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FREE VIBRATION ANALYSIS OF BEAMS ON VARIABLE WINKLER ELASTIC FOUNDATION BY USING THE DIFFERENTIAL TRANSFORM METHOD

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Abstract- In this study, free vibration of an Euler-Bernoulli beam resting on a variable Winkler foundation is considered. Structures which are supported along their length such as beams or pipelines resting on elastic soil are very commonly modeled with a Winkler foundation. In this problem, the elastic coefficient of the foundation is variable along the beam major axis. Constant, linear and parabolic variations are considered. The problem is handled for three different boundary conditions: simply supported-simply supported, clamped-clamped and cantilever (clamped-free) beams. The governing differential equations of the beam are solved by using Differential Transform Method (DTM). DTM is an easy transformation technique based on Taylor expansion series, providing high accuracy.
Key Words- Differential Transform Method, DTM, Elastic soil, Vibration, Beam, Pipeline

1. INTRODUCTION

During the past decades, various models have been introduced for beams resting on elastic foundations such as soil etc. Those structures supported along their main axis are represented by several approaches such as Winkler, Pasternak or Vlasov, Flonenko - Borodich foundations. The Winkler modeling, one of the most fundamental methods was suggested in 1867 by Winkler. The approach introduces a linear algebraic relationship between the normal displacement of the structure and the contact pressure [1]. The Winkler Model represents the soil medium by a set of mutually independent spring elements. Such an approach grants simplicity in obtaining closed-form solutions [2,3]. Moreover, it gives the chance of obtaining a nonlinear behavior with lower computational effort compared to other methods [4-8].

There are numerous studies on the Winkler elastic foundation modeling in literature. Zhou [9] and Eisenberger [27] studied a general solution to vibrations of beams on a variable Winkler elastic foundation. Auersch [10] carried out a study about infinite beams on half-space compared with finite and infinite beams on a Winkler support. Eisenberger and Clastornik [11] examined the vibrations and buckling of a beam on a variable Winkler elastic foundation. Gupta et al. [12] presented buckling and vibrational behavior of polar orthotropic circular plates with linearly varying thickness. Also, Ruge and Birk [13] studied the dynamic behavior of infinite beam models, giving importance on asymptotic behavior at high frequencies. Dynamic response of a Timoshenko beam with a moving concentrated mass was solved by Lee [14]. Huang and Thambiratnam [15] who worked on the deflection of plates with moving accelerated loads by using Winkler model and the finite strip method. Oz and Pakdemirli has studied on resonances of shallow beams resting on elastic foundations [26]. Also, some researchers [16-17] studied the analysis of elastic foundations with Winkler-Pasternak models. In addition to differential transform method for structures on elastic foundation, Differential Quadrature Method (DQM) and Harmonic DQ methods are also widely used, where some of the studies of this method by Civalek [28-29] could be examined.

This study covers the free vibration of an Euler-Bernoulli beam resting on a variable Winkler elastic foundation. The elastic variation through the beam is handled in three cases: Constant, linear and parabolic variations. Boundary conditions of the beam are taken to be simply supported-simply supported, clamped-clamped and clamped-free ends, respectively. In order to find the natural frequencies, DTM is applied to the governing differential equations and boundary conditions. By using this method, these equations are transformed to a set of algebraic equations whose solutions give the desired results with an excellent accuracy compared with the exact results in open literature.

2. EQUATION of MOTION and BOUNDARY CONDITIONS

The governing differential equation of motion for an Euler-Bernoulli beam is expressed as follows;

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+k(x)w(x,t)=0, \quad 0\le x\le l \tag{1}$$

where l is the length, EI is the bending rigidity, ρA is the mass per unit length, $k(x)$ is the elastic coefficient of Winkler foundation and $w(x,t)$ is the displacement.

Figure 1 represents the beam with constant cross-section laying on elastic Winkler foundation.

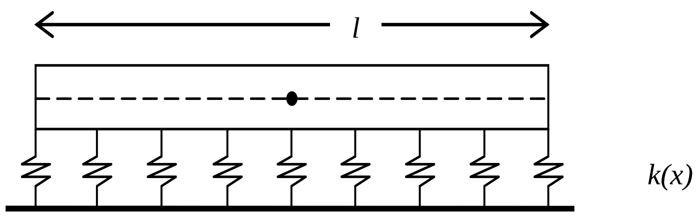

Figure 1: Beam structure resting on Winkler foundation

The relevant boundary conditions are;

Simply supported-simply supported: $w=\frac{\partial^2 w}{\partial x^2}=0$ at $x=0,l$ (2)

Clamped-clamped: $w=\frac{\partial w}{\partial x}=0$ at $x=0,l$ (3)

Cantilevered: $w=\frac{\partial w}{\partial x}=0$ at $x=0$ and $\frac{\partial^2 w}{\partial x^2}=\frac{\partial^3 w}{\partial x^3}=0$ at $x=l$ (4)

Assuming the displacement function as follows, where ω is the circular natural frequency;

$$w(x,t) = \overline{w}(x)e^{i\omega t} \tag{5}$$

and substituting this into the governing differential equation Eq. 1 takes the form:

$$EI\frac{\partial^4 \overline{w}}{\partial x^4} - \rho A\omega^2 \overline{w} + k(x)\overline{w} = 0 \tag{6}$$

The elastic foundation is represented by a set of linear springs in Winkler modeling. In this study, Winkler elastic foundation can vary linearly or parabolically or even constantly through the length of the beam. Variation is given below for constant, linear and parabolic cases, respectively:

Constant: $$k(x) = k_0 \tag{7}$$

Linear: $$k(x) = k_0(1-\alpha x),\quad 0 \le \alpha \le 1 \tag{8}$$

Parabolic: $$k(x) = k_0(1-\beta x^2),\quad 0 \le \beta \le 1 \tag{9}$$

3. NON-DIMENSIONALIZATION

Defining a non-dimensional coordinate $\xi = \dfrac{x}{l}$, the equation of motion is obtained as;

$$\frac{\partial^4 \tilde{w}}{\partial \xi^4} + (K(\xi) - \lambda^4)\tilde{w}(\xi) = 0 \tag{10}$$

Where dimensionless parameters are:

$$\xi = \frac{x}{l},\quad \tilde{w} = \frac{\overline{w}}{l},\quad K(\xi) = \frac{k(\xi)}{EI}\lambda^4,\quad \lambda^4 = \frac{\rho A\omega^2 l^4}{EI} \tag{11}$$

Additionally, dimensionless boundary conditions can be expressed as follows;

Simply supported $$\tilde{w} = \frac{\partial^2 \tilde{w}}{\partial \xi^2} = 0 \quad \text{at } \xi = 0,1 \tag{12}$$

Clamped-clamped: $$\tilde{w} = \frac{\partial \tilde{w}}{\partial \xi} = 0 \quad \text{at } \xi = 0,1 \tag{13}$$

Cantilever: $$\tilde{w} = \frac{\partial \tilde{w}}{\partial \xi} = 0 \text{ at } \xi = 0 \text{ and } \frac{\partial^2 \tilde{w}}{\partial \xi^2} = \frac{\partial^3 \tilde{w}}{\partial \xi^3} = 0 \text{ at } \xi = 1 \tag{14}$$

and variation of elastic coefficient of Winkler foundation is expressed as;

Constant: $$K(\xi) = K_0 \tag{15}$$

Linear: $$K(\xi) = K_0(1-\alpha\xi) \tag{16}$$

Parabolic: $$K(\xi) = K_0(1-\beta\xi^2) \tag{17}$$

4. DIFFERENTIAL TRANSFORM METHOD

Differential transform method is an efficient technique for solving differential equations by iteration with considerable accuracy and easiness. It was first introduced by Zhou in 1986 [18], who applied differential transformation not only to linear but also to non-linear initial value problems in electrical circuit analysis. Many scientists [19-24] have studied DTM in order to examine a variety of applications, such as the solution of non-linear systems, eigen-value, and initial value or boundary value problems.

Differential transform is a simulation method which depends on Taylor Series expansion. The prime advantages of DTM, which make the method superior to others, are its accuracy, simplicity and rapidity. In contrast to higher order Taylor series method, DTM does not need the symbolic calculation of derivatives. It uses some transformation rules to transform original functions, including boundary conditions, into a set of algebraic functions [20]. Solving the algebraic set by iteration, desired results are obtained with great accuracy.

A function *f(x)*, analytical in domain D could be represented by a power series around any $x=x_0$ point in the current domain. The differential transform of the function *f(x)* is given by

$$F(r)=\frac{1}{r!}\left(\frac{d^r f(x)}{dx^r}\right)_{x=x_0} \tag{18}$$

where *F(r)* is the differentially transformed function and r is the member of the non-negative integer domain Z^+. Then the inverse transformation is described as;

$$f(x)=\sum_{r=0}^{\infty}(x-x_0)^r F(r) \tag{19}$$

Combining Eqs. 18 and 19, the following equation is obtained:

$$f(x)=\sum_{r=0}^{\infty}\frac{(x-x_0)^r}{r!}\left(\frac{d^r f(x)}{dx^r}\right)_{x=x_0} \tag{20}$$

Considering *f(x)* by a series of finite terms, Eq. 20 is arranged as follows, with assuming the residual terms to be negligibly small. The increase of convergence is determined by the value q.

$$f(x)=\sum_{r=0}^{q}\frac{(x-x_0)^r}{r!}\left(\frac{d^r f(x)}{dx^r}\right)_{x=x_0} \tag{21}$$

Basic transformation rules depending on the DTM for differential equations and boundary conditions are tabulated in Tables 1 and 2, respectively.

Table 2: Theorems of differential transform method for equation of motion

Original Function	Transformed Function
$f(x)=g(x)\pm h(x)$	$F(k)=G(k)\pm H(k)$
$f(x)=\beta g(x)$	$F(k)=\beta G(k)$
$f(x)=g(x)h(x)$	$F(k)=\sum_{l=0}^{k}G(l)h(k-l)$
$f(x)=\frac{d^n g(x)}{dx^n}$	$F(k)=\frac{(k+n)!}{k!}G(k+n)$

$f(x)=x^n$	$F(k)=\delta(k-n)=\begin{cases}0 \text{ if } k\neq n\\ 1 \text{ if } k=n\end{cases}$

Table 3: Theorems of differential transform method for boundary conditions

x=0		***x=1***	
Original B.C.	**Transformed B.C.**	**Original B.C.**	**Transformed B.C.**
$f(0)=0$	$F(0)=0$	$f(1)=0$	$\sum_{k=0}^{\infty}F(k)=0$
$\frac{df(0)}{dx}=0$	$F(1)=0$	$\frac{df(1)}{dx}=0$	$\sum_{k=0}^{\infty}kF(k)=0$
$\frac{d^2f(0)}{dx^2}=0$	$F(2)=0$	$\frac{d^2f(1)}{dx^2}=0$	$\sum_{k=0}^{\infty}k(k-1)F(k)=0$
$\frac{d^3f(0)}{dx^3}=0$	$F(3)=0$	$\frac{d^3f(1)}{dx^3}=0$	$\sum_{k=0}^{\infty}k(k-1)(k-2)F(k)$

5. FORMULATION WITH DTM

By using the DTM rules tabulated in Tables 1 and 2, below analytical expressions are obtained.

Constant modulus:

For constant elastic coefficient of Winkler foundation, $K(\xi)=K(0)$, the differential equation takes the form:

$$\frac{d^4\tilde{w}}{d\xi^4}+(K_0-\lambda^4)\tilde{w}=0 \tag{22}$$

Applying DTM to the above equation, following recurrence relation is obtained:

$$W(r+4)=\frac{\lambda^4-K_0}{(r+4)(r+3)(r+2)(r+1)}W(r) \tag{23}$$

Linear modulus:

$$\frac{d^4\tilde{w}}{d\xi^4}+(K_0-\lambda^4)\tilde{w}-\alpha K_0\xi\tilde{w}=0 \tag{24}$$

$$W(r+4)=\frac{\lambda^4-K_0}{(r+4)(r+3)(r+2)(r+1)}(W(r)+\alpha K_0 W(r-1)), \qquad r\geq 1 \tag{25}$$

Parabolic variation:

$$\frac{d^4\tilde{w}}{d\xi^4}+(K_0-\lambda^4)\tilde{w}-\beta K_0\xi^2\tilde{w}=0 \tag{26}$$

$$W(r+4)=\frac{\lambda^4-K_0}{(r+4)(r+3)(r+2)(r+1)}W(r)+\frac{\beta W(r-2)}{(r+4)(r+3)(r+2)(r+1)}, \qquad r\geq 2 \tag{27}$$

For example, *W(0)* and *W(2)* coefficients of the beam with both ends simply supported, are obtained by using boundary conditions, and *W(1)* and *W(3)* are set to unknown constants, namely;

$$W(0)=0\,, W(1)=c_1\,, W(2)=0\,, W(3)=c_2 \tag{28}$$

Hereby, the calculation procedure is given only for parabolic variation. Using Eq. 27 and Eq. 28, *W(r)*'s are evaluated in terms of λ, K_0, c_1 and c_2.

$$W(4)=0 \tag{29}$$

$$W(5)=\frac{(\lambda_4-K_0)c_1}{240} \tag{30}$$

$$W(6)=0 \tag{31}$$

$$W(7)=\frac{(\lambda_4-K_0)c_2}{840}+\frac{\beta c_1}{840} \tag{32}$$

$$\vdots$$

Similar procedure is followed for the other two boundary conditions; clamped-clamped and cantilever beam.

In order to give an idea to the reader about the accuracy, free vibration case of an Euler-Bernoulli beam without foundation is compared to the Reference [25]. The Winkler elastic parameter is set to zero and the above DTM equations are ran. The results are given below:

Table 1: Comparison of DTM with exact solutions

Mode	DTM	[25]
1	1.875104069	1.8751014
2	4.694091133	4.694091
3	7.854757438	7.854757
4	10.99554073	10.995541

As seen from the above table, differential transform method provides the satisfying accuracy. The method also is also accurate for the free vibrational modes of the beam lying on Winkler foundation, which is the main subject of the study and where the results are given with comparison to the open literature in the Tables 2-7. Through the study, 50 iteration has been used and for the solutions.

The convergence of the first six natural frequency set is introduced at Figure 2. At least 55 terms should be evaluated for five-digit precision for the 6th natural frequency. By definition of differential transform method, as the higher terms are evaluated, the more natural frequencies are obtained.

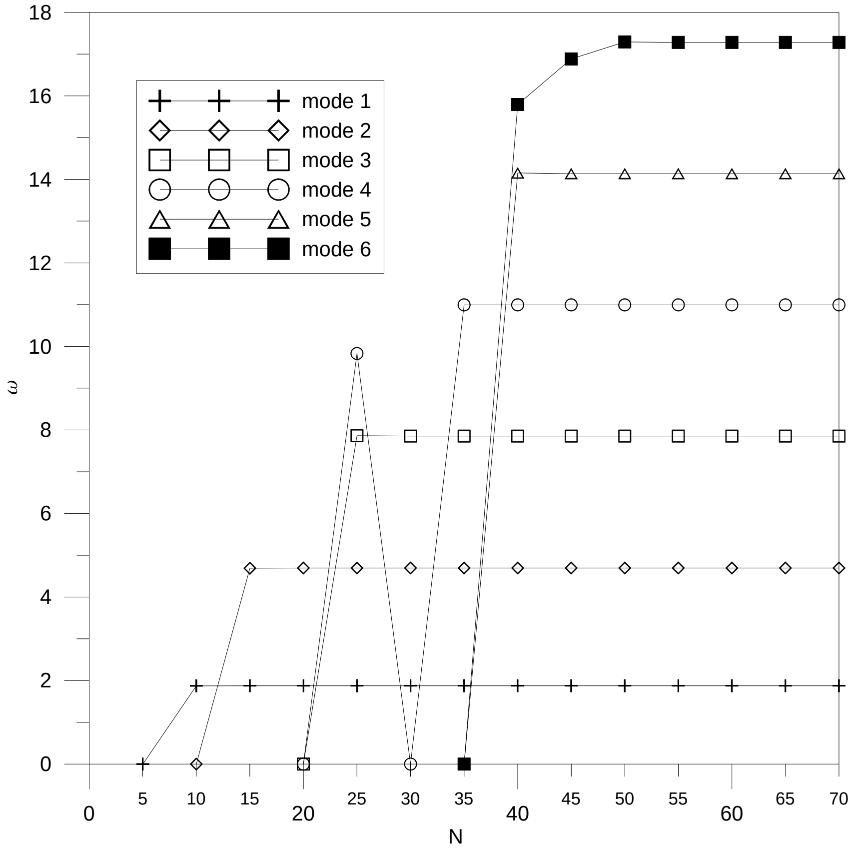

Figure 2: Convergence of natural frequencies with N

6. NUMERICAL RESULTS

Frequency parameters λi for different boundary conditions are tabulated in this section. In particular the results of the simply supported-simply supported case at different K0 values are given extensively. The present results are compared with various results in open literature.

6.1 Simply Supported-Simply Supported Beam

Frequency variation for simply supported-simply supported beam at constant elastic, linear and parabolic modulus is shown at Figure3, 4 and 5, respectively.

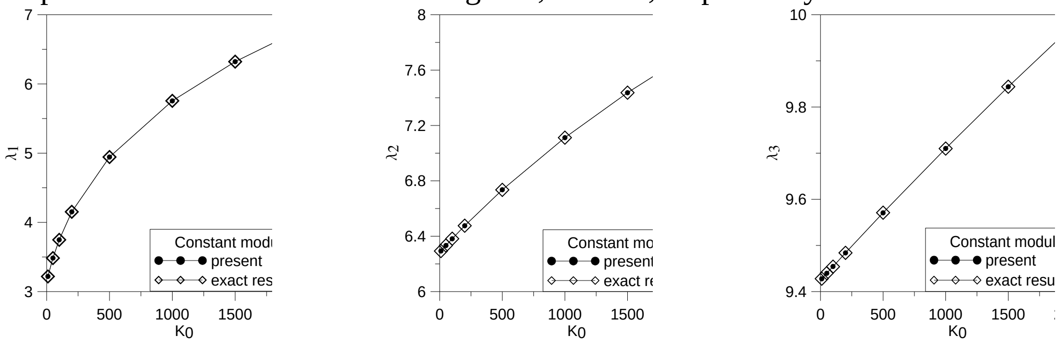

Figure 3: Frequency variation of S-S beam with constant modulus

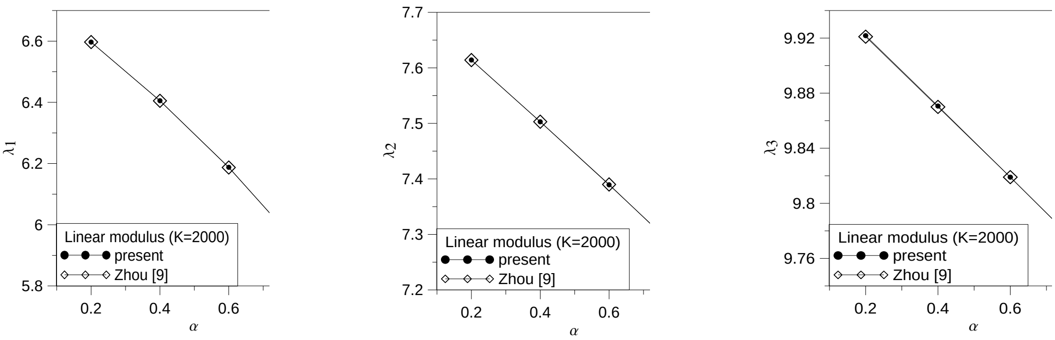

Figure 4: Frequency variation of S-S beam with linear modulus

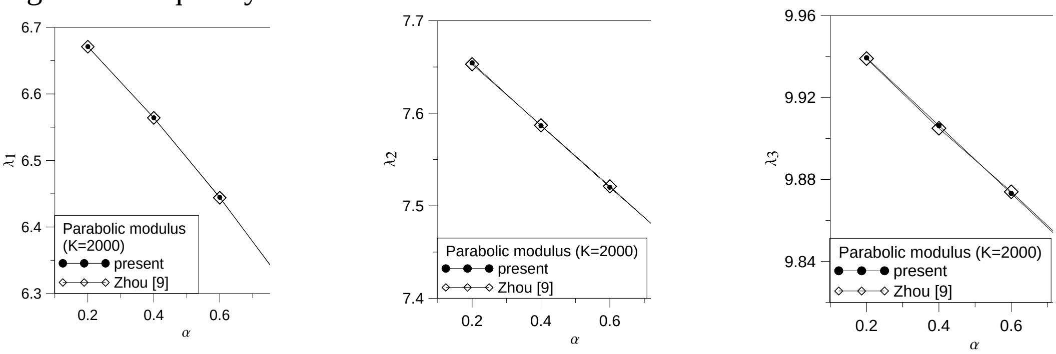

Figure 5: Frequency variation of S-S beam with parabolic modulus

Frequency parameters for above conditions are listed at Table 4, 5, and6, respectively.

Table 4: Frequency parameters for S-S beam at constant elastic modulus

K_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
10	3,219291184	6,293239752	9,427762796	12,56763025	15,70860826	18,84992919	21,99138364	25,13289871
	(3,219)*	(6,293)	(9,428)	(12,568)	(15,709)	(18,850)	(21,991)	(25,133)
50	3,484424567	6,33298318	9,439673875	12,57266501	15,71118743	18,85142205	21,99232383	25,13352858
	(3,484)	(6,333)	(9,440)	(12,573)	(15,711)	(18,851)	21,992	(25,134)
100	3,748364250	6,381633293	9,454499603	12,57894997	15,71440961	18,85328763	21,99349889	25,13431586
	(3,748)	6,382	(9,454)	(12,579)	15,714	18,853	(21,993)	(25,134)
200	4,152776516	6,475725032	9,483943557	12,59149170	15,72084802	18,85701712	21,99584846	25,13589020
	(4,153)	6,476	(9,484)	(12,591)	(15,721)	(18,857)	(21,996)	25,136
500	4,943880409	6,735814452	9,570668085	12,62889372	15,74011595	18,86819235	22,00289264	25,14061144
	(4,944)	(6,736)	(9,571)	(12,629)	(15,740)	(18,868)	(22,003)	(25,141)
1000	5,755620336	7,112107040	9,710176091	12,69050177	15,77207279	18,88677371	22,01461793	25,14847427
	(5,756)	7,112	(9,710)	12,691	(15,772)	(18,887)	(22,015)	25,149
1500	6,321993397	7,436673846	9,843917717	12,75122540	15,80383655	18,90530040	22,02632451	25,15632973
	(6,322)	7,437	(9,844)	(12,751)	15,804	(18,905)	(22,026)	(25,156)
2000	6,767383474	7,723570755	9,972420206	12,81109369	15,83540994	18,92377278	22,03801246	25,16417784
	(6,767)	(7,724)	(9,972)	(12,811)	(15,835)	(18,924)	(22,038)	(25,164)

* Exact results [9]

Table 5: Frequency parameters for S-S beam at linear elastic modulus

α=0,2 K_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
10	3,211771150	6,292236540	9,427464444	12,56750430	15,70854376	18,84989187	21,99136013	25,13288352
50	3,454480416	6,328057675	9,438187563	12,57203601	15,71086511	18,85123547	21,99220627	25,13345065
100	3,699921549	6,371998414	9,451540469	12,57769379	15,71376534	18,85291457	21,99326387	25,13415880
200	4,080996862	6,457257919	9,478078411	12,58898659	15,71956102	18,85627142	21,99537858	25,13557555
500	4,836530578	6,694670148	9,556387790	12,62268461	15,73690986	18,86633128	22,00171909	25,13982521
1000	5,618515814	7,042035261	9,682794448	12,67825778	15,76569823	18,88306219	22,01227444	25,14690373
1500	6,165920336	7,344688366	9,804455230	12,73311253	15,79433039	18,89974892	22,02281476	25,15397413
2000	6,596856775	7,614150729	9,921767296	12,78727024	15,82280832	18,91639176	22,03334004	25,16103879

Table 6: Frequency parameters for S-S beam at parabolic elastic modulus

β=0.2 K_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
10	3,215045955	6,292596434	9,427567258	12,56754708	15,70856552	18,84990441	21,99136804	25,132886950
50	3,467585253	6,329826163	9,438699854	12,57224968	15,71097386	18,85129819	21,99224569	25,133476400
100	3,721190961	6,375461338	9,452560641	12,57812054	15,71398272	18,85303997	21,99334274	25,134209970
200	4,112601888	6,463908152	9,480101377	12,58983775	15,71999529	18,85652210	21,99553618	25,135681680
500	4,883849037	6,709562392	9,561319893	12,62479524	15,73799190	18,86695697	22,00211273	25,140088566
1000	5,678785297	7,067574639	9,692271528	12,68242292	15,76785029	18,88431020	22,01306060	25,147429051
1500	6,234265490	7,378411599	9,818140700	12,73927864	15,79754069	18,90161591	22,02399227	25,154762587
2000	6,671219285	7,654476911	9,939366318	12,79538616	15,82706529	18,91887441	22,03490765	25,162090591

6.2 Clamped-Clamped Beam

Table 7 shows that the frequency parameters for clamped-clamped beam.

Table 7: Frequency parameters for C-C beam

K_0	Constant			Linear (α=0.2)			Parabolic (β=0.2)		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	4,7324	7,85372	10,9958	4,73217	7,85367	10,9958	4,73227	7,85369	10,9958
[24]*	4,730042	7,853203	10,99559	-	-	-	-	-	
10	4,75349	7,85836	10,9975	4,75116	7,85785	10,9973	4,75222	7,85805	10,9974
100	4,95039	7,90432	11,0144	4,92965	7,89925	11,0125	4,93914	7,90125	11,0132
1000	6,22391	8,32512	11,179	6,11724	8,28151	11,1611	6,16646	8,29879	11,1677

* Squared results are tabulated in the reference for K_0=1

6.3 Cantilever Beam

Frequency parameters for cantilever (C-F) beam can be seen in the Table 8.

Table 8: Frequency parameters for cantilever (C-F) beam

K_0	Constant			Linear (α=0.2)			Parabolic (β=0.2)		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	1,91192	4,69651	7,85527	1,90613	4,69622	7,85522	1,90708	4,69631	7,85524
[24]*	1,911926	4,696509	7,855272	-	-	-	-	-	-
10	2,1746	4,71808	7,85991	2,13427	4,71525	7,85936	2,14102	4,71611	7,85954
100	3,25578	4,9191	7,90584	3,13189	4,89398	7,90045	3,15299	4,90169	7,9022
1000	5,64071	6,20825	8,32642	5,39828	6,08174	8,28005	5,43684	6,12325	8,29529

* Squared results are tabulated in the reference for K_0=1

7. CONCLUSIONS

The study covers the dynamic response of an Euler-Bernoulli beam in free vibration. The beam with constant cross-sectional area is supported along its length by variable Winkler elastic foundation. Winkler modeling is frequently applied to the beams and pipelines resting on an elastic soil. Such modeling introduces the elastic foundation by a set of mutually independent spring elements. In the present work, the elastic coefficient of the spring set is variable throughout the major axis of the beam. Three cases are studied: Constant, linear and parabolic variations. Also, three boundary cases are concerned: Simple support-Simple Support, Clamped-Clamped and Clamped-Free (cantilever) beams. Although the governing differential equations and boundary conditions are determined, they are not easily solvable. Hereof, corresponding studies which aim to solve the system equations with different approaches can be found in literature. In this study, the system equations are solved by Differential Transform Method, which is a succeeding and easy transformation technique. By solving the algebraic functions set, which are the transforms of differential equations, natural frequencies are obtained. The results are tabulated and compared with the former studies and a great accuracy to exact results is obtained.

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