



THE CURVATURE OF A BÉZIER CONTROL POLYLINE

Bahadır Tantay¹, Ferhat Taş²

¹Ege University, Department of Mathematics, Izmir, 35100 Turkey,
bahadir.tantay@ege.edu.tr

² Istanbul University, Department of Mathematics, Istanbul, 34134 Turkey
tasf@istanbul.edu.tr

Abstract- The role of differential geometry in describing a curve can not be denied. The differential forms defined for Bézier curves which are widely used in computer aided geometric design, plays a significant role in classification and image processing of curves. For this reason, the definitions such as Serret-Frenet frame, curvature and torsion which are described for Bézier curves are very important in computer aided geometric design. In this paper, in addition to these definitions we have also defined a new classification by applying angular curvature used for planar curves in computational geometry to Bézier control polygon.

Key Words- Bézier curves, blossom, line curvature.

1. INTRODUCTION

1.1 Representation of curves

Among the representations used for curves parametric representation is the most commonly used one owing to advantages in representing curves. In the parametric method a curve is represented as a function of one parameter.

$$\alpha(t) : I \subset \mathbf{R} \rightarrow \mathbf{R}^n, \quad n = 2, 3, \dots$$

We look for class of functions which is as simple as possible and yet diverse enough to represent a wide variety of curves. Polynomial functions to large extent, satisfy this requirement. A general polynomial function is shown below:

$$\alpha(t) = \sum_{i=0}^n p_i t^i \quad (1)$$

where n is the degree of the polynomial [1].

The variety of curves that can be obtained by using polynomials depends on the maximum allowed degree. The higher the degree, the greater variety of shapes one can represent. For example, to define a curve with n wiggles, we need a polynomial of degree $n+1$. But higher degrees result in some problems. The higher the degree of a curve, the less controllable it is, in a sense that small changes in coefficients are likely to result in large changes in the shape of the curve [1].

Consider the functional case only. Degree 1 polynomials can only be used to represent straight lines; degree 2 curves are parabolas in non-degenerate cases. The parabola does not have a point of inflection, where the curve turns from being concave to convex or vice-versa. Cubic curves may have inflection points [1].

To represent arbitrarily complex curves one usually uses piecewise polynomials, stitching together many polynomial pieces.

To represent the curves in space we commonly use:

$$\alpha(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

where p_i 's are points in three dimensional space [1].

2. BÉZIER CURVES

2.1 Definition

Generally, a Bézier curve is a polynomial curve with degree n and shown in vector form as:

$$B(t) = \sum_{i=0}^n b_i B_i^n(t), \quad t \in [0,1] \quad (2)$$

where

$$B_i^n(t) = \begin{cases} \frac{n!}{i!(n-i)!} (1-t)^{n-i} t^i & , 0 \leq i \leq n \\ 0 & , \text{otherwise} \end{cases}$$

is Bernstein bases function and $\frac{n!}{i!(n-i)!}$ is binomial coefficient [1].

2.2 Properties of Bézier curves

- a) Interpolation: One can easily see that $B(0) = b_0$ and $B(1) = b_n$ [1].
- b) Affine Invariance: This property can be easily verified by considering an affine map $\varphi(x) = Ax + v$ where A is a $n \times n$ matrix and v is in \mathbb{R}^n [1].

$$\varphi\left(\sum_{i=0}^n b_i B_i^n(t)\right) = A \sum_{i=0}^n b_i B_i^n(t) + v = \sum_{i=0}^n (Ab_i + v) B_i^n(t) = \sum_{i=0}^n \varphi(b_i) B_i^n(t)$$

- c) Tangent to the curve at point b_0 is the vector $\overline{b_0 b_1}$ and tangent to the curve at b_n is $\overline{b_n b_{n-1}}$. This property can be used to test whether two Bézier curves are joined smoothly [1].
- d) Convex Hull: It is clear that $B_i^n(t) \geq 0$ and $\sum B_i^n(t) = 1$, for $t \in [0,1]$. This means that all points on the Bézier curve lie inside the convex hull of points b_0, b_1, \dots, b_n [1].

3. POLAR FORMS AND BLOSSOMING

Polar form of a polynomial of $\alpha(t): \mathbb{R} \rightarrow \mathbb{R}^n$ of degree n is a multi-affine symmetric function $\alpha(t_1, t_2, \dots, t_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that [1]

$$\alpha(\underbrace{t, t, \dots, t}_n) = \alpha(t) \quad (3)$$

3.1 Properties of polar forms

- a) A function $f(t_1, t_2, \dots, t_n)$ is symmetric if the value of the function is the same for any permutation of the arguments t_1, t_2, \dots, t_n [1].

b) A function $f(t_1, t_2, \dots, t_n)$ is multi-affine if for any arguments $t_1, \dots, t_j^0, t_j^1, \dots, t_n$ and any s ;

$$f(t_1, \dots, st_j^0 + (1-s)t_j^1, \dots, t_n) = sf(t_1, \dots, t_j^0, t_{j+1}, \dots, t_n) + (1-s)f(t_1, \dots, t_j^1, \dots, t_n) \quad [1].$$

For any polynomial, there exists a unique symmetric and multi-affine polar form. For example, a straight line given like this form: $F = ax + b$. Then if we polarize this first order polynomial we obtain $f = at_0 + b$. The general quadratic polynomial function form is $G = ax^2 + bx + c$. If we polarize it, then we obtain that $g = at_0t_1 + b(t_0 + t_1)/2 + c$. Also, the elementary symmetric functions of the cubic polynomial function are [1]

$$\left[1 \quad \frac{t_0 + t_1 + t_2}{3} \quad \frac{t_0t_1 + t_1t_2 + t_2t_0}{3} \quad t_0t_1t_2 \right] \quad (4)$$

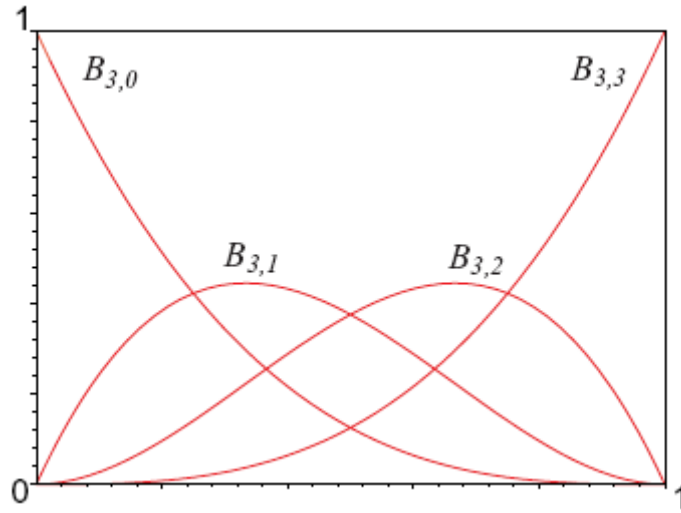
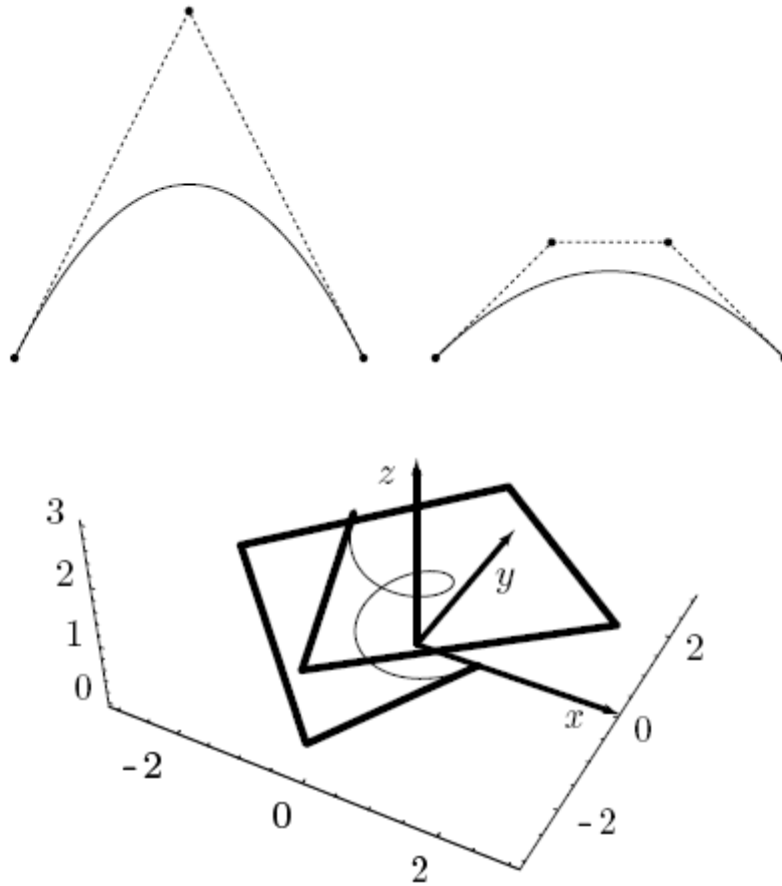


Figure 1: Bernstein basis function for $n = 3$

Figure 2: Bézier curves for $n = 2, 3$

3.2 Example

Suppose we want to construct a parabola, $y = x^2$ with the simple parameterization of,

$$F_0(t) = t,$$

$$F_1(t) = t^2$$

First we find the polar forms of F_s .

$$f_0(t_0, t_1) = (t_0 + t_1) / 2,$$

$$f_1(t_0, t_1) = t_0 t_1$$

Next, we find the control points defining the 3 2-affine mappings.

$$b_0 = (f_0(0, 0), f_1(0, 0)) = (0, 0),$$

$$b_1 = (f_0(0, 1), f_1(0, 1)) = (1/2, 0),$$

$$b_2 = (f_0(1, 1), f_1(1, 1)) = (1, 1)$$

Finally we get the constructive parabola as in Figure 3 [2].

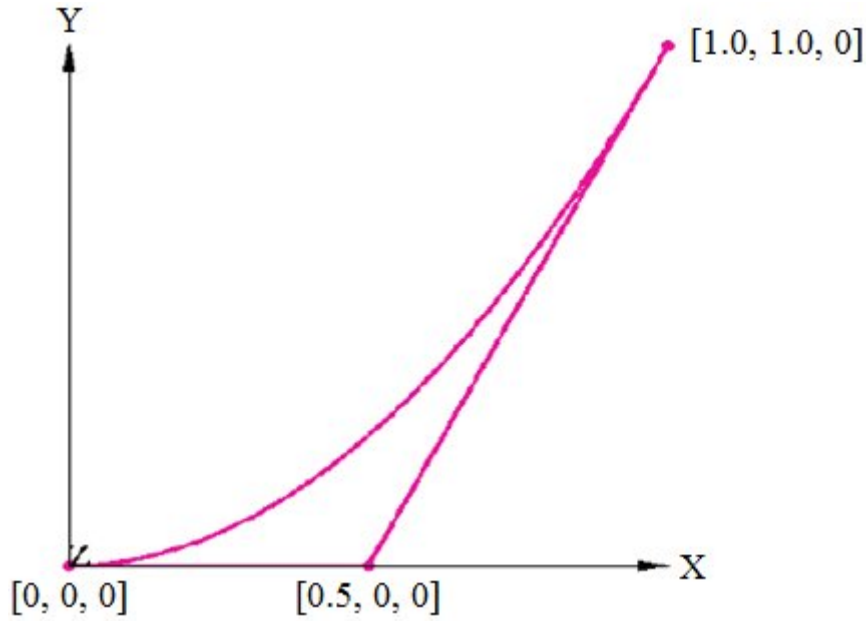
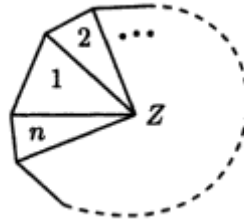


Figure 3 : Quadratic curve and its control points.

4. THE CURVATURE OF A POLYGONAL LINE

Consider a convex n -sided polygon or n -gon. From interior point z we draw lines connecting z to the vertices of the n -gon. Since the n -gon is convex, these connecting lines are all interior and they divide the n -gon into n triangles, as indicated schematically in Figure 4 [5].

Figure 4: n -gon.

Let the angles of the i th triangle be $\alpha_i, \beta_i, \gamma_i$ in clockwise order around the triangle such that γ_i is the vertex at z . An (interior) angle of the n -gon is of the form $\alpha_{i+1} + \beta_i$, except for the last, which is $\alpha_1 + \beta_n$. Hence the sum of the angles of the n -gon is $\sum_{i=1}^n (\alpha_i + \beta_i)$. Since the sum of the angles γ_i at z is 2π , we can write sum of the angles of the n -gon = $\sum (\alpha_i + \beta_i + \gamma_i) - 2\pi$ [5].

If δ_i denotes the effect of the i th triangle, we have $\alpha_i + \beta_i + \gamma_i = \pi - \delta_i$. We therefore make the following definition.

4.1 Definition

The angular defect of the n -gon is $(n-2)\pi$ – the sum of its angles [5]. Let G be a polygonal line and p one of its interior vertices. Let $\angle(p) \in [0, \pi)$ be the turning angle between the two consecutive edges k_1 and k_2 adjacent to p , of length N_1 and N_2 . The curvature $c(p)$ of G at p is defined by the angular defect

$$c(p) = \frac{2(\pi - \angle(p))}{N_1 + N_2} \quad (5)$$

So, $c(p)$ vanishes when the two edges are collinear [3].

5. THE CURVATURE OF A BÉZIER CONTROL POLYLINE

Let a Bézier curve with degree n is determined by $\{b_0, b_1, \dots, b_n\} (n \geq 1)$ control vertices. So the set which includes the control vertices can be defined as an oriented simplex. Each consecutive pair of points determine an edge. So we can define the curvature of a Bézier control polyline at each pair of consecutive edges based on definition in 4. As in the general geometry formula

$$\arccos\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}\right) = \theta$$

where \vec{u} and \vec{v} are any vectors in E^n without their magnitude is zero. So each consecutive control vertex determines a vector and an angle in E^n .

Let b_{i-1}, b_i, b_{i+1} be three consecutive control points. Angular curvature at b_i is determined by

$$A(b_i) = \frac{2(\pi - \angle(b_{i-1}b_i)(b_ib_{i+1}))}{\|b_{i-1}b_i\| + \|b_ib_{i+1}\|} \quad (6)$$

5.1 Example

Consider a quadratic Bézier curve given with control points $b_0 = (0, 0)$, $b_1 = (0, 5)$ ve $b_2 = (6, 7)$. If we use the Bernstein basis form;

$$\alpha(t) = (1-t)^2 b_0 + (1-t)t b_1 + t^2 b_2$$

then equation of the curve is

$$\alpha(t) = (6t^2, 2t^2 + 5t)$$

Approximate values of the curvature of the curve at $t = 0$ and $t = 1$ are 0.48 and 0.017, respectively.

On the other hand, we can compute the curvature of the Bézier polyline at b_0 , b_1 and b_2 .

$$A(b_0) \cong 3.315$$

$$A(b_1) \cong 1.384$$

$$A(b_2) \cong 3.031$$

b_1 is not on the curve and if we wish to change it;

For example let $b_1 = (1, 3)$. Then the equation of the curve is $\beta(t) = (5t^2 + t, 4t^2 + 3t)$. Approximate values of the curvature of the curve at $A(b_1) \cong 1.641$. So the curvature of the Bézier polyline has changed by changing the control points. Also the curvature of the curve has changed. Approximate values of the curvature of the curve at $t = 0$ and $t = 1$ are 0.695 and 0.005, respectively. Figure 5 shows the difference between $\alpha(t)$ and $\beta(t)$.

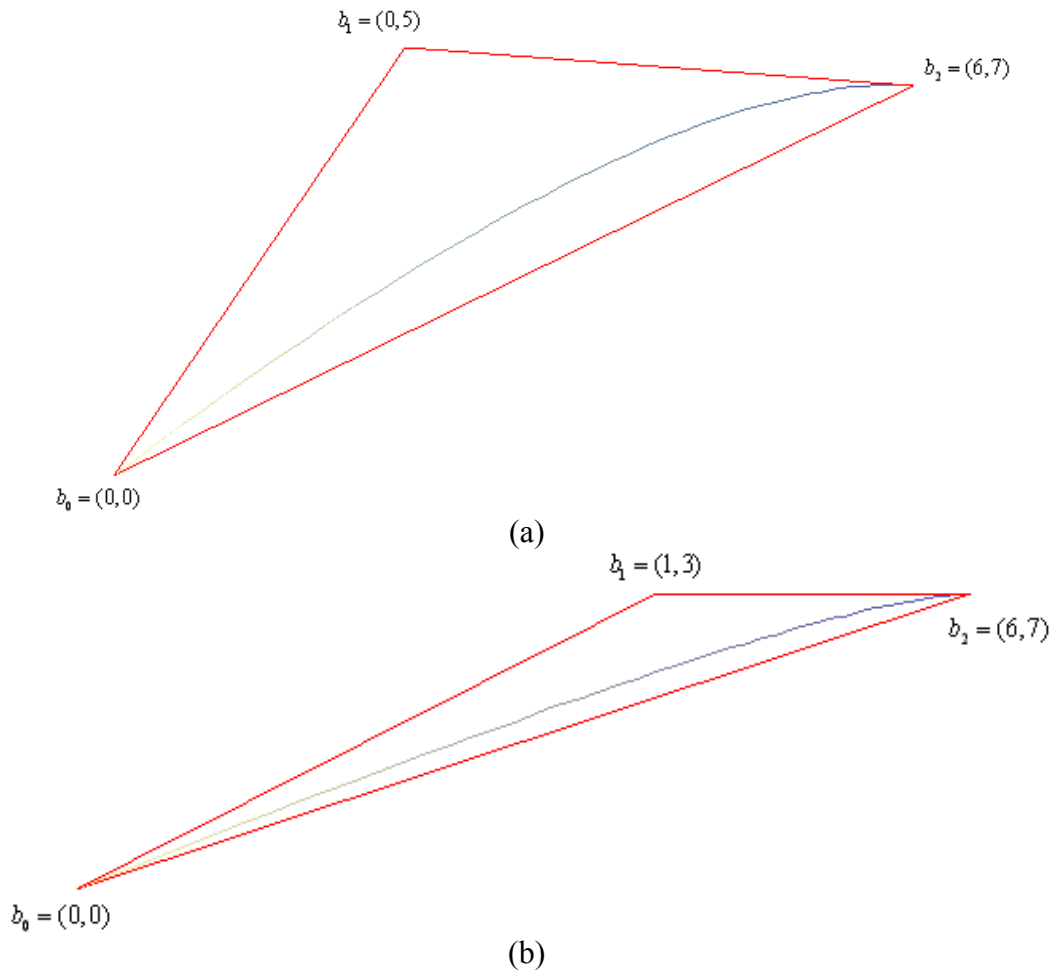


Figure 5: (a) $\alpha(t)$ and its Bézier control polygon. (b): $\beta(t)$ and its Bézier control polygon.

6. CURVATURE OF THE BÉZIER POLYLINE OF A PARAMETRIC CURVE WITH DEGREE N

Consider a parametric curve with degree n represented by $\alpha(t) = \sum_{i=0}^n p_i t^i$. Its polar form is

$$\alpha(t_0, t_1, \dots, t_n) = \left(1, \frac{t_0 + t_1 + \dots + t_n}{n}, \dots, \frac{t_0 t_1 \dots t_{n-2} t_{n-1} + t_0 t_1 \dots t_{n-2} t_n + \dots + t_1 t_2 \dots t_{n-1} t_n}{n}, t_0 t_1 \dots t_n\right)$$

So the curve has $n+1$ control points b_0, b_1, \dots, b_{n+1} , and we can determine the angular curvature of the curve by using these control points.

Example 3

Let the curve be $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$. Since the curve is cubic, there are four control points. As we explained in section 3 we find the control points via using the blossoming as

$b_0 = (0, 0, 0), b_1 = (1, 0, 1), b_2 = (2, 1, 2)$ and $b_3 = (2, 3, 4)$.

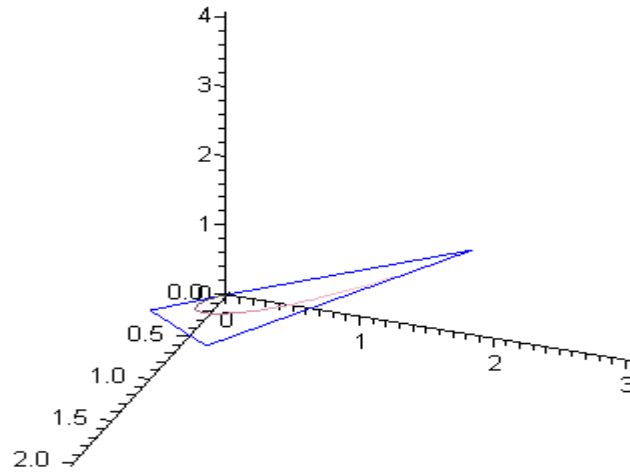


Figure 6: The curve $\alpha(t)$ and its control polygon

The curvature of the curve at b_0 and b_3 are $1/3$ and $1/12$, respectively. The approximate value of the curvatures of the Bézier control polyline at b_1 and b_2 are 4.703 and 3.418, respectively.

7. CONNECTION BETWEEN CURVATURE AND CURVATURE

We know that the curvature of a Bézier curve is formulated with its control points as follow [4]:

$$C_E = \frac{2}{3} \frac{\left| (\overrightarrow{b_0 b_1}) \times (\overrightarrow{b_1 b_2}) \right|}{\left| \overrightarrow{b_0 b_1} \right|^3} \quad (7)$$

Also we have the curvature of a Bézier control polyline of the corresponding curve with its control points as:

$$A_E = 2 \frac{\left[\pi - \arcsin \left(\frac{\left| (\overrightarrow{b_0 b_1}) \times (\overrightarrow{b_1 b_2}) \right|}{\left| \overrightarrow{b_0 b_1} \right| \left| \overrightarrow{b_1 b_2} \right|} \right) \right]}{\left| \overrightarrow{b_0 b_1} \right| + \left| \overrightarrow{b_1 b_2} \right|} \quad (8)$$

$$\Rightarrow C_E = \frac{2}{3} \frac{|\overrightarrow{b_1 b_2}|}{|\overrightarrow{b_0 b_1}|^2} \sin \left[\pi - \left(\frac{|\overrightarrow{b_0 b_1}| + |\overrightarrow{b_1 b_2}|}{2} \right) \cdot A_E \right] \quad (9)$$

So we have a connection between angular curvature and curvature.

8. CONNECTION BETWEEN ANGULAR CURVATURE AND TORSION

We know that the torsion of a Bézier curve is formulated with its control points as follow [4]:

$$B_E = \frac{1}{3} \frac{[(\overrightarrow{b_1 - b_0}), (\overrightarrow{b_2 - b_1}), (\overrightarrow{b_3 - b_2})]}{|\overrightarrow{(b_1 - b_0)} \times \overrightarrow{(b_2 - b_1)}|^2} = \frac{1}{3} \frac{[(\overrightarrow{b_0 b_1}), (\overrightarrow{b_1 b_2}), (\overrightarrow{b_2 b_3})]}{|\overrightarrow{(b_0 b_1)} \times \overrightarrow{(b_1 b_2)}|^2} \quad (10)$$

Also we have curvature of a Bézier control polyline as mentioned above, then we have;

$$B_E = \frac{1}{3} \frac{[(\overrightarrow{b_0 b_1}), (\overrightarrow{b_1 b_2}), (\overrightarrow{b_2 b_3})]}{\left(|\overrightarrow{b_0 b_1}| |\overrightarrow{b_1 b_2}| \sin \left(\pi - G_E \cdot \frac{|\overrightarrow{b_0 b_1}| + |\overrightarrow{b_1 b_2}|}{2} \right) \right)^2} \quad (11)$$

Therefore, we have a connection between curvature of the Bézier control polyline and torsion.

9. CONCLUSION

For a Bézier curve to obtain the control points which allow us to make some changes on the curve given in its parametric form, we can use the blossoming method. Bézier curves are very important in the manufacturing of automobile and ship parts. Curves can be used in computer aided geometric design in a similar way that they are used in differential geometry. While in classical differential geometry, a curve is defined by using its curvature, here we do it by using control points. This idea resulted using them in the Bézier curves and their control polygon. Finally, we obtain the characterization of the curves.

10. REFERENCES

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