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NON-LINEAR TRANSVERSE VIBRATIONS and 3:1 INTERNAL RESONANCES OF A TENSIONED BEAM ON MULTIPLE SUPPORTS

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Abstract – In this study, nonlinear transverse vibrations of a tensioned Euler-Bernoulli beam resting on multiple supports are investigated. The immovable end conditions due to simple supports cause stretching of neutral axis and introduce cubic nonlinearity to the equations of motion. Forcing and damping effects are included in the analysis. The general arbitrary number of support case is investigated and 3, 4, and 5 support cases analyzed in detail. A perturbation technique (the method of multiple scales) is applied to the equations of motion to obtain approximate analytical solutions. 3:1 internal resonance case is also considered. Natural frequencies and mode shapes for the linear problem are found for the tensioned beam. Nonlinear frequencies are calculated; amplitude and phase modulation figures are presented for different forcing and damping cases. Frequency-response and force-response curves are drawn. Different internal resonance cases between modes are investigated.

Keywords – *vibration, multiply supported beam, axial tension, perturbation method*

1. INTRODUCTION

Transverse vibrations of beams are of importance in engineering systems and investigated in detail. A literature survey up to 1979 is done by Nayfeh and Mook [1]. Nonlinear free vibrations of multispan beams on elastic supports were studied by Lewandowski [2] using dynamic stiffness method to find frequencies and nonlinear modes of vibrations by considering effects of support flexibility on the frequency amplitude relations. Özkaya [3] discussed the effects of different end conditions for beam-mass systems. More recent works on this type are due to [4-16]. These studies include restrained beams [4, 5], in-span support [6,7], different boundary conditions for nonlinear vibrations [8-10], stepped beam systems using artificial neural networks[9] and finite element methods [12], single, multiple mass on simply supports [13-16], and non ideal support cases for three different simply supported beams [16], infinite mode analysis was performed [17]. Nonlinear vibrations and 3:1 internal resonances on multiple supports were investigated and excitation frequency-frequency response curves drawn for different support numbers [18] and Tekin et al. studied on three-to one resonance in multi stepped beam systems [19]. For slightly curved beams with stretching, one may refer to Rehfield [20]. There are also some studies about axially moving beams composed of viscoelastic materials [17-23]. Beams simply-supported in span were discussed and frequency response functions are determined [23]. Varadan et al. [24] studied nonlinear behavior of a beam in bending with immovable ends for various loadings and edge conditions. The authors concluded that it was enough to consider only the nonlinearity arising from the axial force in the nonlinear analysis of beams with immovable ends since the hardening effect due to axial force predominates over that resulting from the use of an actual nonlinear expression for curvature. Da Silva [25] derived nonlinear equations for a class of inextensible flexible multibeam structures having arbitrary cross section varying along its span and also having supports and lumped masses. Cheng et al. [26] investigated nonlinear random response of internally hinged beams using finite element method. Main and Jones [27, 28] formulated exact analytical solutions for free vibrations of tensioned beams with an intermediate viscous damper and a viscous damper attached transversely near a support using dynamic stiffness method to obtain characteristic equations for both clamped and pinned supports. Mazzilli et al. [29] constructed nonlinear normal modes and nonlinear multi modes using the method of multiple scales for a beam with uniformly distributed axial and a thrust force and compared with finite element method simulations.

In this study, nonlinear transverse vibrations of a tensioned Euler–Bernoulli beam with multiple simple supports are considered. The beam is stretched during vibration due to immovable supports. This introduces cubic nonlinearity to the equations of motion. Transverse forcing and damping are also included in the problem. The equations of motion are derived for general case: arbitrary number of supports, and then solved for 3, and 4 support cases by using the method of multiple scales. Natural frequencies are calculated and mode shapes are presented. The effect of support number on the natural frequencies is investigated for the nonlinear vibrations. Amplitude and phase modulation relations are presented for different forcing and damping cases. 3:1 internal resonance cases are investigated between different modes of vibration.

2. EQUATIONS OF MOTION

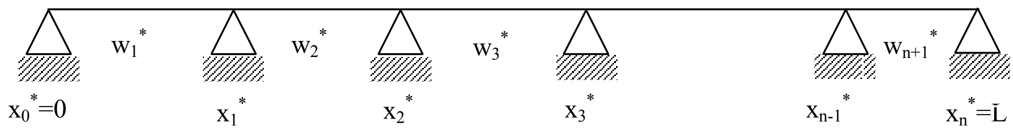

Figure 1. Tensioned beam on multiple simple supports.

For the system shown in Fig. 1, In Figure 1, x_{m+1}^* denotes location of the m^{th} support and w_{m+1}^* denotes the transverse displacement of the beam section between supports *m* and *m+1*. L is the length of the beam. t^* is the time. The total number of supports is *n*. ρA is the mass per unit length, *EA* is longitudinal rigidity, *EI* is flexural rigidity and *P* is the axial tension force on the beam. The Lagrangian can be written as follows

$$
\begin{gathered}
£ = \frac{1}{2}\sum_{m=0}^{n}\left[\int_{x_m^*}^{x_{m+1}^*} \rho A \dot{w}_{m+1}^{*\,2}\, dx^* - \int_{x_m^*}^{x_{m+1}^*} EI w_{m+1}''^{*\,2}\, dx^* \right. \\
\left. - \int_{x_m^*}^{x_{m+1}^*} EA\left(u_{m+1}'^{*} + \frac{1}{2} w_{m+1}'^{*\,2}\right)^2 dx^* - \int_{x_m^*}^{x_{m+1}^*} P\left(u_{m+1}'^{*} + \frac{1}{2} w_{m+1}'^{*\,2}\right) dx^* \right], \quad x_0^* = 0,\ x_{n+1}^* = L
\end{gathered}
\tag{1}
$$

where dot denotes derivative with respect to t^* and prime denotes derivative with respect to x^*. The first integral is the kinetic energy of the beam section between any successive supports. The second integral is the elastic energy in bending, the third integral is the elastic energy in extension due to stretching of the neutral axis and the last one is the elastic energy due to axial tension. Applying Hamilton's principle and performing the necessary algebra, the equations of motion and boundary conditions for the general case for the tensioned beam in dimensional form is obtained as follows

$$\rho A\ddot{w}_{m+1} + EIw^{iv}_{m+1} - P\,w''^{*}_{m+1} = \frac{EA}{2L}\left[\sum_{r=0}^{n}\int_{x_r^*}^{x_{r+1}^*} w'^{*\,2}_{r+1}\,dx^*\right]w''_n \tag{2}$$

$$w_1^*(0,t^*) = w_1^{*''}(0,t^*) = 0\,,\ w^*_{m+1}(x^*_{m+1},t^*) = w^*_{m+2}(x^*_{m+1},t^*) = 0\,,$$

$$w_{m+1}^{*'}(x_{m+1}^{*},t^*) = w_{m+2}^{*'}(x_{m+1}^{*},t^*)\,, w_{m+1}^{*''}(x_{m+1}^{*},t^*) = w_{m+2}^{*''}(x_{m+1}^{*},t^*)$$

$$w_{n+1}^{*}(L,t^*) = w_{n+1}^{''}(L,t^*) = 0 \tag{3}$$

The equations are made dimensionless using the following definitions

$$x = \frac{x^*}{L}, w_{m+1} = \frac{w^*_{m+1}}{R}, \eta_{m+1} = \frac{x_{m+1}}{L}, t = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}\,t^*, v_p^2 = \frac{PL^2}{EI} \tag{4}$$

where R is the radius of gyration of the beam cross-section with respect to the neutral axis. Substituting the dimensionless parameters into the equations of motion yields

$$\ddot{w}_{m+1} + w^{iv}_{m+1} - v_p^2\ w''_{m+1} = \frac{1}{2}\left[\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} w'^{2}_{r+1}dx\right]w''_{m+1} \tag{5}$$

3. METHOD OF MULTIPLE SCALES

The method of multiple scales will be applied to the partial differential equation system and boundary conditions directly. There is no quadratic non-linearities, that's why one can write an expansion of the form

$$w_{m+1}(x,t;\varepsilon) = \varepsilon\, w_{m+1,1}(x,T_0,T_2) + \varepsilon^3\, w_{m+1,3}(x,T_0,T_2) + \ldots\ldots \tag{6}$$

where ε is a small book-keeping parameter representing that the deflections are small. This procedure models a weak non-linear system. $T_0=t$ and $T_2=\varepsilon^2 t$ are the fast and slow time scales. Here only the primary resonance case is considered. The forcing and damping terms are ordered as shown below so that they are included in the cubic order of expansion, $\mu = \varepsilon^2\mu$, $F_{m+1} = \varepsilon^3 F_{m+1}$, the time derivatives are written as $(\dot{\ }) = D_0 + \varepsilon^2 D_2$, $(\ddot{\ }) = D_0^2 + 2\varepsilon^2 D_0 D_2$, where $D_n = \partial/\partial T_n$. After expansion, one obtains equations of motion and boundary conditions at different orders as follows

Order (ε):

$$D_0^2 w_{m+1,1} + w_{m+1,1}^{iv} - v_p^2 w_{m+1,1}'' = 0\,,\; w_{1,1}(0,t) = w_{1,1}''(0,t) = 0\,,\; w_{m+1,1}(\eta_{m+1},t) = w_{m+2,1}(\eta_{m+1},t) = 0$$

$$w_{m+1,1}'(\eta_{m+1},t) = w_{m+2,1}'(\eta_{m+1},t)\,,\; w_{m+1,1}''(\eta_{m+1},t) = w_{m+2,1}''(\eta_{m+1},t)\,,\; w_{n+1,1}(1,t) = w_{n+1,1}''(1,t) = 0 \quad (7)$$

Order (ε^3):

$$D_0^2 w_{m+1,3} + w_{m+1,3}^{iv} - v_p^2 w_{m+1,3}'' = -2D_0 D_2 w_{m+1,1}$$

$$-2\mu D_0 w_{m+1,1} + \frac{1}{2}\left[\sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} \mathrm{w}_{r+1,1}'^{2} dx\right] w''_{\mathrm{m+1,1}} + F_{m+1}\cos\Omega T_0 \quad (8)$$

$$w_{1,3}(0,t) = w_{1,3}''(0,t) = 0\,,\; w_{m+1,3}(\eta_{m+1},t) = w_{m+2,3}(\eta_{m+1},t) = 0\,,\; w_{m+1,3}'(\eta_{m+1},t) = w_{m+2,3}'(\eta_{m+1},t)$$

$$w_{m+1,3}''(\eta_{m+1},t) = w_{m+2,3}''(\eta_{m+1},t)\,,\; w_{n+1,3}(1,t) = w_{n+1,3}''(1,t) = 0$$

Solution of the first order of expansion gives natural frequency values and a solvability condition is obtained from the second order of expansion.

3.1. Exact Solution to the Linear Problem

For Eq. (8) one can assume solutions of the form for any beam segment

$$w_{m+1,1} = [A(T_2)e^{i\omega T_0} + cc]Y_{m+1,1}(x) \quad (9)$$

where cc stands for complex conjugate of the preceding terms. Eqs. (7) and (9) give

$$Y_{m+1}^{iv} - v_p^2 Y_{m+1}'' - \omega^2 Y_{m+1} = 0\,,\; Y_1(0) = Y_1''(0) = 0\,,\; Y_{m+1}(\eta_{m+1}) = Y_{m+2}(\eta_{m+1}) = 0 \quad (10)$$

$$Y_{m+1}'(\eta_{m+1}) = Y_{m+2}'(\eta_{m+1})\,,\quad Y_{m+1}''(\eta_{m+1}) = Y_{m+2}''(\eta_{m+1})\,,\quad Y_{n+1}(1) = Y_{n+1}''(1) = 0$$

The solution of the equations can be sought by assuming the following shape function for any beam segment

$$Y_{m+1}(x) = C_{4m+1}e^{\beta_1 x} + C_{4m+2}e^{\beta_2 x} + C_{4m+3}e^{\beta_3 x} + C_{4m+4}e^{\beta_4 x} \quad (11)$$

Frequency equations can be obtained when the boundary conditions are applied.

3.2. Approximate Solution to the Non-linear Problem

Solution of nonlinear Eq. (8) gives corrections to the problem. They will have a solution only if a solvability condition is satisfied as explained in reference [30]. The secular and nonsecular terms are separated to find the solvability condition by assuming a solution of the form

$$w_{m+1,3} = \phi_{m+1}(x,T_2)e^{i\omega T_0} + W_{m+1}(x,T_0,T_2) + cc \quad (12)$$

and inserting it into Eq. (8), the terms related with secularities are discarded. Here $W_{m+1}(x,T_0,T_2)$ stands for the solution related with non-secular terms. One obtains

$$\phi_{m+1}^{iv} - \omega^2\phi_{m+1} - v_p^2\phi_{m+1}'' = -2i\omega(D_2A + \mu A)Y_{m+1} + \frac{3}{2}A^2\bar{A}\left[\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'^{\,2}dx\right]Y_{m+1}'' + \frac{1}{2}F_{m+1}e^{i\sigma T_2} \tag{13}$$

$$\phi_1(0) = \phi_1''(0) = 0\,, \quad \phi_{m+1}(\eta_{m+1}) = \phi_{m+2}(\eta_{m+1}) = 0$$

$$\phi_{m+1}'(\eta_{m+1}) = \phi_{m+1}'(\eta_{m+1})\,, \quad \phi_{m+1}''(\eta_{m+1}) = \phi_{m+1}''(\eta_{m+1})\,,\ \phi_{n+1}(1) = \phi_{n+1}''(1) = 0 \tag{14}$$

Assuming that excitation frequency is close to one of the natural frequencies of the system as shown below

$$\Omega = \omega + \varepsilon^2\sigma(T_2) \tag{15}$$

where σ is a detuning parameter of order 1, the solvability condition for Eqs. (13) and (14) is obtained as follows

$$2i\omega(D_2A + \mu A) + \frac{3}{2}b^2A^2\bar{A} - \frac{1}{2}fe^{i\sigma T_2} = 0 \tag{16}$$

where $\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}^2dx = 1, \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}'^2dx = b^2, \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} F_{r+1}Y_{r+1}dx = f$. The complex amplitude A in Eq. (16) can be written in terms of a real amplitude a and a phase θ

$$A = \frac{1}{2}a(T_2)e^{i\theta T_2} \tag{17}$$

Then amplitude and phase modulation equations can be obtained as follows

$$\omega a D_2\gamma = \omega a\sigma - \frac{3}{16}b^2a^3 + \frac{1}{2}f\cos\gamma\,,\ \omega D_2a = -\omega\mu a + \frac{1}{2}f\sin\gamma \tag{18}$$

where, $\gamma = \sigma T_2 - \theta$. Eq. (18) will be solved for steady-state case in the next section and variation of nonlinear amplitude with forcing will be discussed.

4. NUMERICAL RESULTS

In this section numerical examples for frequencies will be presented for different cases. Firstly, the linear natural frequencies for different tension forces and support locations (η) will be calculated. Then, the non-linear frequencies for free, undamped vibrations will be calculated. For this case, by taking $\mu = f = \sigma = 0$, one obtains

$$D_2a = 0 \text{ and } a = a_0 \text{ (constant)} \tag{19}$$

from Eq. (19). Here a_0 is the steady-state real amplitude. The non-linear frequency is

$$\omega_{nl} = \omega + D_1\theta = \omega + \lambda a_0^2 \tag{20}$$

where $\lambda = (3/16)b^2/\omega$ is the correction coefficient due to nonlinear terms. Up to the second order of approximation, the non-linear frequencies have a parabolic relation with the maximum amplitude of vibration. The linear frequencies and nonlinear correction terms for the first five frequencies are presented in Table 1 for 3 support case, in Table 2 for 4 support case for different axial tension values. The effects of locations of supports and axial tension are given. For the same support locations, the tension increases the frequencies, as expected, as shown in Table 1. For a given value of tension, the effect on the frequencies at different modes varies depending on the location of support. As the support is moved to the center of the beam, the frequencies increase in the first modes for all tension values. In other modes, it depends on the location of the node and closeness of the support to the node.

Table 1. The first five frequencies and corrections due to nonlinear terms for v_p=1, 10 and different η values for three support case.

v_p	η	ω_1	ω_2	ω_3	ω_4	ω_5	λ
1.00	0.1	18.235343	58.671204	122.572995	210.121541	321.332650	1.863198
	0.3	26.743873	86.548540	139.217694	204.494168	339.221550	3.297357
	0.5	39.975291	62.044961	158.412881	200.288248	355.805407	7.310198
10.00	0.1	40.282610	91.944202	161.705195	252.596838	365.883295	0.762287
	0.3	53.251780	125.653101	177.151732	244.979963	384.280263	1.586446
	0.5	74.205035	91.264311	201.812027	238.892053	402.209843	4.090697

Table 2. The first five frequencies and corrections due to nonlinear terms for v_p=1, 10 and different η values for four support case.

v_p	η_1	η_2	ω_1	ω_2	ω_3	ω_4	ω_5	λ
1.00	0.1	0.2	23.000017	74.190483	155.088058	265.891120	406.559884	2.370243
		0.3	28.362907	92.699965	194.072026	321.310065	403.423882	3.114322
		0.4	36.459731	117.944906	182.802245	274.384326	451.627198	4.433622
		0.5	48.759392	104.552386	180.798089	306.735884	396.611039	6.888531
		0.6	58.472939	89.277249	199.013802	282.991786	417.120754	7.763814
		0.7	47.640061	121.011061	163.836619	291.755537	454.602796	3.980270
		0.8	37.611300	105.905053	207.229104	301.928783	385.695187	2.542881
		0.9	30.814771	85.704355	169.469074	282.126340	423.412296	1.844953
	0.3	0.4	39.723634	128.254623	150.488184	272.056753	462.242046	4.245237
		0.5	52.993323	136.331770	177.459798	334.512733	415.990173	6.402350
		0.6	75.163601	123.969190	189.081660	285.850006	459.035825	11.387395
		0.7	82.167472	134.202371	160.811222	307.943490	470.094119	12.710612
10.00	0.1	0.2	46.578915	108.874884	195.279228	309.143078	451.694780	1.067535
		0.3	54.033760	129.789552	236.662486	365.622080	441.079562	1.552245
		0.4	64.892609	157.928266	215.694876	315.984284	497.691033	2.415834
		0.5	80.879356	134.334602	220.656272	348.060667	439.195150	4.057833
		0.6	90.058699	119.213991	238.239592	324.455643	459.663458	4.405195
		0.7	73.090648	160.342677	197.881535	333.011392	500.575273	2.386336
		0.8	60.153227	140.755204	248.819925	343.775104	426.082274	1.509860
		0.9	51.352917	117.829483	207.654395	323.956005	467.646583	1.028337
	0.3	0.4	67.060424	166.398950	185.310096	314.701280	507.952437	2.372516
		0.5	83.566988	174.401463	218.006961	377.578547	457.464405	3.936503
		0.6	110.758293	162.739537	222.041658	326.537094	504.776860	7.439908
		0.7	118.361161	172.726353	192.733377	348.837860	516.101396	8.254400

Nonlinear frequency versus amplitude curves are plotted in Figs. 2-3 for v_p= 1, 10 respectively. In these figures, the variation of nonlinear frequency is plotted for three-support case for the first mode when η=0.1, 0.2, 0.3, 0.4 and 0.5. Nonlinearities are of hardening type. As the third support is located close to the midpoint the frequencies increase which means the beam becomes stiff. In Figs. 2-3, v_p values changed between 1 and 10, an increase in v_p caused an increase in nonlinear frequencies. The behavior in all figures is of hardening type. In Figs 4 and Figs. 5, four-support case and five-support cases are shown respectively for v_p= 1, 10. Again the behaviors are of hardening type.

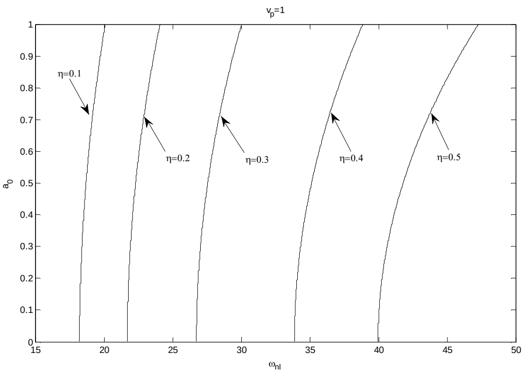

Figure 2. Nonlinear frequency-amplitude variation for v_p= 1 and for different η values.

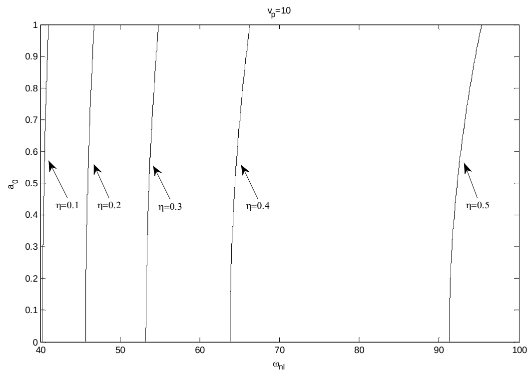

Figure 3. Nonlinear frequency-amplitude variation for v_p= 10 and for different η values.

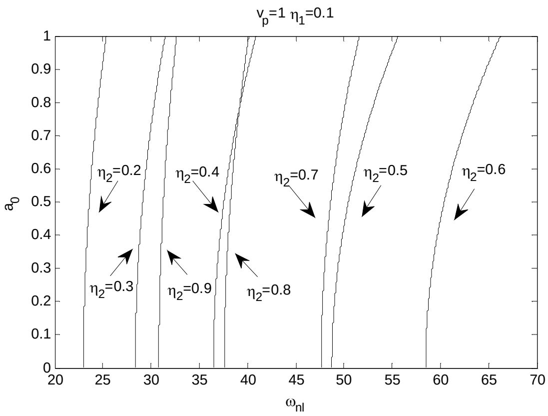

Figure 4. Nonlinear frequency-amplitude variation for v_p= 1 and for η_1=0.1 and η_2=0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 values.

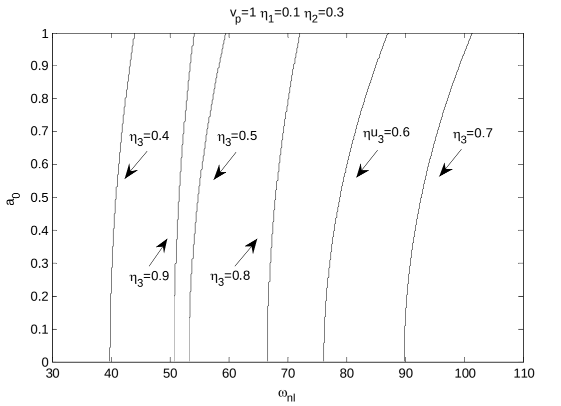

Figure 5.Nonlinear frequency-amplitude variation for v_p= 1 and for η_1=0.1, η_2=0.3 and η_3=0.4, 0.5, 0.6, 0.7, 0.8, 0.9 values.

At the steady state, a′ and γ' become zero. The frequency detuning parameter is as follows

$$\sigma = \frac{3}{16}\frac{a^2 b^2}{\omega} \mp \sqrt{\frac{f^2}{4\omega^2 a^2} - \mu^2} \tag{21}$$

For three-support case, frequency response curves, are plotted in Fig. 6 for v_p=1, respectively. This term is a measure for nearness of the forcing frequency to the natural frequency. In all figures f=1, μ=1 are assumed. In Fig. 6 for v_p=1, vibration amplitudes decrease as the support becomes closer to the midpoint and the behavior is of hardening type. Maximum vibration amplitude is obtained when σ>0 for all mid support locations but at different σ values. Their maxima are obtained at lower frequencies when the support location changed towards ¼ of the beam, then it is obtained at higher frequencies when the support is moved towards midpoint. The jump region becomes smaller as the mid support is located towards middle of the beam. Comparison of Fig.6 shows that, maximum vibration amplitude is obtained at higher frequencies when the support location is close to the ends or midpoint. In Figs. 7-8 and in Fig.9 the curves are shown for four-support and five-support cases respectively. The behavior is of hardening type in all figures and tension decreases the jump region.

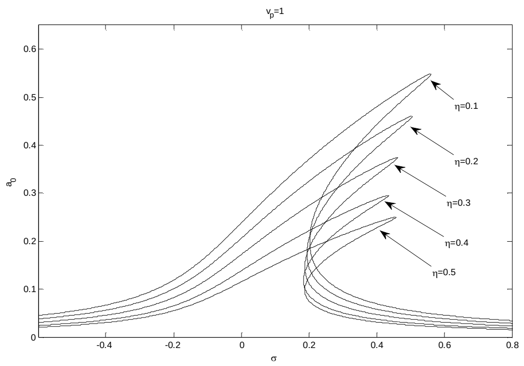

Figure 6. Forcing frequency-amplitude variation for v_p=1 and for different η values.

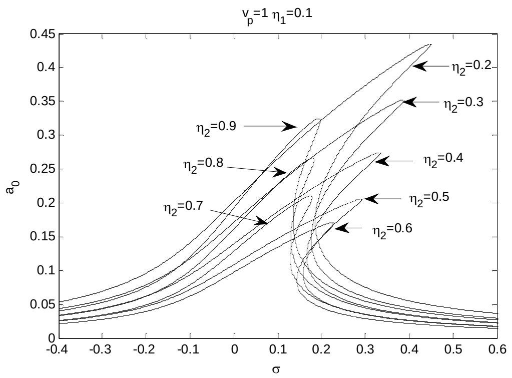

Figure 7. Forcing frequency-amplitude variation for v_p= 1 and for η_1=0.1 and η_2=0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 values.

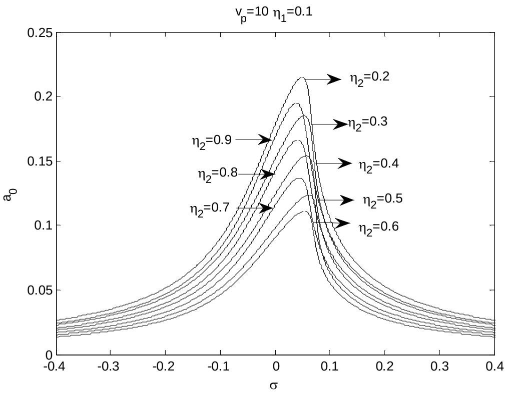

Figure 8. Forcing frequency-amplitude variation for v_p= 10 and for η_1=0.1 and η_2=0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 values.

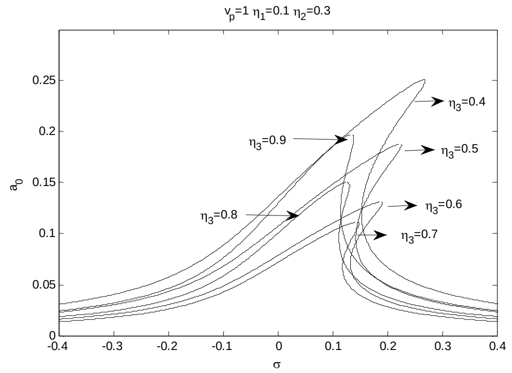

Figure 9. Forcing frequency-amplitude variation for v_p= 1 and for η_1=0.1, η_2=0.3 and η_3=0.4, 0.5, 0.6, 0.7, 0.8, 0.9 values.

5. INTERNAL RESONANCE AND STABILITY

In this section, 3:1 internal resonance case among natural frequencies will be investigated. The solution was carried out for the three-support case only. The following detuning parameters were defined,

$$\Omega=\omega_1+\varepsilon^2\sigma(T_2),\ \ \omega_2=3\omega_1+\varepsilon^2\rho(T_2) \tag{22}$$

The ratios of natural frequencies at different modes for different v_p values are plotted in Figs 10, 11. 3:1 internal resonances between $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ are discussed in these figures and support location necessary for internal resonances are presented. This resonance is possible for v_p=1 and 5, it is not possible for v_p=10. In Fig. 10, for v_p=1, when the mid support is located about η=0.34, 3:1 internal resonance is possible for ω_2/ω_1. The effect of mid support is slightly increasing in the interval η=0-0.32. Replacing the support towards middle of the beam decreases frequency ratio continuously η>0.32. 3:1 is not possible for $\omega_3/\ \omega_2$. Similarly 3:1 internal resonance is possible for $\omega_4/\ \omega_2$ around η=0.23 and η=0.38. In all curves, up to a specific support location there is very slight effect ($\omega_2/\ \omega_1$, η=0-0.30), ($\omega_3/\ \omega_2$, η= 0-0.22), ($\omega_4/\ \omega_2$, η=0-0.18), ($\omega_5/\ \omega_3$, η=0-0.18). In Fig. 11, for v_p=5, 3:1 internal resonance is not possible for $\omega_2/\ \omega_1$. 3:1 internal resonance is possible for $\omega_4/\ \omega_2$ around η=0.21, η=0.40 and η=0.50. In some curves, up to a specific support location there is a very slight effect ($\omega_3/\ \omega_2$, η=0-0.20), ($\omega_5/\ \omega_3$, η=0- 0.15). In Figs. 12, 13, the effect of axial tension is depicted for single frequency ratios. Variation of ω_2/ω_1 with η for different v_p values is depicted in Fig. 12. Increasing v_p values decrease the frequency ratio for all mid support locations for approximately η>0.3. 3:1 internal resonance is possible for v_p=0.1,0.5 and 1 at η= 0.34, but no 3:1 internal resonance is possible for v_p=10. Variation of ω_4/ω_2 with η for different v_p values is depicted in Fig. 13. No 3:1 internal resonances is possible for ω_4/ω_2 for v_p=10. For v_p=0.1, 0.5 and 1, 3:1 internal resonances is possible about η=0.24 and 0.38. No 3:1 internal resonances is possible for ω_3/ω_2 and ω_5/ω_3 for v_p values between 0.1 and 10.

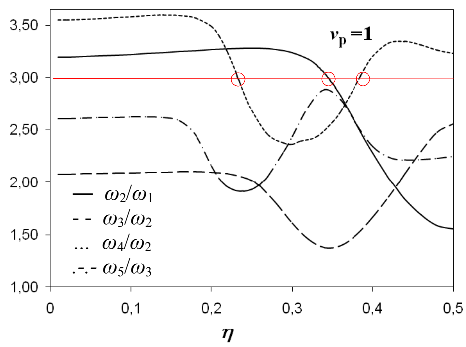

Figure 10. Ratios of different frequencies for v_p=1.

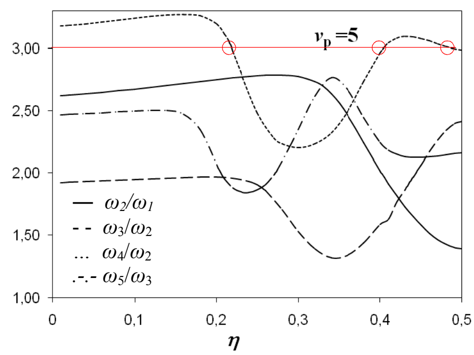

Figure 11. Ratios of different frequencies for v_p=5.

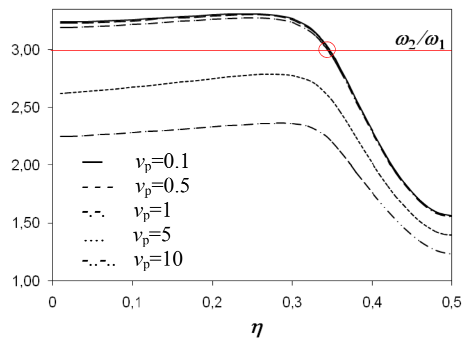

Figure 12. Variation of ω_2/ω_1 with η for different v_p values.

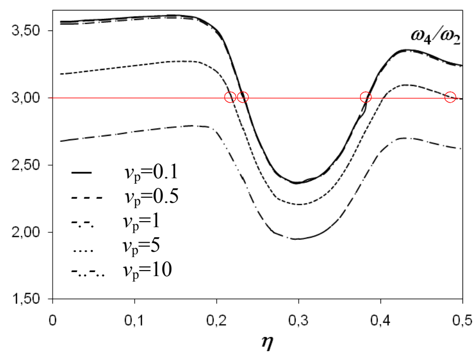

Figure 13. Variation of ω_4/ω_2 with η for different v_p values.

Under the assumption of three-to-one internal resonances, the mode directly excited (ω_1) and indirectly excited through internal resonance (ω_2) will survive and all other modes decay over time due to the damping. The amplitudes can be written as follows

$$w_{1,1}(x,T_0,T_1)=\left(A_1(T_1)e^{i\omega_1 T_0}+cc\right)Y_1(x)+\left(A_2(T_1)e^{i\omega_2 T_0}+cc\right)Y_2(x)$$
$$w_{1,2}(x,T_0,T_1)=\left(A_1(T_1)e^{i\omega_1 T_0}+cc\right)Y_3(x)+\left(A_2(T_1)e^{i\omega_2 T_0}+cc\right)Y_4(x) \tag{23}$$

Inserting this into the O(ε^3), one obtains

$$D_0^2 w_{1,3}+w_{1,3}^{iv}-v_p^2\, w_{1,3}''=-2D_0D_2w_{1,1}-2\mu D_0 w_{1,1}+\frac{1}{2}\left[\int_0^\eta w_{1,1}'^2dx+\int_\eta^1 w_{2,1}'^2dx\right]w_{1,1}''+F_1\cos\Omega T_0$$

$$D_0^2 w_{2,3}+w_{2,3}^{iv}-v_p^2\, w_{2,3}''=-2D_0D_2w_{2,1}-2\mu D_0 w_{2,1}+\frac{1}{2}\left[\int_0^\eta w_{1,1}'^2dx+\int_\eta^1 w_{2,1}'^2dx\right]w_{2,1}''+F_2\cos\Omega T_0 \tag{24}$$

Solution of the equations at this order is as follows

$$w_{1,3}(x,T_0,T_2)=\phi_1(x,T_2)e^{i\omega_1 T_0}+\phi_3(x,T_2)e^{i\omega_2 T_0}+cc+W(x,T_0,T_2)$$
$$w_{2,3}(x,T_0,T_2)=\phi_2(x,T_2)e^{i\omega_1 T_0}+\phi_4(x,T_2)e^{i\omega_2 T_0}+cc+W(x,T_0,T_2) \tag{25}$$

The solvability conditions can be calculated as follows,

$$2i\omega_1(D_2A_1+\mu A_1)+\frac{3}{2}b_1^{\,2}A_1^{\,2}\overline{A}_1+\frac{3}{2}\overline{A}_1^{\,2}A_2e^{i\rho T_2}b_1b_3+A_1A_2\overline{A}_2(b_2b_1+2b_3^{\,2})-\frac{1}{2}fe^{i\sigma T_2}=0$$
$$2i\omega_2(D_2A_2+\mu A_2)+\frac{3}{2}b_2^{\,2}A_2^{\,2}\overline{A}_2+A_1\overline{A}_1A_2(b_2b_1+2b_3^{\,2})+\frac{1}{2}A_1^{\,3}A_2e^{-i\rho T_2}b_1b_3=0 \tag{26}$$

where,

$$b_1=\int_0^\eta Y_1'^2dx+\int_\eta^1 Y_3'^2dx,\ b_2=\int_0^\eta Y_2'^2dx+\int_\eta^1 Y_4'^2dx, b_3=\int_0^\eta Y_1'Y_2'dx+\int_\eta^1 Y_3'Y_4'dx,$$
$$\int_0^\eta F_1Y_1dx+\int_\eta^1 F_2Y_2dx=f,\int_0^\eta Y_1^2dx+\int_\eta^1 Y_3^2dx=1,\ \int_0^\eta Y_2^2dx+\int_\eta^1 Y_4^2dx=1, \tag{27}$$

The complex amplitudes can be written in polar form

$$A_1=\frac{1}{2}a_1(T_2)e^{i\theta_1(T_2)},\ \ A_2=\frac{1}{2}a_2(T_2)e^{i\theta_2(T_2)} \tag{28}$$

One finally obtains frequency modulation equations for steady state solutions, $D_2a_1=D_2a_2=D_2\gamma=D_2\beta=0$, then Eq. (26) becomes

$$\omega_1a_1\sigma-\frac{3}{16}a_1^{\,3}b_1^{\,2}-\frac{3}{16}a_1^{\,2}a_2b_1b_3\cos\beta-\frac{1}{8}a_1a_2^{\,2}(b_1b_2+2b_3^{\,2})+\frac{1}{2}f\cos\gamma=0$$
$$-\mu\omega_1a_1-\frac{3}{16}a_1^{\,2}a_2b_1b_3\sin\beta+\frac{1}{2}f\sin\gamma=0$$
$$\frac{3}{16}a_2^{\,3}b_2^{\,2}+\frac{1}{8}a_1^{\,2}a_2(b_1b_2+2b_3^{\,2})+\frac{1}{16}a_1^{\,3}b_1b_3\cos\beta=\omega_2a_2(3\sigma-\rho)$$
$$\frac{1}{16}a_1^{\,3}b_1b_3\sin\beta-\omega_2\mu a_2=0 \tag{29}$$

where $\gamma=\sigma T_2-\theta_1, \beta=\theta_2-3\theta_1+\rho T_2, D_2\theta_2=D_2\beta+3(\sigma-D_2\gamma)-\rho$

By evaluating the eigenvalues of the jacobian matrix, stability can be determined. Eigenvalues should not have positive real parts to maintain stability. External excitation frequency response graphs are given in Figs. 15, 16 for ω_1=29.613795, ω_2=8.16346, μ=0.075 and f=1 for three-support case (3:1 internal resonance: ω_2/ω_1= 3.0027). External excitation frequency is applied to the first mode and responses are calculated for the first and second modes. The second mode is activated for $\sigma \geq 0$.The frequency amplitude response curves are shown in Figs. 17, 18 for ω_1=29.613795, ω_2=88.921622, μ=0.075 and σ=0.2 for three-support case. As shown in the figures, different forcing amplitudes are applied to the first mode, the second mode amplitude is determined. In these figures v_p=1 and η=0.344. The parameters defined in Eq. (22) and Eq. (27) and used here are ρ=0.0802, b_1=41.2196, b_2=59.4933, b_3= - 21.5592

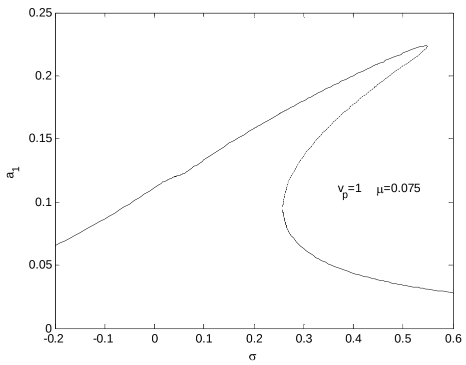

Figure 14. Forcing frequency-1st mode amplitude variation for v_0=1 and μ=0.075.

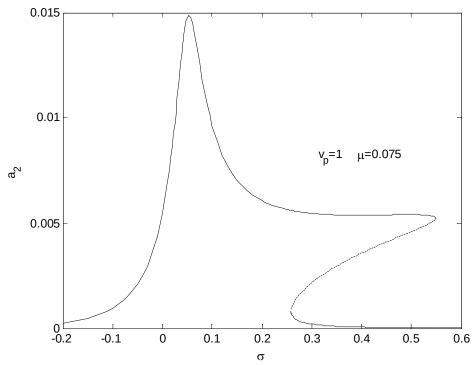

Figure 15. Forcing frequency-2st mode amplitude variation for v_0=1 and μ=0.075.

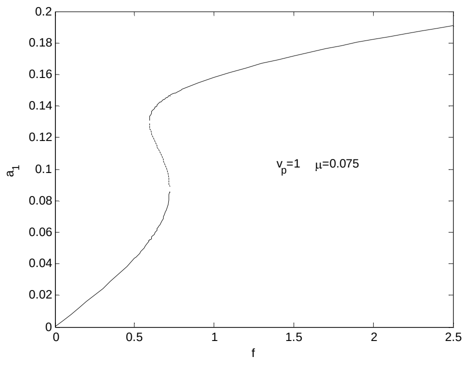

Figure 16. Forcing amplitude-1st mode amplitude variation for v_0=1 and μ=0.075.

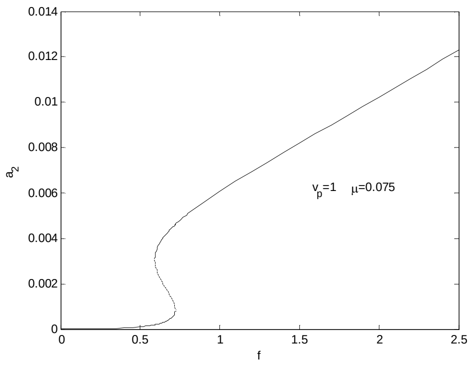

Figure 17. Forcing amplitude-2nd mode amplitude variation for v_0=1 and μ=0.075.

6. CONCLUDING REMARKS

The transverse vibrations of a tensioned Euler-Bernoulli beam having multiple supports are investigated. The non-linear equations of motion including stretching of the neutral axis due to immovable end conditions are derived. The method of multiple scales is applied to obtain approximate solutions. Exact solutions and numerical values for natural frequencies are given for linear problem. For the non-linear problem, correction terms to linear problem are obtained. Non-linear free and forced vibrations are investigated in detail. The effects of the positions of supports and axial tension are determined. The corrections increase as the number of supports increase and natural frequencies increase always. Stretching of the neutral axis causes a non-linearity of hardening type. For forced and damped vibrations, since the non-linearity is of hardening type, the frequency-response curves are bent to right, causing an increase in the multi–valued regions. When support number is increased, the multi–valued regions and maximum amplitude decrease. Then 3:1 internal resonances are investigated. Support locations and tension values producing internal resonances for three-support case are determined. No 3:1 internal resonance is possible for ω_3/ω_2 and ω_5/ω_3 for v_p values between 0.1 and 10. For other cases, it is possible for some values. Frequency-response and force-response curves are plotted. External excitation frequency is applied to the first mode and responses are calculated for the first and second modes. Finally stability analysis is performed and the borders are drawn.

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