Mathematical and Computational Applications, Vol. 16, No. 2, pp. 437-442, 2011.

CONTINUOUS DEPENDENCE FOR THE DAMPED NONLINEAR HYPERBOLIC EQUATION

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Abstract- This paper gives the continuous dependence of solutions for the damped nonlinear hyperbolic equation.

Key Words- Damped nonlinear hyperbolic equation, Continuous Dependence

1.INTRODUCTION

In this paper, we are concerned with the following initial boundary value problem for the damped nonlinear hyperbolic equation:

$$u_{tt} + \alpha\Delta^2 u + \beta\Delta^2 u_t + \Delta g(\Delta u) = 0, \quad x \in \Omega, t > 0 \tag{1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \tag{2}$$

$$u(x,t) = 0, \; x \in \partial\Omega, t > 0 \tag{3}$$

where α and β are positive constants, Ω is bounded domain in R^n with smooth boundary $\partial\Omega$. Δ and Δ^2 denotes Laplacian and biharmonic operators respectively, $g(s)$ is the given nonlinear function.

This problem describes the motion of the neo-Hookean elastomer rod; for more physical interpretation of problems (1)-(3) we refer to [1].

There are some studies about this problem. For example, Uniform stabilization of the energy of a nonlinear damped hyperbolic equation is studied in [2]. Blow up results to the IBVP problem (1)-(3) is given by [3]. The authors of [1] studied a general class of abstract evolution equations

$$\begin{aligned} &u_{tt} + A_1 u + A_2 u_t + N^* g(Nu) = f(t) \\ &u(0) = \varphi_0 \\ &u_t(0) = \varphi_1 \end{aligned} \tag{4}$$

where A_1, A_2, N and f satisfy certain assumptions(see[1]).

Global in time existence, uniqueness, regularity and continuous dependence on the initial data φ_0 and φ_1 of a generalized solution of problem (4) are proven in [1].

The spatial decay estimates for a class of nonlinear damped hyperbolic equations investigated in [4]. Also they compared the solutions of two-dimensional wave equations with different damped coefficients.

The aim of the present paper is to prove the continuous dependence of solutions to the problem (1)-(3) on coefficients α and β.

Throughout this paper we use the notation $\|.\|_p$ for the norm in $L^p(\Omega)$. We denote $\|.\|$ the norm in $\|.\|_2$. $H^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are the usual Sobolev spaces.

The following existence theorem is proved [1].

Theorem 1. Let (u_0,u_1) belong to $H_0^2(\Omega)\times L^2(\Omega)$. Assume that there exist positive constants c_i for $i=1,2,3$ such that

$$\frac{-1}{2}\left(k_1+k_2-\varepsilon\right)|x|^2-c_1\le G(x)\le c_2|x|^2+c_3$$

for $\varepsilon>0$, where we set $G(x)=\int_0^x g(t)dt$. There are positive constants d_i for $i=1,2$ such that

$$|g(x)|\le d_1|x|+d_2$$

$$g'(x)\ge -a,\ \text{for } a>0$$

Then (1)-(3) admits a unique solution $u\in C\left(R^+;H_0^2(\Omega)\right)\cap C^1\left(R^+;L^2(\Omega)\right)$.

Firstly, let us obtain some a priori estimates which we will use next sections. We multiply (1) by u_t in $L^2(\Omega)$ we get

$$\frac{d}{dt}E(t)+\beta\|\Delta u_t\|^2=0 \tag{A}$$

where

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{\alpha}{2}\|\Delta u\|^2+\int_\Omega G(\Delta u)dx.$$

We integrate (A) from 0 to t we have,

$$E(0)-E(t)=\beta\int_0^t\|\Delta u_s\|^2 ds \tag{B}$$

Thus,

$$\|\Delta u\|^2 \le \frac{2}{\alpha} E(t) \le D_1 \tag{C}$$

and

$$\int_0^t \|\Delta u_s\|^2 \, ds \le D_2 \tag{D}$$

where $D_1 = \frac{2}{\alpha} E(0)$ and $D_2 = \frac{E(0)}{\beta}$.

2. CONTINUOUS DEPENDENCE ON THE COEFFICIENT α

In this section we prove that the solution of the problem (1)-(3) depends continuously on the coefficient α.

Now assume that u and v are the solutions of the problems respectively

$$u_{tt} + \alpha_1 \Delta^2 u + \beta \Delta^2 u_t + \Delta g\left(\Delta u\right) = 0, \quad x \in \Omega, t > 0 \tag{5}$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \tag{6}$$
$$u(x,t) = 0, \ x \in \partial\Omega, t > 0 \tag{7}$$

$$v_{tt} + \alpha_2 \Delta^2 v + \beta \Delta^2 v_t + \Delta g\left(\Delta v\right) = 0, \quad x \in \Omega, t > 0 \tag{8}$$
$$v(x,0) = u_0(x), v_t(x,0) = u_1(x), \quad x \in \Omega \tag{9}$$
$$v(x,t) = 0, \ x \in \partial\Omega, t > 0 \tag{10}$$

Let us define the difference variables w and α by $w = u - v$ and $\alpha = \alpha_1 - \alpha_2$ then w satisfy following the initial boundary value problem

$$w_{tt} + \alpha_1 \Delta^2 w + \alpha \Delta^2 v + \beta \Delta^2 w_t + \Delta g\left(\Delta u\right) - \Delta g\left(\Delta v\right) = 0, \quad x \in \Omega, t > 0 \tag{11}$$
$$w(x,0) = 0, w_t(x,0) = 0, \quad x \in \Omega \tag{12}$$
$$w(x,t) = 0, \ x \in \partial\Omega, t > 0 \tag{13}$$

The main result of this section is the following theorem.

Theorem 2. Assume that

$$\left|g(s) - g(t)\right| \le K\left|s - t\right| \tag{14}$$

for some K. Let w be the solution of the problem (11)-(13). Then w satisfies the estimate

$$\|w_t\|^2 + \alpha_1 \|\Delta w\|^2 \le D_3 \left(\alpha_1 - \alpha_2\right)^2 e^{M_1 t} t$$

where D_3 and M_1 are constants.

Proof. Multiplying (11) by w_t in $L^2(\Omega)$ we get

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|^2 + \frac{\alpha_1}{2}\|\Delta w\|^2\right] + \beta \|\Delta w_t\|^2 \le |\alpha| \|\Delta v\| \|\Delta w_t\| + \int_\Omega \left|g\left(\Delta u\right) - g\left(\Delta v\right)\right| |\Delta w_t| \, dx \qquad (15)$$

From (15) and (14) we obtain,

$$\frac{d}{dt} E_1(t) + \beta \|\Delta w_t\|^2 \le |\alpha| \|\Delta v\| \|\Delta w_t\| + K \|\Delta w\| \|\Delta w_t\| \qquad (16)$$

where

$$E_1(t) = \frac{1}{2}\|w_t\|^2 + \frac{\alpha_1}{2}\|\Delta w\|^2 .$$

Using Cauchy-Schwarz inequality at the right hand side of (16) we get,

$$\frac{d}{dt} E_1(t) \le \frac{|\alpha|^2}{2\varepsilon} \|\Delta v\|^2 + M_1 E_1(t) \qquad (17)$$

where $M_1 = \max\left\{\frac{K^2}{\alpha_1 \varepsilon}, 1\right\}$. Applying Gronwall 's inequality with (C) we obtain

$$E_1(t) \le e^{M_1 t} \frac{D_1 t}{2\varepsilon} |\alpha|^2 \qquad (18)$$

which is desired result.

3. CONTINUOUS DEPENDENCE ON THE COEFFICIENT β

In this section we prove that the solution of the problem (1)-(3) depends continuously on the coefficient β.

Now assume that u and v are the solutions of the problems respectively

$$u_{tt} + \alpha \Delta^2 u + \beta_1 \Delta^2 u_t + \Delta g\left(\Delta u\right) = 0, \quad x \in \Omega, t > 0 \qquad (19)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \qquad (20)$$

$$u(x,t)=0,\ \ x\in\partial\Omega, t>0 \tag{21}$$

$$v_{tt}+\alpha\Delta^2 v+\beta_2\Delta^2 v_t+\Delta g\left(\Delta v\right)=0, \quad x\in\Omega, t>0 \tag{22}$$

$$v(x,0)=u_0(x), v_t(x,0)=u_1(x), \quad x\in\Omega \tag{23}$$

$$v(x,t)=0,\ \ x\in\partial\Omega, t>0 \tag{24}$$

Let us define the difference variables w and β by $w=u-v$ and $\beta=\beta_1-\beta_2$ then w satisfy following the initial boundary value problem

$$w_{tt}+\alpha\Delta^2 w+\beta_1\Delta^2 w_t+\beta\Delta^2 v_t+\Delta g\left(\Delta u\right)-\Delta g\left(\Delta v\right)=0, \quad x\in\Omega, t>0 \tag{25}$$

$$w(x,0)=0, w_t(x,0)=0, \quad x\in\Omega \tag{26}$$

$$w(x,t)=0,\ \ x\in\partial\Omega, t>0 \tag{27}$$

The main result of this section is the following theorem.

Theorem 3. Assume that (14) is satisfied and let w be the solution of the problem (25)-(27). Then w satisfies the estimate

$$\|w_t\|^2+\alpha\|\Delta w\|^2\le D_4\left(\beta_1-\beta_2\right)^2 e^{M_2 t}$$

where D_4 and M_2 are constants.

Proof. Multiplying (25) by w_t in $L^2(\Omega)$ we get

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|^2+\frac{\alpha}{2}\|\Delta w\|^2\right]+\beta_1\|\Delta w_t\|^2\le|\beta|\|\Delta v_t\|\|\Delta w_t\|+\int_\Omega\left|g\left(\Delta u\right)-g\left(\Delta v\right)\right|\left|\Delta w_t\right|dx \tag{28}$$

From (28) and (14) we get,

$$\frac{d}{dt}E_2(t)+\beta_1\|\Delta w_t\|^2\le|\beta|\|\Delta v_t\|\|\Delta w_t\|+K\|\Delta w\|\|\Delta w_t\| \tag{29}$$

where

$$E_2(t)=\frac{1}{2}\|w_t\|^2+\frac{\alpha}{2}\|\Delta w\|^2 .$$

Cauchy-Schwarz inequality and from (29) we obtain,

$$\frac{d}{dt}E_2(t) \le \frac{|\beta|^2}{2\varepsilon}\|\Delta v_t\|^2 + M_2 E_2(t) \tag{30}$$

where $M_2 = \max\left\{\frac{K^2}{\alpha\varepsilon}, 1\right\}$. Applying Gronwall 's inequality with (D) we obtain,

$$E_2(t) \le e^{M_2 t}\frac{D_2 t}{2\varepsilon}|\beta|^2 \tag{18}$$

Hence proof is completed.

4. REFERENCES

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