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THE VARIATIONAL APPROACH COUPLED WITH AN ANCIENT CHINESE MATHEMATICAL METHOD TO THE RELATIVISTIC OSCILLATOR

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Abstract- This paper applies the variational approach to the relativistic oscillator. In order to effectively deal with the irrational term, an ancient Chinese mathematics is introduced. Comparison of the obtained result with the numerical one elucidates the efficiency of the present treatment.

Keywords: Variational principle, nonlinear oscillator, He Chengtian's interpolation, homotopy

1. INTRODUCTION

In this paper, we consider the following relativistic oscillator:

$$u'' + \frac{u}{\sqrt{1+u^2}} = 0 \tag{1}$$

with initial conditions $u(0) = A, u'(0) = 0$.
Recently many analytical methods were proposed to solve various nonlinear oscillators, such as the parameter-expansion method[1-3], the energy balance method[4,5], the harmonic balance method[6,7,8], the homotopy perturbation method[9], He's amplitude-frequency formulation[10,11], a complete review on analytical approach to nonlinear oscillators was given in Ref.[12]. In this paper, we will couple the variational approach[13] with an ancient Chinese mathematics called the He Chengtian's interpolation[14] to Eq.(1).

2. HE'S VARIATIONAL APPROACH

The variational approach to nonlinear oscillators was first proposed by Ji-Huan He[12,13], and widely used to search for periodic solutions of various nonlinear oscillators[16-17]. According to Ref.[13], a variational principle for Eq.(1) can be easily established, which reads

$$J(u) = \int_0^{T/4} \{-\frac{1}{2}u'^2 + \sqrt{1+u^2}\}dt \tag{2}$$

where T is the period of the nonlinear oscillator.
We assume that its approximate solution can be expressed as:

$$u = A\cos\omega t \tag{3}$$

where A and ω are the amplitude and frequency of the oscillator, respectively.
Substituting Eq.(3) into Eq.(2), and setting $dJ/dA = 0$, we can obtain an inexplicit amplitude-frequency relationship of Eq.(1). In order to obtain a simple amplitude-frequency relationship, an ancient Chinese mathematics is adopted.

3. HE CHENGTIAN'S INTERPOLATION

For convenience we set

$$y = \int_0^{T/4} \sqrt{1+u^2}\,dt \tag{4}$$

Substituting (3) into (4) leads to

$$y = \int_0^{T/4} \sqrt{1+A^2\cos^2\omega t}\,dt = \frac{1}{\omega}\int_0^{\pi/2} \sqrt{1+A^2\cos^2 t}\,dt \tag{5}$$

Hereby we will introduce He Chengtian's interpolation[14] to approximate the above integral. He Chengtian's interpolation, an ancient Chinese mathematics, was developed to the max-min method[18-22] for nonlinear oscillators.
By a simple analysis, we know that

$$\frac{\pi}{2\omega} \le y \le \frac{\pi}{2\omega}\sqrt{1+A^2} \tag{6}$$

We rewrite Eq.(6)in the following form in order to apply He Chengtian's interpolation easily.

$$\left(\frac{\pi}{2\omega}\right)^2 \le y^2 \le \left(\frac{\pi}{2\omega}\right)^2 \left(1+A^2\right) \tag{7}$$

According to He Chengtian's interpolation, we obtain

$$y^2 = \left(\frac{\pi}{2\omega}\right)^2 \frac{m+n(1+A^2)}{m+n} = \left(\frac{\pi}{2\omega}\right)^2 \left(1+\frac{n}{m+n}A^2\right) = \left(\frac{\pi}{2\omega}\right)^2 \left(1+kA^2\right) \tag{8}$$

where m and n are positive parameters, k=n/(m+n).
We, therefore, can approximate y in the form

$$y = \frac{\pi}{2\omega}\sqrt{1+kA^2} \tag{9}$$

From Eq.(9) we can easily obtain

$$\lim_{A\to 0} y = \frac{\pi}{2\omega} \tag{10}$$

and

$$\lim_{A\to\infty} y = \frac{\pi}{2\omega}\sqrt{k}\,A \tag{11}$$

From Eq.(5),we know that

$$\lim_{A\to\infty} y = \frac{1}{\omega}\int_0^{\pi/2} A\cos t dt = \frac{A}{\omega} \tag{12}$$

Comparing Eq.(11) with Eq.(12), we obtain

$$\frac{\pi}{2\omega}\sqrt{k}A = \frac{A}{\omega} \tag{13}$$

from which the value of k in Eq.(9) can be determined, which is

$$k = \frac{4}{\pi^2} \tag{14}$$

We obtain the approximate value of y :

$$y = \frac{\pi}{2\omega}\sqrt{1+\frac{4}{\pi^2}A^2} \tag{15}$$

Substituting $u = A\cos\omega t$ into Eq.(2), and making the resultant equation stationary with respect to A , we obtain the following equation

$$\int_0^{\pi/2} \left(-A\omega\sin^2 t\right)dt + \frac{\pi}{2\omega}\frac{\frac{4}{\pi^2}\cdot 2A}{2\cdot\sqrt{1+\frac{4}{\pi^2}A^2}} = 0 \tag{16}$$

or

$$-\frac{\pi}{4}A\omega + \frac{\pi}{2\omega}\frac{\frac{4}{\pi^2}\cdot 2A}{2\cdot\sqrt{1+\frac{4}{\pi^2}A^2}} = 0 \tag{17}$$

The frequency-amplitude relationship is, therefore, obtained:

$$\omega = \frac{2\sqrt{2}}{\pi}\left(1+\frac{4}{\pi^2}\cdot A^2\right)^{-\frac{1}{4}} \tag{18}$$

The approximate solution can be expressed in the form:

$$u(t) = A\cos\left(\frac{2\sqrt{2}}{\pi}\left(1+\frac{4}{\pi^2}\cdot A^2\right)^{-\frac{1}{4}}t\right) \tag{19}$$

When comparing with the exact solution, we find Eq.(19) is valid for large A, error arises for small A. This is due to the fact that we identify k in Eq.(9) approximately for $A\to\infty$. However, we can identify k in Eq.(9) approximately for the case when $A\to 0$. For small A, Eq.(5) can be approximated as follows

$$y=\frac{1}{\omega}\int_0^{\frac{\pi}{2}}\sqrt{1+A^2\cos^2 t}\,dt=\frac{1}{\omega}\int_0^{\frac{\pi}{2}}\left(1+\frac{1}{2}A^2\cos^2 t\right)dt=\frac{\pi}{2\omega}\left(1+\frac{1}{4}A^2\right) \tag{20}$$

Expanding Eq.(9) for $A<<1$, we have

$$y=\frac{\pi}{2\omega}\sqrt{1+kA^2}=\frac{\pi}{2\omega}(1+\frac{1}{2}kA^2) \tag{21}$$

Comparison of Eq.(20) with Eq.(21) results in k=1/2. Thus, for small A, Eq.(9) can be approximated as

$$y=\frac{\pi}{2\omega}\sqrt{1+\frac{1}{2}A^2} \tag{22}$$

By the same solution procedure as illustrated above, we obtain

$$-\frac{\pi}{4}A\omega+\frac{\pi}{2\omega}\cdot\frac{A}{2\sqrt{1+\frac{1}{2}A^2}}=0 \tag{23}$$

The frequency-amplitude relationship reads

$$\omega=\left(1+\frac{1}{2}A^2\right)^{-\frac{1}{4}} \tag{24}$$

Eq.(24) agrees well with the exact solution for the case $A<<1$.

4. HOMOTOPY MATCHING

Eq.(18) is valid for the case when $A\to\infty$; while Eq.(24) is valid for the case when $A\to 0$. In order to match the both cases $A\to 0$ and $A\to\infty$, we construct the following homotopy

$$\tilde{\omega}=e^{-\alpha A}f(A)+(1-e^{-\alpha A})g(A) \tag{25}$$

where $f(A)=\frac{2\sqrt{2}}{\pi}\left(1+\frac{4}{\pi^2}\cdot A^2\right)^{-\frac{1}{4}}$, $g(A)=\left(1+\frac{1}{2}A^2\right)^{-\frac{1}{4}}$, and α is a free parameter. Now considering the case when A=1, we have exact frequency, which is $\tilde{\omega}=0.8736$. From this relationship, we can identify α as follows

$$\alpha=0.4962 \tag{26}$$

Finally we obtain the following result

$$\tilde{\omega}=e^{-0.4962A}\cdot\frac{2\sqrt{2}}{\pi}\left(1+\frac{4}{\pi^2}\cdot A^2\right)^{-\frac{1}{4}}+(1-e^{-0.496A})\cdot\left(1+\frac{1}{2}A^2\right)^{-\frac{1}{4}} \tag{27}$$

Fig. 1 shows that the approximate solution, $u=A\cos\tilde{\omega}t$, agrees very well with the exact solution for all A>0.

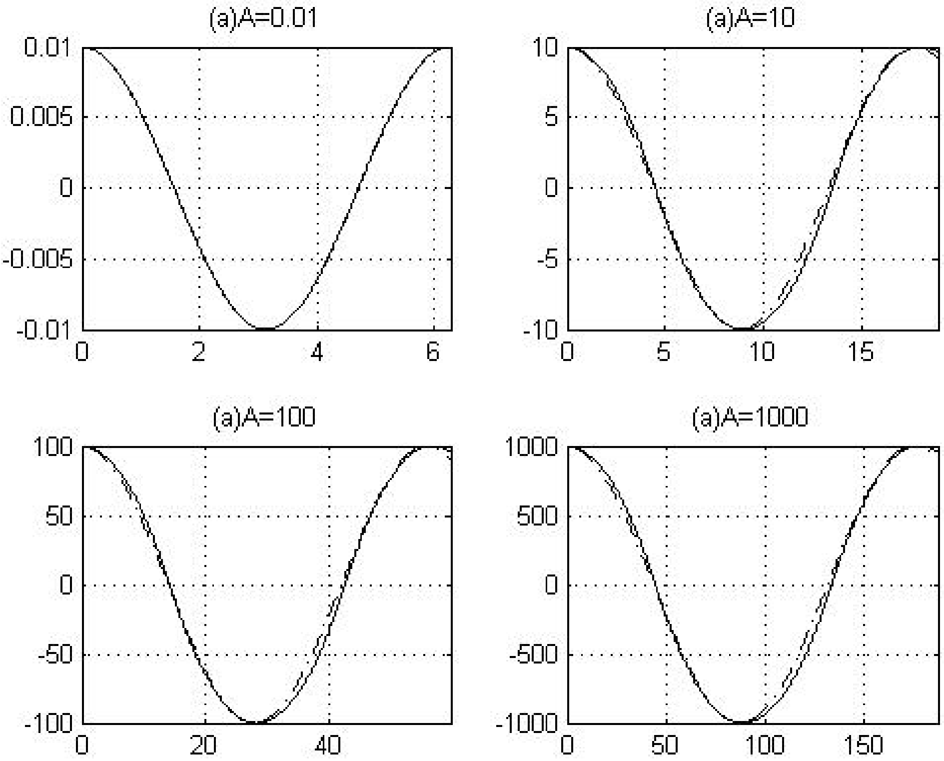

Fig.1 Comparison of the approximate solution, $u = A\cos\tilde{\omega}t$,with the numerical solution. Continuous line: exact solution; dashed line: approximate one.

5. CONCLUSIONS

In this paper we successfully incorporate the He Chengtian's interpolation into the variational approach. the He Chengtian's interpolation leads to thresholds of the frequency of the oscillators, the homotopy technology is then used to match the two thresholds, and the solution is valid for all solution domain.

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