SOLUTION OF NONLINEAR OSCILLATORS WITH FRACTIONAL NONLINEARITIES BY USING THE MODIFIED DIFFERENTIAL TRANSFORMATION METHOD

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Abstract- In this paper, an aproximate analytical method called the differential transform method (DTM) is used as a tool to give approximate solutions of nonlinear oscillators with fractional nonlinearites. The differential transformation method is described in a nuthsell. DTM can simply be applied to linear or nonlinear problems and reduces the required computational effort. The proposed scheme is based on the differential transform method (DTM), Laplace transform and Padé approximants. The results to get the differential transformation method (DTM) are applied Padé approximants. The reliability of this method is investigated by comparison with the classical fourth-order Runge–Kutta (RK4) method and Cos-AT and Sine-AT method. Our the presented method showed results to analytical solutions of nonlinear ordinary differential equation. Some plots are gived to shows solutions of nonlinear oscillators with fractional nonlinearites for illustrating the accurately and simplicity of the methods.
Key words- Padé approximants, Modified differential transform method, Nonlinear oscillators with fractional nonlinearites.

1. INTRODUCTION

The modified differential transform method (MDTM) will be employed in a straightforward manner without any need of linearization or smallness assumptions. DTM was first applied in the engineering domain by [1,2]. DTM provides an efficient explicit and numerical solution with high accuracy, minimal calculations, avoidance of physically unrealistic assumptions. However, DTM has some drawbacks. By using DTM, we obtain a series solution, in practice a truncated series solution. This series solution does not exhibit the periodic behavior which is characteristic of oscillator equations and gives a good approximation to the true solution in a very small region. In order that improve the accuracy of DTM, we use an alternative technique which modifies the series solution for non-linear oscillatory systems as follows: we first apply the Laplace transformation to the truncated series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants[3], and finally adopt an inverse Laplace transform to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution. Ebaid [6] have developed a so-called Cosine-AT and Sine-AT method for solutions of nonlinear oscillators with fractional nonlinearites.

The aim of this paper is to extend the differential transformation method proposed by Zhou[1] to solve nonlinear oscillators with fractional nonlinearites. The results of the modified differential transformation method are numerically compared with conclusions acquired by Cosine-AT and Sine-AT method and the fourth-order Runge–Kutta method. The MDTM is benefical to obtain exact and approximate solutions of linear and nonlinear oscillations equations. No necessity to linearization or discretization, large computational work and round-off errors is avoid. It has been used to solve efficiently, easily and accurately a large class of nonlinear problems with approximations. These approximations converge rapidly to exact solutions [6–19, 24-28].

2. DIFFERENTIAL TRANSFORM METHOD

As in [6–19, 24-28]. the basic definition of the differential transformation method are gived as follows:

Differential transform of function $y(t)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]\Bigg|_{t=0}, \tag{1}$$

where $y(t)$ is the original function and $Y(k)$ is the transformed function, which is also called the T-function. In this paper, the lowercase and uppercase letters represent the original and transformed functions respectively.

The inverse differential transform of $Y(k)$ is defined as:

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k. \tag{2}$$

Combining Eqs. (1) and (2), we obtain:

$$y(t) = \sum_{k=0}^{\infty}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}\frac{t^k}{k!}. \tag{3}$$

From the definitions (1) and (3), it is easy to obtain the following mathematical operations:

Table 1 The fundamental operations of the differential transformed method

Original function	Transformed function
$y(t) = cw(t)$	$Y(k) = cW(k)$
$y(t) = \dfrac{dw(t)}{dt}$	$Y(k) = (k+1)W(k+1)$
$y(t) = \dfrac{d^i w(t)}{dt^i}$	$Y(k) = (k+1)(k+2)...(k+i)W(k+i)$
$y(t) = u(t)v(t)$	$Y(k) = \sum\limits_{r=0}^{k} U(r)V(k-r)$
$y(t) = u(t)v(t)w(t)$	$Y(k) = \sum\limits_{s=0}^{k}\sum\limits_{m=0}^{k-s} U(s)V(m)W(k-s-m)$
$y(t) = u(t)\int\limits_{0}^{t} v(t)dt$	$Y(k) = \sum\limits_{r=1}^{k} U(k-r)\dfrac{V(r-1)}{r}, k \geq 1$

2.1. Padé Approximation

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations to a function $f(x)$ are given by [3]

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b. \tag{4}$$

The method of Padé requires that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (4) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + ... + p_N x^N \tag{5}$$
$$Q_M(x) = 1 + q_1 x + q_2 x^2 + ... + q_M x^M \tag{6}$$

The polynomials in (5) and (6) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of Q_M is $q_0 = 1$. This is permissible, because it notice be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ..., \tag{7}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{M} q_i x^i - \sum_{i=0}^{N} p_i x^i = \sum_{i=N+M+1}^{\infty} c_i x^i \tag{8}$$

The lower index $j = N + M + 1$ in the summation on the right side of (8) is chosen because the first $N + M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

Equating the coefficients of the powers of x^k are set equal to zero for $k = 0, 1, 2, ..., N + M$, we obtain a system of linear equations. Maple can be used to solve these linear equations.

3. APPLICATIONS

In this section, we will apply the differential transformed method to nonlinear oscillators with fractional nonlinearites.

3.1. Example 1

Consider the nonlinear differential equation[20]

$$\frac{d^2 u}{dt^2} + \frac{u}{1+u^2} = 0, \tag{9}$$

with the initial conditions

$$u(0) = \mu, \quad u'(0) = 0. \tag{10}$$

Firstly, we suppose that

$$f(u) = \frac{u}{1+u^2}, \tag{11}$$

then we take a differentiation of $f(u)$ with respect to t to obtain

$$\frac{df}{dt} + u^2 \frac{df}{dt} + 2uf \frac{du}{dt} = \frac{du}{dt}. \tag{12}$$

Now, the application of the differential transform to Eq.(9), (10) and (12) give the following recurrence relations for $k \geq 0$:

$$U(0) = \mu, \quad U(1) = 0.$$

$$F(0) = \frac{U(0)}{1+U(0)^2} = \frac{\mu}{1+\mu^2}. \tag{13}$$

$$(k+1)(k+2)U(k+2) + F(k) = 0.$$

$$(k+1)F(k+1) = (k+1)U(k+1) - \sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}(k-k_1+1)U(k_1-k_2)\Big[U(k_2)F(k-k_1+1) + 2U(k-k_1+1)F(k_2)\Big].$$

Using these recurrence relations by taking $N = 6$, we obtain a system of algebraic equations for $k = 0, ..., 4$. By solving this equations fort pense he values $U(2), U(3), ..., U(6)$ by using MAPLE, we get

$$U(2) = \frac{-\mu}{2(1+\mu^2)}, U(3) = 0, U(4) = \frac{\mu(1-\mu^2)}{24(1+\mu^2)^3}, U(5) = 0, U(6) = \frac{-\mu(1-20\mu^2+7\mu^4)}{720(1+\mu^2)^5}. \tag{14}$$

$$u(t) = \mu - \frac{\mu}{2\left(1+\mu^2\right)}t^2 + \frac{\mu\left(1-\mu^2\right)}{24\left(1+\mu^2\right)^3}t^4 - \frac{\mu\left(1-20\mu^2+7\mu^4\right)}{720\left(1+\mu^2\right)^5}t^6. \tag{15}$$

Substituting $\mu = 0.1$ into Eq.(15) we obtain the equation,

$$u(t) = 0.1 - .0495049505t^2 + .00400368436t^4 - .0001058109133t^6. \tag{16}$$

We apply Laplace transformation to (16), which yields

$$L\left(u(s)\right) = \frac{0.3}{s} - \frac{.099009901}{s^3} + \frac{.09608842464}{s^5} - \frac{.07618385758}{s^7}. \tag{17}$$

For simplicity, let $s = \frac{1}{t}$; then

$$L\left(u(t)\right) = 0.1t - .099009901t^3 + .09608842464t^5 - .07618385758t^7. \tag{18}$$

The [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right]_{u(t)} = \frac{.09999999998t + .8773649552t^3}{.9999999998 + 9.763748562t^2 + 8.706193539t^4}. \tag{19}$$

Recalling $t = \frac{1}{s}$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right]_{u(t)} = \frac{.4999999999*10^9 s^3 + .4386824776*10^9 s}{4999999999s^4 + .4881874281*10^{11}s^2 + .435309677*10^{11}}. \tag{20}$$

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified solution

$$u(t) = -.00003207215013\cos(2.961613547t) + .1000320721\cos(.9962899989t). \tag{21}$$

[6] the cosine-AT approximate periodic solution obtain in following:

$$u_{\text{approx}}(t) = -.000032\cos(2.96161t) + .100032\cos(.99629t). \tag{22}$$

Table 2. Approximate periodic solution using different Padé approximants at $\mu = 0.1$

$[3/3]$	$-.002020202085 + .1020202021\cos(.9851360760t)$
$[3/4]$	$.1\cos(.9950371903t)$
$[4/4]$	$-.3207215013*10^{-4}\cos(2.961613547t) + .1000320721\cos(.9962899989t)$
$[4/5]$	$.1\cos(.9950371903t)$
$[5/4]$	$-.001000334325\cos(1.812894486t) + .09789910457\cos(1.022217551t) + .0031012\mathrm{2}$
$[5/5]$	$-.001000334265\cos(1.812894486t) + .09789910456\cos(1.022217551t)$ $+ .003101229705$
$[5/6]$	$.1\cos(.9950371903t)$

3.2. Example 2

Consider the Duffing-harmonic oscillator[21]

$$\frac{d^2 u}{dt^2} + \frac{u^3}{1+u^2} = 0,$$ (23)

with the initial conditions

$$u(0) = 0, \quad u'(0) = \xi.$$ (24)

Firstly, we suppose that

$$f(u) = \frac{u^3}{1+u^2},$$ (25)

then we take a differentiation of $f(u)$ with respect to t to obtain

$$\frac{df}{dt} + u^2 \frac{df}{dt} + 2uf \frac{du}{dt} = 3u^2 \frac{du}{dt}.$$ (26)

Now, the application of the differential transform to Eq.(23), (24) and (26) give the following recurrence relations for $k \geq 0$:

$$U(0) = 0, \quad U(1) = \xi.$$

$$F(0) = \frac{U(0)^3}{1+U(0)^2} = 0, \; F(1) = \frac{U(1)^3}{1+U(1)^2} = \frac{\xi^3}{1+\xi^2}.$$ (27)

$$(k+1)(k+2)U(k+2) + F(k) = 0.$$

$$(k+1)F(k+1) = (k+1)U(k+1) - \sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}(k-k_1+1)U(k_1-k_2)[3U(k_2)U(k-k_1+1) - F(k-k_1+1)U(k_2) - 2U(k-k_1+1)F(k_2)].$$

Using these recurrence relations by taking $N = 11$, we obtain a system of algebraic equations for $k = 0,...,9$. By solving this equations fort pense he values $U(2), U(3),..., U(11)$ by using MAPLE, we get

$$U(3) = 0, U(4) = 0, U(5) = \frac{-\xi^3}{20}, U(6) = 0, U(7) = \frac{\xi^5}{42},$$

$$U(8) = 0, U(9) = \frac{\xi^5(3-20\xi^2)}{1440}, U(10) = 0, U(11) = -\frac{\xi^7(9-28\xi^2)}{3080}.$$ (28)

$$u(t) = \xi t - \frac{\xi^3}{20}t^5 + \frac{\xi^5}{42}t^7 + \frac{\xi^5(3-20\xi^2)}{1440}t^9 - \frac{\xi^7(9-28\xi^2)}{3080}t^{11}.$$ (29)

Substituting $\xi = 0.3$ into Eq.(29) we obtain the equation,

$$u(t) = 0.3t - .00135t^5 + .00005785714286t^7 + .000002025t^9 - .4601220779e\text{-}6t^{11}.$$ (30)

We apply Laplace transformation to (32), which yields

$$L(u(s)) = \frac{0.3}{s^2} - \frac{.162}{s^6} + \frac{.2916}{s^8} + \frac{.734832}{s^{10}} - \frac{18.36660096}{s^{12}}.$$ (31)

For simplicity, let $s = \frac{1}{t}$; then

$$L(u(t)) = 0.3t^2 - .162t^6 + .2916t^8 + .734832t^{10} - 18.36660096t^{12}.$$ (32)

The [6/6] Padé approximant gives

$$\left[\frac{6}{6}\right]_{u(t)} = \frac{.3t^2 + 4.321611941t^4 + 8.977701492t^6}{.9999999999 + 14.40537314t^2 + 30.46567165t^4 + 6.806901496t^6}.$$ (33)

Recalling $t = \dfrac{1}{s}$, we obtain [6/6] in terms of s

$$\left[\frac{6}{6}\right]_{u(t)} = \frac{300000000.s^4 + 4321611941s^2 + 8977701492}{9999999999.s^6 + .1440537314*10^{12}s^4 + .3046567165*10^{12}s^2 + .6806901496*10^{11}}. \quad (34)$$

By using the inverse Laplace transform to the [6/6] Padé approximant, we obtain the modified solution

$$u(t) = .00002515875810\sin(3.448412631t) - .02550319495\sin(1.50353282t) + .6722110609\sin(.5032022717t). \quad (35)$$

[6] the sine-AT approximate periodic solution obtain in following:

$$u_{\text{approx}}(t) = .000025\sin(3.44841t) - .0255032\sin(1.50353t) + .672211\sin(.503202t). \quad (36)$$

Table 3. Approximate periodic solution using different Padé approximants at $\xi = 0.3$

$[3/3]$	$-.09216099531\sin(1.191476077t) + .6644594427\sin(.6167534011t)$
$[3/4]$	$2474616\cosh(.6061546512t)\sin(.6061546512t)$ $+.2474616\sinh(.6061546512t)\cos(.6061546512t)$
$[4/4]$	$-.09216099531\sin(1.191476077t) + .6644594427\sin(.6167534011t)$
$[4/5]$	$-.09216099531\sin(1.191476077t) + .6644594427\sin(.6167534011t)$
$[5/5]$	$-.09216099531\sin(1.191476077t) + .6644594427\sin(.6167534011t)$
$[6/6]$	$.2515875810*10^{-4}\sin(3.448412631t) - .02550319495\sin(1.50353282t)$ $+.6722110609\sin(.5032022717t)$
$[7/7]$	$.2515875810*10^{-4}\sin(3.448412631t) - .02550319495\sin(1.50353282t)$ $+.6722110609\sin(.5032022717t)$

3.3. Example 3

Consider the relativistic harmonic oscillator [22]

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1+u^2}} = 0, \quad (37)$$

with the initial conditions

$$u(0) = \eta, \quad u'(0) = 0. \quad (38)$$

Firstly, we suppose that

$$f(u) = \frac{u}{\sqrt{1+u^2}}, \quad (39)$$

then we take a differentiation of $f(u)$ with respect to t to obtain

$$u^3 \frac{df}{dt} = f^3 \frac{du}{dt}. \tag{40}$$

Now, the application of the differential transform to Eq.(37), (38) and (40) give the following recurrence relations for $k \geq 0$:

$$U(0) = \eta, \quad U(1) = 0.$$

$$F(0) = \frac{U(0)}{1 + U(0)^2} = \frac{\eta}{\sqrt{1 + \eta^2}}. \tag{41}$$

$$(k+1)(k+2)U(k+2) + F(k) = 0.$$

$$\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}(k-k_1+1)\left[U(k_1-k_2)U(k_2-k_3)U(k_3)F(k-k_1+1) - F(k_1-k_2)F(k_2-k_3)F(k_3)U(k-k_1+1)\right] = 0.$$

Using these recurrence relations by taking $N = 6$, we obtain a system of algebraic equations for $k = 0,...,4$. By solving this equations fort pense he values of $U(2), U(3),...,U(6)$ by using MAPLE, we get

$$U(2) = \frac{-\eta}{2\sqrt{1+\eta^2}}, \quad U(3) = 0, \quad U(4) = \frac{\eta}{24(1+\eta^2)^2}, \quad U(5) = 0, \quad U(6) = \frac{-\eta(1-9\eta^2)}{720(1+\eta^2)^{7/2}}.$$

$$u(t) = \eta - \frac{\eta}{2\sqrt{1+\eta^2}}t^2 + \frac{\eta}{24(1+\eta^2)^2}t^4 - \frac{\eta(1-9\eta^2)}{720(1+\eta^2)^{7/2}}t^6. \tag{42}$$

Substituting $\eta = 0.5$ into Eq.(42) we obtain the equation,

$$u(t) = 0.5 - .2236067977t^2 + .01333333333t^4 + .000397523196t^6. \tag{43}$$

We apply Laplace transformation to (43), which yields

$$L(u(s)) = \frac{0.5}{s} - \frac{.4472135954}{s^3} + \frac{.3199999999}{s^5} + \frac{.2862167011}{s^7}. \tag{44}$$

For simplicity, let $s = \frac{1}{t}$; then

$$L(u(t)) = 0.5t - .4472135954t^3 + .3199999999t^5 + .2862167011t^7. \tag{45}$$

The [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right]_{u(t)} = \frac{.4999999998t + 3.130495169t^3}{.9999999998 + 7.155417528t^2 + 5.759999999t^4}. \tag{46}$$

Recalling $t = \frac{1}{s}$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right]_{u(t)} = \frac{2499999999.s^3 + .1565247584e11s}{4999999999.s^4 + .3577708764e11s^2 + .2880000000e11}. \tag{47}$$

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified solution

$$u(t) = -.002824948426\cos(2.496198830t) + .5028249481\cos(.9614618722t). \quad (48)$$

[6] the cosine-AT approximate periodic solution obtain in following:

$$u_{approx}(t) = -.00282495\cos(2.4962t) + .502825\cos(.961462t). \quad (49)$$

Table 4

Approximate periodic solution using different Padé approximants at $\eta = 0.5$

$[3/3]$	$-.1249999996 + .6249999996\cos(.8458970109t)$
$[3/4]$	$-.002824948485\cos(2.49619883t) + .5028249485\cos(.961461872t)$
$[4/4]$	$-.002824948426\cos(2.49619883t) + .5028249481\cos(.9614618722t)$
$[4/5]$	$-.002824948485\cos(2.49619883t) + .5028249485\cos(.961461872t)$
$[5/5]$	$e^{-.6304944061t}\left[-.6999999983\cos(.4458268703t) + .1502601915\sin(.4458268703t)\right]$ $+e^{.6304944061t}\left[-.6999999983\cos(.4458268703t) - .1502601915\sin(.4458268703t)\right]$ $+1.899999996$

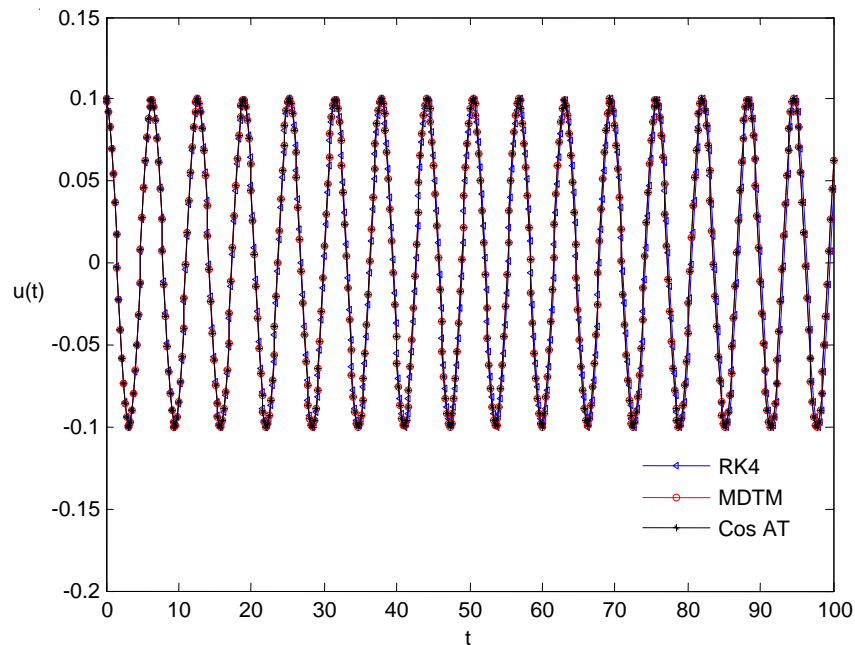The following comparison of the results obtained from three methods are given.

Figure 1. The comparison of the results of the three methods for Eq.(9) at $\mu = 0.1$.
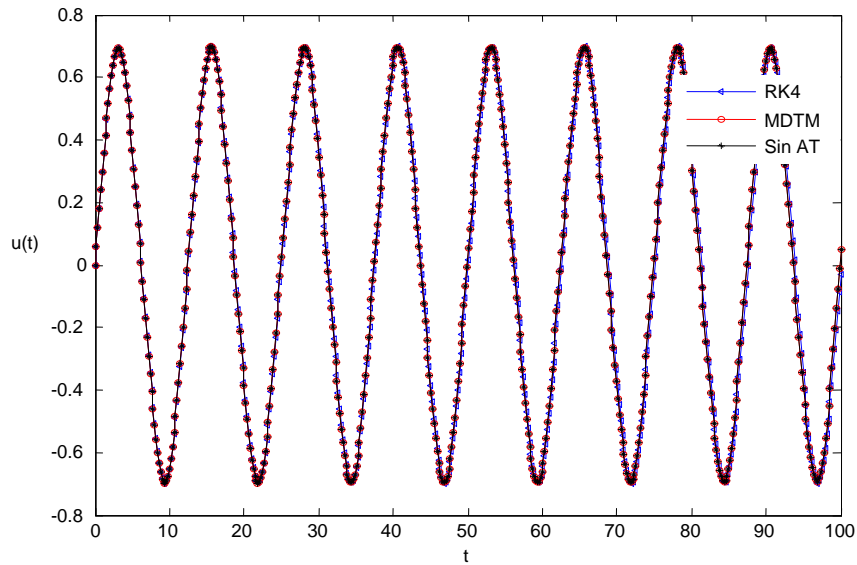
Figure 2. The comparison of the results of the three methods for Eq.(23) at $\xi = 0.3$.
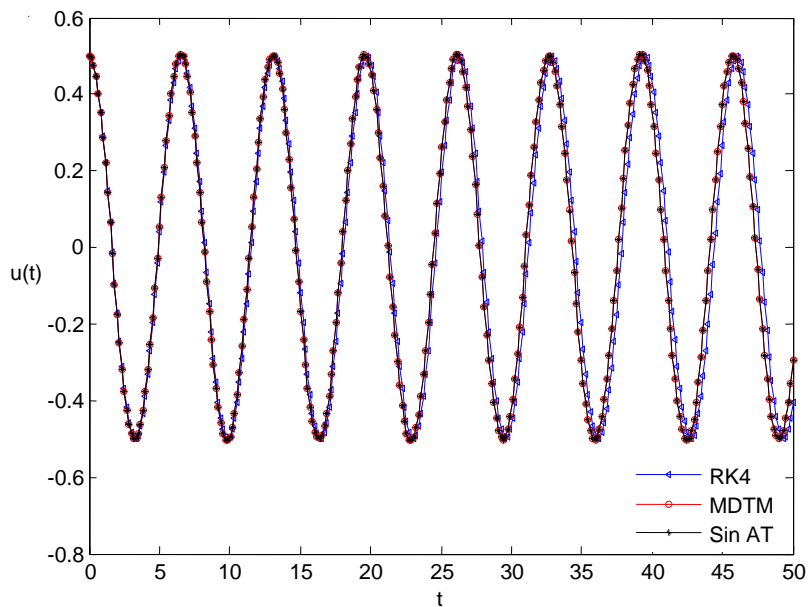

Figure 3. The comparison of the results of the two methods for Eq.(37) at $\eta = 0.5$.

Comparison of the modified approximate solutions for Eq.(9), Eq.(23) and Eq.(37) and the solutions obtained by the fourth-order Runge–Kutta method in Fig. 1, Fig. 2 and Fig. 3 show that the modified DTM greatly improves the differential transform truncated series in the convergence rate and the accuracy.

5. CONCLUSIONS

In this article, the application of differential transform method was extended to obtain approximate analytical and numerical solutions of nonlinear oscillators with fractional nonlinearites. The differential transform method generates the Taylor series of the exact solution. For the oscillatory systems, Laplace transformation of the differential transform series solution has some specific properties, so we applied Laplace transformation and Pade´ approximant to obtain an analytic solution and to develop the accuracy of differential transform method. The modified DTM is an efficient method for calculating periodic solutions of nonlinear oscillators with fractional nonlinearites. It is seen from the results of the modified DTM, Sine-AT and Cosine-AT techniques and the results of the fourth-order Runge–Kutta(RK4) solution that rate of convergence and accuracy of the modified DTM is very good.

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