\tag{7}
$$

where $k, c_0, c_1 \neq 0$, and β are arbitrary constants and the exact solutions are given by

$$
\begin{cases}
u(x,t) = \dfrac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2\left[k(x+\beta t)\right], \\[2mm]
v(x,t) = -\dfrac{4k^2 c_0\left(\beta + k^2\right)}{3c_1^2} + \dfrac{4k^2\left(\beta + k^2\right)}{3c_1}\tanh\left[k(x+\beta t)\right], \\[2mm]
w(x,t) = c_0 + c_1 \tanh\left[k(x+\beta t)\right],
\end{cases}
\tag{8}
$$

which is bell-type for $u(x,t)$ and kink-type for $v(x,t)$ and $w(x,t)$. According to the original VIM, we have the following variational iteration formula (1):

$$
\begin{cases}
u_{n+1} = u_n - \int_0^t \left\{ u_{n_\tau} - \dfrac{1}{2} u_{n_{xxx}} + 3u_n u_{n_x} - 3(v_n w_n)_x \right\} d\tau, \\[2mm]
v_{n+1}(x,t) = v_n - \int_0^t \left\{ v_{n_\tau} + v_{n_{xxx}} - 3u_n v_{n_x} \right\} d\tau, \\[2mm]
w_{n+1}(x,t) = w_n - \int_0^t \left\{ w_{n_\tau} + w_{n_{xxx}} - 3u_n w_{n_x} \right\} d\tau.
\end{cases}
\tag{9}
$$

Fig. 5 shows the absolute error of $u_3(x,t)$ by the original VIM. The same situations exist for $v_3(x,t)$ and $w_3(x,t)$ which confirm that the obtained results by original VIM is not valid for large values of x and t in example 3.1. Now, using the recursive scheme (6a), and by selecting $u_0 = u(x,0)$, $v_0 = v(x,0)$, and $w_0 = w(x,0)$, we successively obtain

$$
\begin{cases}
u_0 = \dfrac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2(kx), \\[2mm]
v_0 = -\dfrac{4k^2 c_0\left(\beta + k^2\right)}{3c_1^2} + \dfrac{4k^2\left(\beta + k^2\right)}{3c_1}\tanh(kx) \\[2mm]
w_0 = c_0 + c_1 \tanh(kx),
\end{cases}
\tag{10}
$$

$$\begin{cases} u_1(x,t,h)=u_0-h\int_0^t\left\{u_{n_\tau}-\dfrac{1}{2}\sum_{n=0}^{\infty}u_{n_{xxx}}+3\sum_{n=0}^{\infty}A_n-3\sum_{n=0}^{\infty}B_n\right\}d\tau \\[2mm] \qquad =\dfrac{1}{3}\dfrac{\beta\cosh^3(kx)+4k^2\cosh^3(kx)-6k^2\cosh(kx)+12hk^3t\sinh(kx)\beta}{\cosh^3(kx)} \\[3mm] v_1(x,t,h)=v_0-h\int_0^t\left\{v_{n_\tau}+\sum_{n=0}^{\infty}v_{n_{xxx}}-3\sum_{n=0}^{\infty}C_n\right\}d\tau \\[2mm] \qquad =\dfrac{4}{3}\dfrac{k^2\left(\beta+k^2\right)\left(-c_0\cosh^2(kx)+c_1\sinh(kx)\cosh(kx)+hkt\beta c_1\right)}{c_1^2\cosh^3(kx)} \\[3mm] w_1(x,t,h)=w_0-h\int_0^t\left\{w_{n_\tau}+\sum_{n=0}^{\infty}w_{n_{xxx}}-3\sum_{n=0}^{\infty}D_n\right\}d\tau \\[2mm] \qquad =\dfrac{c_0\cosh^2(kx)+c_1\sinh(kx)\cosh(kx)+hkt+\beta c_1}{\cosh^2(kx)}, \end{cases} \tag{11}$$

$\vdots$

where A_n, B_n, C_n, D_n are the so-called Adomian's polynomials and can be generated by using the specific algorithm defined in Abbasbandy [1-3]. First, to find the proper value of h for the approximate solutions (11), we plot the so-called h-curve of $\dfrac{\partial^4}{\partial t^4}u_3(x,t,h)$, $\dfrac{\partial^4}{\partial t^4}v_3(x,t,h)$ and $\dfrac{\partial^4}{\partial t^4}w_3(x,t,h)$ for the case $x=1$ and $t=1$ as shown in Fig. 1. According to these h-curves, it is easy to discover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. Here, we select $h=0.01$.
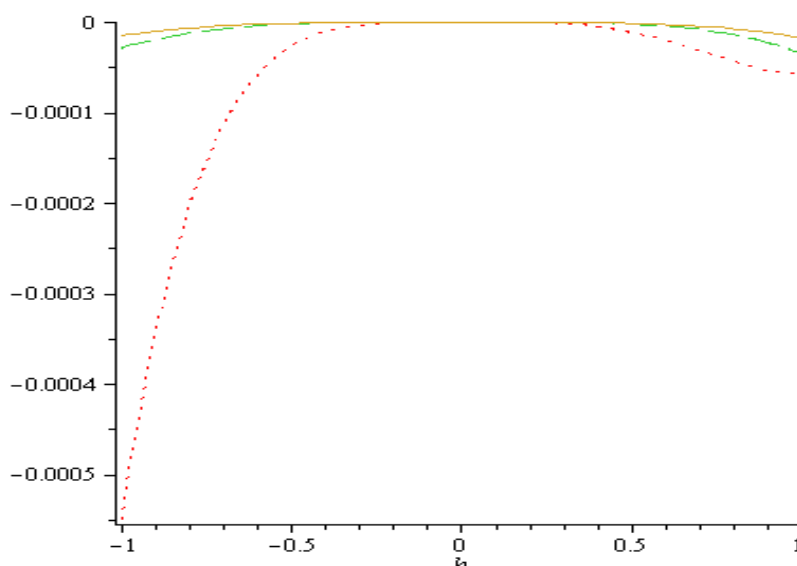

Fig.1. The h-curve of $\dfrac{\partial^4}{\partial t^4}u_3(x,t,h)$, (dot), $\dfrac{\partial^4}{\partial t^4}v_3(x,t,h)$, (dash), and $\dfrac{\partial^4}{\partial t^4}w_3(x,t,h)$, (solid) given by (11) when $x=1$ and $t=1$.

The absolute error of coupling of VIM and auxiliary parameter results for $u_3(x,t)$, $v_3(x,t)$ and $w_3(x,t)$ when $c_0 = 1.5$, $c_1 = 0.1$, $k = 0.1$, $\beta = 1.5$, $x \in [0,100]$ and $t \in [0,100]$ are plotted in Figs. 2, 3 and 4, respectively.
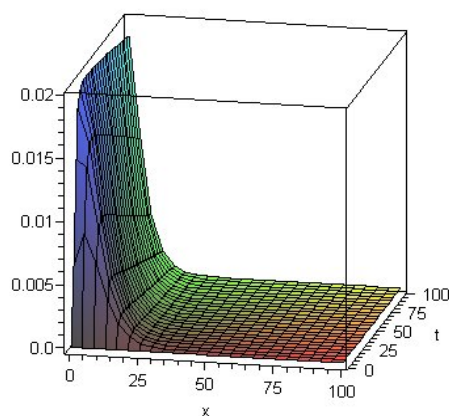
Fig. 2. Absolute error for the 3rd-order approximation by the coupled form of VIM for $u(x,t)$ and $h = 0.01$.
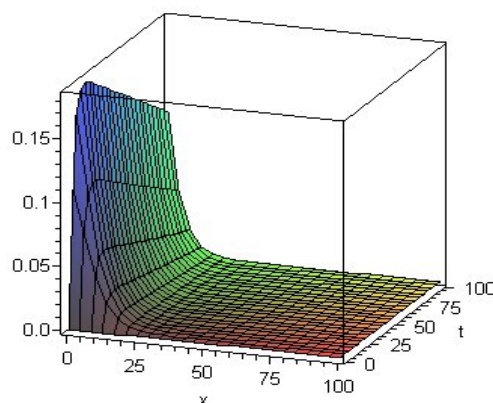
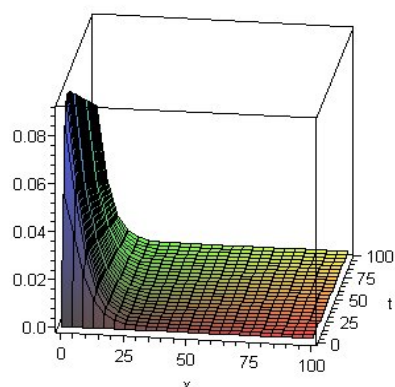Fig. 3. Absolute error for the 3rd-order approximation by coupled form of VIM for $v(x,t)$ and $h = 0.01$.

Fig. 4. Absolute error for the 3rd-order approximation by coupled form of VIM for $w(x,t)$ and $h = 0.01$.
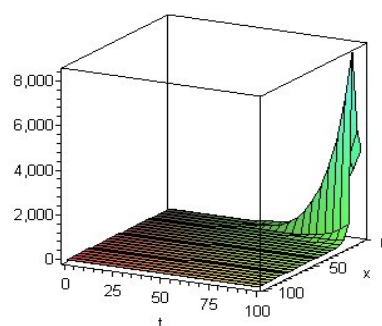
Fig. 5. Absolute error for the 3rd-order approximation by original VIM for $u(x,t)$.

Example 3.2 Consider the generalized Hirota–Satsuma coupled KdV equation (1) with the initial conditions (Wu [29], Yu [30], Yong [31])

$$\begin{cases} u(x,0) = \dfrac{1}{3}\left(\beta - 8k^2\right) + 4k^2 \tanh^2[k(x)], \\[2mm] v(x,0) = -\dfrac{4k^2\left(3k^2 c_0 - 2\beta c_2 + 4k^2 c_2\right)}{3c_2^2} + \dfrac{4k^2}{c_2}\tanh^2[k(x)], \\[2mm] w(x,t) = c_0 + c_2 \tanh^2[k(x)], \end{cases}$$

where $k, c_0, c_1 \neq 0$, and β are arbitrary constants and the exact solutions are

$$\begin{cases} u(x,t) = \dfrac{1}{3}\left(\beta - 8k^2\right) + 4k^2 \tanh^2[k(x+\beta t)], \\[2mm] v(x,t) = -\dfrac{4k^2\left(3k^2 c_0 - 2\beta c_2 + 4k^2 c_2\right)}{3c_2^2} + \dfrac{4k^2}{c_2}\tanh^2[k(x+\beta t)], , \\[2mm] w(x,t) = c_0 + c_2 \tanh^2[k(x+\beta t)], \end{cases}$$

which is bell-type for $u(x,t)$ and kink-type for $v(x,t)$ and $w(x,t)$. Proceeding as before, the absolute errors are given for $u_3(x,t)$, $v_3(x,t)$ and $w_3(x,t)$ when $c_0 = 1.5$, $c_2 = 0.1$, $k = 0.1$, $\beta = 1.5$, $x \in [0,100]$ and $t \in [0,100]$ are plotted in Figs. 6, 7 and 8, respectively. Furthermore, the absolute error of $u_3(x,t)$ by the original VIM is shown in Fig. 9 which confirms that the obtained results by original VIM is not valid for large values of x and t in example 3.2.
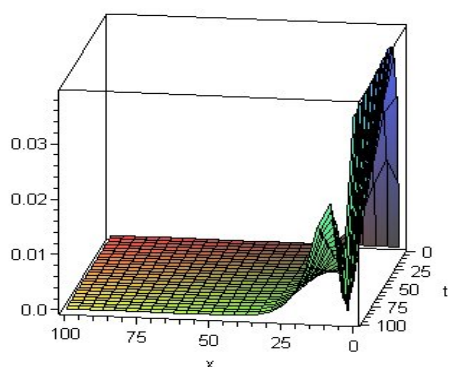

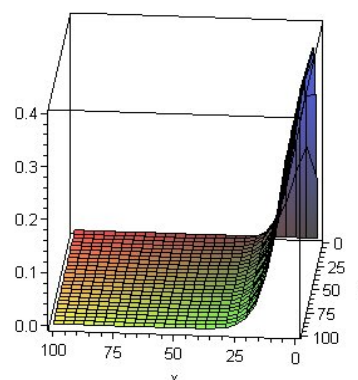Fig. 6. Absolute error for the 3rd-order approximation by coupled form of VIM for $u(x,t)$ and $h = 0.001$.

Fig. 7. Absolute error for the 3th-order approximation by coupled form of VIM for $v(x,t)$ and $h = 0.001$.
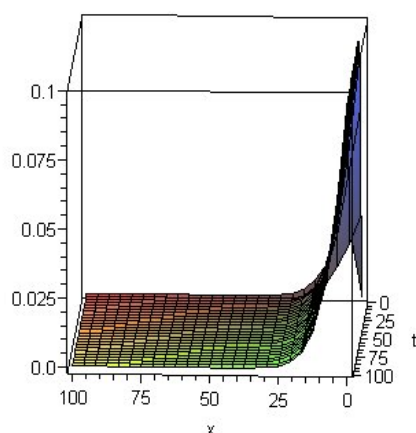
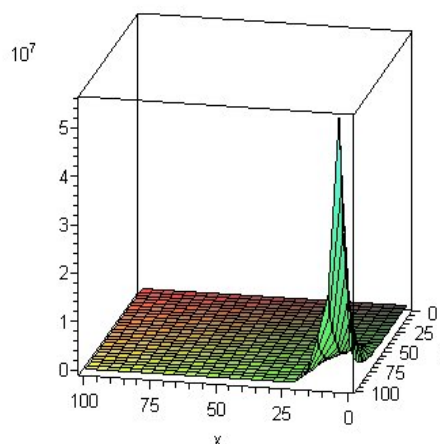Fig. 8. Absolute error for the 3rd-order approximation by coupled form of VIM for $w(x,t)$ and $h = 0.001$.

Fig. 9. Absolute error for the 3rd-order approximation by original VIM for $u(x,t)$.

4. CONCLUSION

In this paper, we coupled an unknown auxiliary parameter and Adomian's polynomials in the correction functional of He's VIM for generalized Hirota–Satsuma coupled KdV equations. Numerical results and graphical representations explicitly reveal the complete reliability of this combination. It is observed that the used coupling can be very effective in solving complicated nonlinear problems of physical nature.

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