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EXPRESSION OF DUAL EULER PARAMETERS USING THE DUAL RODRIGUES PARAMETERS AND THEIR APPLICATION TO THE SCREW TRANSFORMATION

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Abstract- Dual numbers and dual vectors are widely used in spatial kinematics [3,5-15,18]. Plücker line coordinates of a straight line can be represented by a dual unit vector located at the dual unit sphere (DUS). By this way, the trajectory of the screw axis of a rigid body in R^3 (the real three space) corresponds to a dual curve on the DUS. This correspondence is done through Study Mapping [8,9]. Conversely a dual curve on DUS obtained from the rotations of the DUS represents a rigid body motion in R^3 [8]. The dual Euler parameters are used in defining the screw transformation in R^3 [8], but originally in this paper these parameters are constructed from the Rodrigues and the dual Rodrigues parameters [15].

Key Words- Kinematics, Study Mapping, Dual Euler Parameters, Screw Transformation.

1. INTRODUCTION

The dual representation of a line is simply the Plücker vector written as a dual unit vector [9]. For any operation defined on a real vector space, there is a dual version of it with similar interpretation [5].

Olinde Rodrigues, the French mathematician, wrote a paper on rigid body kinematics in 1840. This paper is well known for its contributions to spherical kinematics [17]. Rodrigues revealed that every translation can be represented in an infinite number of ways by composition of two rotations of equal but opposite angle about parallel axes [16]. Similarly Euler showed that every displacement can be described by a rotation followed by a translation.

There is a detailed survey ranging from Chasles motion to the Rodrigues parametrization and also from the theoretical developments of the rigid body displacements to the finite twist in Dai [20].

Regarding the historical developments of the rigid body displacement, the studies in this field are associated with the finite twist in the 1990s. The finite twist representation and transformation and its ordered combination for several manipulators which is based on the Lie group operation are investigated by Dai, Holland and Kerr in 1995[19].

In our paper, the dual Euler parameters are used for defining the transformation of screws in R^3. The dual Euler parameters are constructed from the Rodrigues and the dual Rodrigues parameters (see [15]) which are obtained from the rotations of the DUS. When a dual vector $\vec{\hat{x}}$ is rotated to the dual vector $\vec{\hat{x}}'$ in DUS, this movement corresponds to a screw transformation in R^3. This transformation can be given by the dual Euler parameters. In other words, this paper discusses the usage of dual Euler parameters for the transformations of screws in R^3 and these parameters are defined in terms of the Rodrigues and the dual Rodrigues parameters.

Quaternions and dual numbers were combined and generalized to form what is referred as " Clifford Algebra" as first discussed by Clifford in 1882. Application to kinematic analysis is discussed by [1],[13]. A comprehensive introduction to dual Quaternions can be found in [8].

Assembling the Euler parameters c_0, c_1, c_2, c_3 of a rotation into the quaternion $Z = c_0 + c_1 i + c_2 j + c_3 k$ ($i^2 = j^2 = k^2 = -1$, $jk = -kj = i$, $ki = -ik = j$, $ij = -ji = k$), rotations in real space can be identified. If a vector $\vec{x} = (x_1, x_2, x_3) \in R^3$ is defined as the vector quaternion $\vec{x} = x_1 i + x_2 j + x_3 k$, then the rotation from $\vec{x}$ to $\vec{x}'$ is given by the quaternion equation $\vec{x}' = Z\vec{x}\overline{Z}$, where the conjugate is defined as $\overline{Z} = c_0 - c_1 i - c_2 j - c_3 k$ [8]. If the dual quaternion $\hat{Z} = \hat{c}_0 + \hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k$ is given by the dual Euler parameters $\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3$ and the corresponding spatial displacement is given by the dual vector $\vec{\hat{w}} = \vec{w} + \varepsilon\vec{v}$ (here $\vec{w}$ and $\vec{v}$ define the angular and linear velocities of the spatial displacement respectively) then the transformed screw $\vec{\hat{w}}'$ is obtained by $\vec{\hat{w}}' = \hat{Z}\vec{\hat{w}}\hat{\overline{Z}}$ [8]. Since the transformed screw has the coordinates produced from the dual Rodrigues parameters, it has informative coordinates about the rotations of the DUS.

In section 1, we introduce the dual numbers and the Study mapping. The theoretical background of the dual Euler parameters is developed in section 2 and the application of dual Euler parameters on the screw transformation is discussed by an example in section 3.

1.1. Dual numbers

A dual number is a formal sum $\hat{a} = a + \varepsilon a^*$, where a and a^* are real numbers. Similar to the complex unit $i^2 = -1$, we have here $\varepsilon^2 = 0$. Addition and multiplication are given by

$$(a_1 + \varepsilon a_1^*) + (b_1 + \varepsilon b_1^*) = (a_1 + b_1) + \varepsilon(a_1^* + b_1^*)$$
$$(a_1 + \varepsilon a_1^*).(b_1 + \varepsilon b_1^*) = (a_1 b_1) + \varepsilon(a_1 b_1^* + a_1^* b_1)$$

For a given real analytic function f we can extend its definition to dual numbers by letting

$$f(x + \varepsilon x^*) = \sum_{k=0}^{\infty} a_k (x + \varepsilon x^* - x_0)^k$$

$$=\sum_{k=0}^{\infty} a_k (x-x_0)^k + \varepsilon \sum_{k=0}^{\infty} k a_k (x-x_0)^{k-1} x^* \;=\; f(x)+\varepsilon x^* f'(x)$$

For instance,

$$\mathrm{Sin}\,\hat{x} = \mathrm{Sin}(x+\varepsilon x^*) = \mathrm{Sin}x + \varepsilon x^* \mathrm{Cos}x$$

$$Cos\hat{x} = Cos(x+\varepsilon x^*) = Cosx - \varepsilon x^* Sinx$$
$$e^{\hat{x}} = e^x + \varepsilon x^* e^x$$

1.2. Dual Vectors

A dual vector $\vec{\hat{v}}$ in three dimensional dual space D^3 is defined by $\vec{\hat{v}} = \vec{v} + \varepsilon \vec{v}^*$, where $\vec{v}, \vec{v}^* \in R^3$.

The norm of $\vec{\hat{v}}$, denoted by $\left\|\vec{\hat{v}}\right\| : D^3 \to D$ is;

$$\left\|\vec{\hat{v}}\right\| = (\vec{\hat{v}}\cdot\vec{\hat{v}})^{\frac{1}{2}} = (\vec{v}\cdot\vec{v} + 2\varepsilon\vec{v}\cdot\vec{v}^*)^{\frac{1}{2}} = \left\|\vec{v}\right\|[(1+\varepsilon\frac{\vec{v}\cdot\vec{v}^*}{\left\|\vec{v}\right\|^2})^2]^{\frac{1}{2}} = \left\|\vec{v}\right\| + \varepsilon\frac{\vec{v}\cdot\vec{v}^*}{\left\|\vec{v}\right\|} = (\left\|\vec{v}\right\|, \frac{\vec{v}\cdot\vec{v}^*}{\left\|\vec{v}\right\|})$$

The dual vector with the norm $1=(1,0)$ is called a dual unit vector. Therefore a dual unit vector $\vec{\hat{v}}$ is the vector with $\left\|\vec{v}\right\| = 1$ and $\vec{v}\cdot\vec{v}^* = 0$. The set of dual unit vectors defines the dual unit sphere (DUS), which is also called the Study Sphere (For detailed algebraic properties of dual numbers see also [18]).

1.3. Study Mapping

A point $p \in l$ (p can be written as a vector, $\vec{p}$, from origin to l) and a unit direction vector $\vec{g}$ of l determine the equation of the straight line l in R^3. A unit force with respect to the origin acting to l gives the moment vector $\vec{g}^* = \vec{p}\times\vec{g}$. The norm of the moment vector is the smallest distance from line to the origin [9].

The compenents of $(\vec{g},\vec{g}^*) = (g_1, g_2, g_3, g_1^*, g_2^*, g_3^*) \in R^6$ are called the Plücker coordinates of l. Since $\vec{g}\cdot\vec{g} = 1$ and $\vec{g}\cdot\vec{g}^* = 0$, the dual vector $\vec{\hat{g}} = \vec{g} + \varepsilon\vec{g}^*$ defines a point on DUS. The mapping which assigns to an oriented line of Euclidean space the dual vetor $\vec{\hat{g}} = \vec{g} + \varepsilon\vec{g}^*$ is called the Study mapping.

1.4. The Cayley Formula

Performing the Cayley formula [8] for the dual spherical motion with the dual rotation matrix $\hat{A}$ (it is clear that $\hat{A}$ is orthogonal), we obtain the skew symmetric dual matrix $\hat{B}$ and the dual Rodrigues vector $\vec{\hat{b}} = \vec{b} + \varepsilon\vec{b}^*$ (see section 3). In these computations, similar to the real case (that is $\left\|\vec{b}\right\| = \tan\frac{\phi}{2}$) [8], the norm of dual

Rodrigues vector is the tangent of the half of the rotation angle $\hat{\phi}$, that is $\|\vec{\hat{b}}\| = \tan\frac{\hat{\phi}}{2}$. Using the algebra of dual numbers one can simply obtain

$$\|\vec{\hat{b}}\| = \|\vec{b}\| + \varepsilon\frac{\vec{b}\vec{b}^*}{\|\vec{b}\|} = \tan\frac{\hat{\phi}}{2} = \tan\frac{\phi}{2} + \varepsilon\frac{\phi^*}{2}(1+\tan^2\frac{\phi}{2}) \text{ (See[15])} \tag{1}$$

2. THE DUAL EULER PARAMETERS AND THE SCREW TRANSFORMATION

The dual Rodrigues vector $\vec{\hat{b}}$ is the axis of rotation of DUS. Let us define the dual unit vector $\vec{\hat{s}}$ by $\vec{\hat{s}} = (\hat{s}_1, \hat{s}_2, \hat{s}_3) = (s_1 + \varepsilon s_1^*, s_2 + \varepsilon s_2^*, s_3 + \varepsilon s_3^*) = \frac{\vec{\hat{b}}}{\|\vec{\hat{b}}\|}$. Using the dual rotation angle $\hat{\phi}$ and the dual unit vector $\vec{\hat{s}}$ we get the dual parameters

$$\hat{c}_0 = \cos\frac{\hat{\phi}}{2},\ \hat{c}_1 = \sin\frac{\hat{\phi}}{2}\hat{s}_1,\ \hat{c}_2 = \sin\frac{\hat{\phi}}{2}\hat{s}_2,\ \hat{c}_3 = \sin\frac{\hat{\phi}}{2}\hat{s}_3, \tag{2}$$

which are known as the dual Euler parameters [8].

Reviewing the method of transformation of vectors given in real space, the similar method for the dual case can be proposed. As it is discussed, the rotation from $\vec{x}$ to $\vec{x}'$ in R^3 is given by the quaternion equation $\vec{x}' = Z\vec{x}\overline{Z}$. A spatial displacement can be identified by a coordinate transformation $[T]$ in terms of a rotation matrix $[A]$ and a distance d, $[T]=[A,d]$. This coordinate transformation can be represented by a dual quaternion

$$\hat{Z} = \cos\frac{\hat{\phi}}{2} + \hat{s}_1\sin\frac{\hat{\phi}}{2}i + \hat{s}_2\sin\frac{\hat{\phi}}{2}j + \hat{s}_3\sin\frac{\hat{\phi}}{2}k\,.$$

The dual quaternion $\hat{Z}$ is the sum of real Z and Z^* components. Z is the quaternion obtained from the rotation matrix $[A]$. $Z = c_0 + c_1 i + c_2 j + c_3 k$, where $c_0 = \cos\frac{\phi}{2}$, $c_1 = \sin\frac{\phi}{2}s_1$, $c_2 = \sin\frac{\phi}{2}s_2$, $c_3 = \sin\frac{\phi}{2}s_3$ are the Euler parameters of $[A]$. Z^* is the quaternion produced from $Z^* = \frac{1}{2}DZ$, where D is the quaternion, $D = d_1 i + d_2 j + d_3 k$, relates to translation vector $\vec{d} = (d_1, d_2, d_3)$. The compenents of $\hat{Z}$ are known as the dual Euler parameters of the spatial displacement. Using the dual Euler parameters, the dual orthogonal (dual rotation) matrix $[A]$ is generated by

$[\hat{A}] = I + 2\sin\frac{\hat{\phi}}{2}\cos\frac{\hat{\phi}}{2}[\hat{S}] + 2\sin^2\frac{\hat{\phi}}{2}[\hat{S}^2]$, (see [8]). On the other hand from a given dual rotation matrix $[\hat{A}]$, the dual Euler parameters hence the dual quaternion $\hat{Z}$ can be obtained.

Originally in this paper the dual Euler parameters and the dual quaternion $\hat{Z}$ results from the Rodrigues $b_1 = s_1 Tan\frac{\phi}{2}$, $b_2 = s_2 Tan\frac{\phi}{2}$, $b_3 = s_3 Tan\frac{\phi}{2}$ and the dual Rodrigues parameters $b_1^* = s_1^* Tan\frac{\phi}{2} + s_1\frac{\phi^*}{2}(1+Tan^2\frac{\phi}{2})$, $b_2^* = s_2^* Tan\frac{\phi}{2} + s_2\frac{\phi^*}{2}(1+Tan^2\frac{\phi}{2})$, $b_3^* = s_3^* Tan\frac{\phi}{2} + s_3\frac{\phi^*}{2}(1+Tan^2\frac{\phi}{2})$ [see 15].

If a screw $(\vec{w},\vec{v})$ (where $\vec{w}=(w_1,w_2,w_3)$ is the angular velocity and $\vec{v}=(v_1,v_2,v_3)$ is the translation velocity) is defined by the dual quaternion $\hat{\vec{w}}=(w_1+\varepsilon v_1)i+(w_2+\varepsilon v_2)j+(w_3+\varepsilon v_3)k$ then the final screw (the transformed screw), $\hat{\vec{w}}'=(w_1'+\varepsilon v_1')i+(w_2'+\varepsilon v_2')j+(w_3'+\varepsilon v_3')k$ is obtained by

$$\hat{\vec{w}}' = \hat{Z}\hat{\vec{w}}\bar{\hat{Z}}$$

That is

$$\hat{\vec{w}}' = (\hat{c}_0+\hat{c}_1 i+\hat{c}_2 j+\hat{c}_3 k)\hat{\vec{w}}(\hat{c}_0-\hat{c}_1 i-\hat{c}_2 j-\hat{c}_3 k) \qquad (3)$$

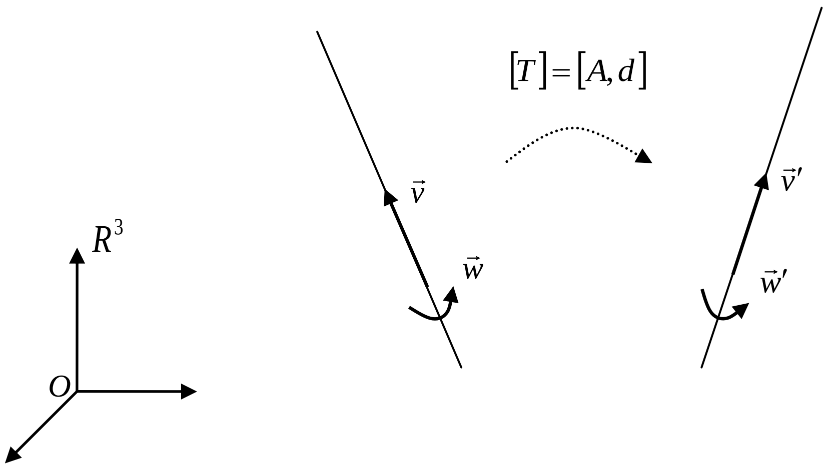

Figure 1. Screw transformation

Let us expand the dual Euler parameters $\hat{c}_i$ given by (2) as follows;

$$\hat{c}_i = c_i + \varepsilon c_i^* = \sin\frac{\hat{\phi}}{2}\hat{s}_i = \sin\frac{\hat{\phi}}{2}\frac{\hat{b}_i}{\|\hat{b}\|}$$

$$= (\sin\frac{\phi}{2} + \varepsilon\frac{\phi^*}{2}\cos\frac{\phi}{2})(\frac{b_i}{\|\vec{b}\|} + \varepsilon(\frac{b_i}{\|\vec{b}\|} - \frac{b_i(\vec{b}\vec{b}^*)}{\|\vec{b}\|^3}))$$

$$= \frac{b_i}{\|\vec{b}\|}\sin\frac{\phi}{2} + \varepsilon(\frac{\phi^*}{2}\cos\frac{\phi}{2}\frac{b_i}{\|\vec{b}\|} + \sin\frac{\phi}{2}(\frac{b_i^*}{\|\vec{b}\|} - \frac{b_i(\vec{b}\vec{b}^*)}{\|\vec{b}\|}))$$

since $\|\vec{b}\| = \tan\frac{\phi}{2}$,

$$\hat{c}_i = b_i \cos\frac{\phi}{2} + \varepsilon \cos\frac{\phi}{2}(b_i \cot\frac{\phi}{2}(\frac{\phi^*}{2} - b_i(\vec{b}\vec{b}^*)\cot\frac{\phi}{2}) + b_i^*), \quad i = 1,2,3. \tag{4}$$

From (1) we get

$$\vec{b}\vec{b}^* = \frac{\phi^*}{2}\tan\frac{\phi}{2}(1 + \tan^2\frac{\phi}{2}) \tag{5}$$

Substituting the equation (5) into (4) yields

$$\hat{c}_i = b_i \cos\frac{\phi}{2} + \varepsilon(b^* \cos\frac{\phi}{2} - b_i \frac{\phi^*}{2}\sin\frac{\phi}{2}), \quad i = 1,2,3. \tag{6}$$

On the other hand, the expansion of (3) gives,

$$\begin{aligned}\vec{\hat{w}}' &= \hat{w}_1' i + \hat{w}_2' j + \hat{w}_3' k = (w_1' + \varepsilon v_1')i + (w_2' + \varepsilon v_2')j + (w_3' + \varepsilon v_3')k \\ &= \{(\hat{c}_0^2 + \hat{c}_1^2 - \hat{c}_2^2 - \hat{c}_3^2)\hat{w}_1 + (2\hat{c}_1\hat{c}_2 - 2\hat{c}_0\hat{c}_3)\hat{w}_2 + (2\hat{c}_1\hat{c}_3 + 2\hat{c}_0\hat{c}_2)\hat{w}_3\}i \\ &\quad + \{((2\hat{c}_1\hat{c}_2 + 2\hat{c}_0\hat{c}_3)\hat{w}_1 + (\hat{c}_0^2 - \hat{c}_1^2 + \hat{c}_2^2 - \hat{c}_3^2)\hat{w}_2 + (2\hat{c}_2\hat{c}_3 - 2\hat{c}_0\hat{c}_1)\hat{w}_3\}j \\ &\quad + \{(2\hat{c}_1\hat{c}_3 - 2\hat{c}_0\hat{c}_2)\hat{w}_1 + (2\hat{c}_2\hat{c}_3 + 2\hat{c}_0\hat{c}_1)\hat{w}_2 + (\hat{c}_0^2 - \hat{c}_1^2 - \hat{c}_2^2 + \hat{c}_3^2)\hat{w}_3\}k\end{aligned} \tag{7}$$

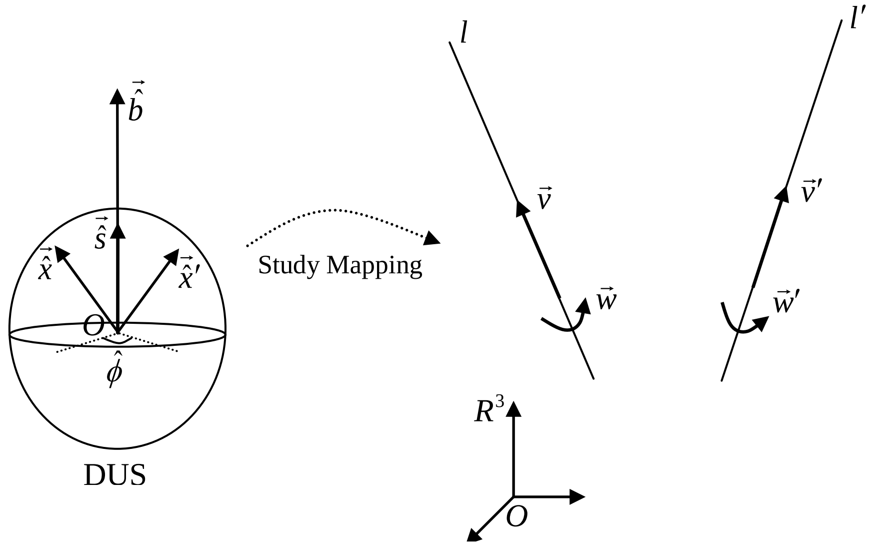

Figure 2. The relation between the rotation of DUS and the Screw Transformation

2.1. The Transformed Velocities

Substituting the dual Euler parameters $\hat{c}_0$ in (2) and $\hat{c}_1, \hat{c}_2, \hat{c}_3$ in (6) into (7) gives the transformed (or the final) angular $\vec{w}' = (w_1', w_2', w_3')$ and linear $\vec{v}' = (v_1', v_2', v_3')$ velocities as,

$$
\begin{aligned}
w_1' &= \cos^2\frac{\phi}{2}((1+b_1^2-b_2^2-b_3^2)w_1+2(-b_3+b_1b_2)w_2+2(b_2+b_1b_3)w_3),\\
w_2' &= \cos^2\frac{\phi}{2}(2(b_3+b_1b_2)w_1+(1+b_2^2-b_1^2-b_3^2)w_2+2(-b_1+b_2b_3)w_3),\\
w_1' &= \cos^2\frac{\phi}{2}(2(-b_2+b_1b_3)w_1+2(b_1+b_2b_3)w_2+(1+b_3^2-b_1^2-b_2^2)w_3).
\end{aligned}
\tag{8}
$$

$$
v_1' = 2\cos\frac{\phi}{2}\left\{\begin{aligned}
&(b_1c_1^* - b_2c_2^* - b_3c_3^* - \frac{\phi^*}{2}\sin\frac{\phi}{2})w_1 + \frac{1}{2}\cos\frac{\phi}{2}(1+b_1{}^2-b_2{}^2-b_3{}^2)v_1\\
&+(b_1c_2^* + b_2c_1^* + b_3\frac{\phi^*}{2}\sin\frac{\phi}{2} - c_3^*)w_2 + \cos\frac{\phi}{2}(b_1b_2-b_3)v_2\\
&+(b_1c_3^* - b_2\frac{\phi^*}{2}\sin\frac{\phi}{2} + b_3c_1^* + c_2^*)w_3 + \cos\frac{\phi}{2}(b_1b_3+b_2)v_3
\end{aligned}\right\},
$$

$$
v_2' = 2\cos\frac{\phi}{2}\left\{\begin{aligned}
&(b_1c_2^* + b_2c_1^* - b_3\frac{\phi^*}{2}\sin\frac{\phi}{2} + c_3^*)w_1 + \cos\frac{\phi}{2}(b_1b_2+b_3)v_1\\
&+(-b_1c_1^* + b_2c_2^* - b_3c_3^* - \frac{\phi^*}{2}\sin\frac{\phi}{2})w_2 + \frac{1}{2}\cos\frac{\phi}{2}(1-b_1{}^2+b_2{}^2-b_3{}^2)v_2\\
&+(b_1\frac{\phi^*}{2}\sin\frac{\phi}{2} + b_2c_3^* + b_3c_2^* - c_1^*)w_3 + \cos\frac{\phi}{2}(b_2b_3-b_1)v_3
\end{aligned}\right\},
\tag{9}
$$

$$
v_3' = 2\cos\frac{\phi}{2}\left\{\begin{aligned}
&(b_1c_3^* + b_2\frac{\phi^*}{2}\sin\frac{\phi}{2} + b_3c_1^* - c_2^*)w_1 + \cos\frac{\phi}{2}(b_1b_3-b_2)v_1\\
&+(-b_1\frac{\phi^*}{2}\sin\frac{\phi}{2} + b_2c_3^* + b_3c_2^* + c_1^*)w_2 + \cos\frac{\phi}{2}(b_2b_3+b_1)v_2\\
&+(-b_1c_1^* - b_2c_2^* + b_3c_3^* - \frac{\phi^*}{2}\sin\frac{\phi}{2})w_3 + \frac{1}{2}\cos\frac{\phi}{2}(1-b_1{}^2-b_2{}^2+b_3{}^2)v_3
\end{aligned}\right\},
$$

where $c_i^* = b_i^*\cos\frac{\phi}{2} - b_i\frac{\phi^*}{2}\sin\frac{\phi}{2}$, $i = 1,2,3$.

It is seen from (8) and (9) that the transformed screw is computed directly from the dual rotation angle $\hat{\phi}$, the Rodrigues parameters and the dual Rodrigues parameters.

3. APPLICATION OF DUAL EULER PARAMETERS TO THE SCREW TRANSFORMATION

Theoretically the formulas (8) and (9) are obtained from the rotations of the DUS. Let us examine (8) and (9) on an example by taking a dual rotation matrix, that is an orthogonal dual matrix $\hat{A} = \begin{pmatrix} -\varepsilon & -1 & 0 \\ 1 & -\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on DUS and ascrew axis l in R^3.

Let l be the screw axis passing through $\vec{p}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\sqrt{2})$ with the direction $\vec{x}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ and let $(\vec{w},\vec{v})$ be the screw with angular velocity $\vec{w}=(\frac{\pi}{8},\frac{\pi}{4},\frac{\pi}{4})$ $rad/\sec$ and the translation velocity $\vec{v}=(1,2,1)$ $cm/\sec$ at that moment. The dual vector $\vec{\hat{w}}=(\vec{w},\vec{v})=\vec{w}+\varepsilon\vec{v}=(\frac{\pi}{8}+\varepsilon,\frac{\pi}{4}+2\varepsilon,\frac{\pi}{4}+\varepsilon)$ defines this screw. l has the moment vector $\vec{x}^*=\vec{p}\times\vec{x}$ of a unit force on l with respect to the origin. Hence $\vec{x}^*=(1,-1,0)$. Therefore the Plücker coordinates of l, that is $(\vec{x},\vec{x}^*)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,1,-1,0)$, defines the point $\vec{\hat{x}}=\vec{x}+\varepsilon\vec{x}^*=(\frac{1}{\sqrt{2}}+\varepsilon,\frac{1}{\sqrt{2}}-\varepsilon,0)$ on DUS. The effect of rotation $\hat{A}$ on DUS causes the transformed screw axis l' and the transformed screw $\vec{\hat{w}}'=(\vec{w}',\vec{v}')$ in R^3.

$\hat{A}$ takes $\vec{\hat{x}}$ to $\vec{\hat{x}}'$ which corresponds to l' in R^3. Hence,

$$\vec{\hat{x}}'^T=\hat{A}\vec{\hat{x}}^T=\begin{pmatrix}-\varepsilon & -1 & 0\\ 1 & -\varepsilon & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}+\varepsilon\\ \frac{1}{\sqrt{2}}-\varepsilon\\ 0\end{pmatrix}=\begin{pmatrix}-\frac{1}{\sqrt{2}}+\frac{\sqrt{2}-1}{\sqrt{2}}\varepsilon\\ \frac{1}{\sqrt{2}}+\frac{\sqrt{2}-1}{\sqrt{2}}\varepsilon\\ 0\end{pmatrix}.$$

Then the Plücker coordinates of l' are $(\vec{x}',\vec{x}^{*\prime})=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\frac{\sqrt{2}-1}{\sqrt{2}},\frac{\sqrt{2}-1}{\sqrt{2}},0)$, where $\vec{\hat{x}}'=\vec{x}'+\varepsilon\vec{x}^{*\prime}$. Hence $\vec{x}'=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ is the unit direction vector to l' and $\vec{x}^{*\prime}=(\frac{\sqrt{2}-1}{\sqrt{2}},\frac{\sqrt{2}-1}{\sqrt{2}},0)$ is moment vector of l' determined for a unit force on l' with respect to the origin. Let $\vec{p}'$ denotes any point on l'. Since $\vec{x}^{*\prime}=\vec{p}'\times\vec{x}'$, by the vectorial division (inverse operation for the vector product) $\vec{p}'=\frac{\vec{x}'\times\vec{x}^{*\prime}}{\|\vec{x}'\|}+\lambda\vec{x}'$, where λ is a real parameter. If we take $\lambda=1$ then $\vec{p}'=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1-\sqrt{2})$ on l'. It is found that l' is passing through $\vec{p}'=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1-\sqrt{2})$ with the direction vector $\vec{x}'=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ .On the other hand, by the Cayley mapping,

$$\hat{B}=(\hat{A}-I)(\hat{A}+I)^{-1} \quad \Rightarrow \quad \hat{B}=\begin{pmatrix} 0 & -1-\varepsilon & 0 \\ 1+\varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so the dual Rodrigues vector is $\vec{\hat{b}}=(0,0,1+\varepsilon)$ [15]. Since $\left\|\vec{\hat{b}}\right\|=\tan\frac{\hat{\phi}}{2}$, we find the dual rotation angle as, $\hat{\phi}=\phi+\varepsilon\phi^{*}=\frac{\pi}{2}+\varepsilon$. Using the formulas (8) and (9) the final angular velocity $\vec{w}'=(-\frac{\pi}{4},\frac{\pi}{8},\frac{\pi}{4})\ rad/\sec$ and the translation velocity $\vec{v}'=(-\frac{\pi}{8}-2,-\frac{\pi}{4}+1,1)\ cm/\sec$ are easily obtained.

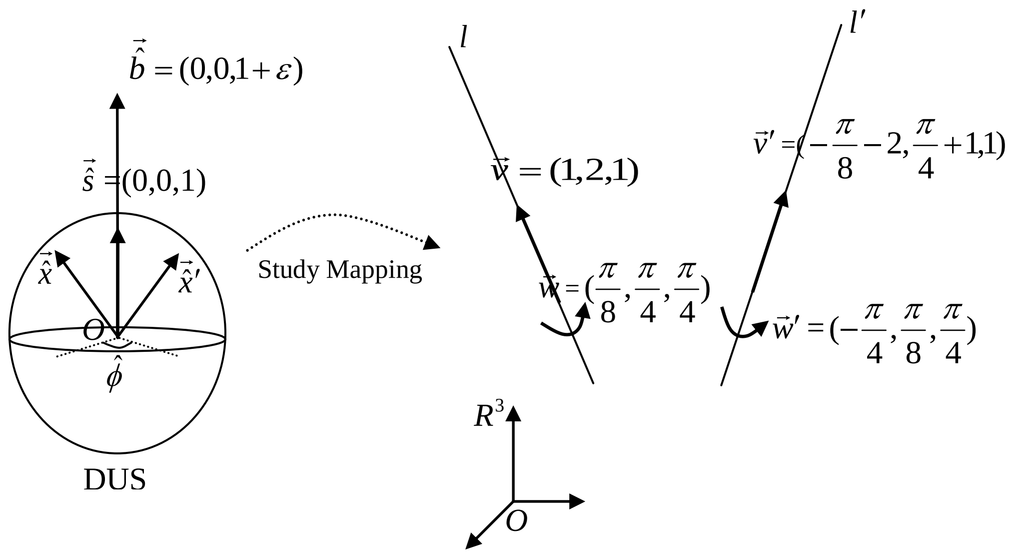

Figure 3. The application of a given rotation $\hat{A}$ to the Screw Transformation

3. CONCLUDING REMARKS

In the classical method, the real part Z of the dual quaternion $\hat{Z}=Z+\varepsilon Z^{*}$ is defined by $Z=\cos\frac{\phi}{2}+s_1\sin\frac{\phi}{2}i+s_2\sin\frac{\phi}{2}j+s_3\sin\frac{\phi}{2}k$, where ϕ is the rotation angle and $s=(s_2,s_2,s_3)$ is the rotation axis of the spatial motion. The dual part Z^{*} is given by the formula $Z^{*}=\frac{1}{2}DZ$, where $D=d_1i+d_2j+d_3k$ is the dual quaternion formed from the translation vector $d=(d_1,d_2,d_3)$. Therefore the transformed screw $\vec{\hat{w}}'=\hat{z}\vec{\hat{w}}\hat{\bar{z}}$ is proposed from the translation and the rotarion of the rigid body in real space. But in this paper instead of working on the real entities of the spatial motion, the quantity $\hat{Z}$ is established using the Rodrigues and the dual Rodrigues parameters of the one parameter motion on DUS which corresponds to the given spatial motion. The formulas (8) and (9) of the screw have informative coordinates about the rotations of the DUS. So for a given orthogonal dual matrix $\hat{A}$ and a screw l one can easily compute the final (the transformed) screw l' by directly using $\hat{A}$.

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