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ON THE REDEFINITION OF THE VARIATIONAL AND 'PARTIAL' VARIATIONAL CONSERVATION LAWS IN A CLASS OF NONLINEAR PDEs WITH MIXED DERIVATIVES

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Abstract- The construction of conserved vectors using Noether's and partial Noether's theorems are carried out for high order PDEs with mixed derivatives. The resultant conserved flows display some interesting 'divergence properties' owing to the existence of the mixed derivatives. These are spelled out for various equations from mathematical physics.

Keywords- Noether's theorem, mixed derivatives, conservation laws.

1. INTRODUCTION

When considering the construction of conservation laws via Noether's theorem using a Lagrangian or a 'partial Lagrangian', an interesting situation arises when the equations under investigation are such that the highest derivative term is mixed; the mixed derivative term is the one that involves differentiation by more than one of the independent variables. When substituting the conserved flow back into the divergence relationship, a number of 'extra' terms (on which the Euler operator vanishes) arise. Thus, we have essentially 'trivial' conserved quantities that need to be fed back into the conserved vectors that are computed initially via Noether's theorem - these are necessary terms that may guarantee the notion of 'association' between conserved flows and symmetries (see [6, 1, 2]) - otherwise, the total divergence of the computed conserved flows are the equations modulo the trivial part. A variety of high order equations are studied. For example, we consider the fourth-order Shallow Water Wave equations and the Camassa-Holms, Hunter-Saxton, Inviscid Burgers and KdV family of equations. These equations have their importance in many areas of physics, and real world applications, e.g., tsunamis which are characterized with long periods and wave lengths as a result they behave as shallow-water waves. We firstly present the notation and preliminaries that will be used.

Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^{\mu}(x, u, u_{(1)}, \ldots, u_{(r)}) = O, \quad \mu = 1, \ldots, \tilde{m}, \tag{1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, …, rth-order partial derivatives, that is, $u_i^{\alpha} = D_i(u^{\alpha}), u_{ij}^{\alpha} = D_j D_i(u^{\alpha}), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \ldots, \quad i = 1, \ldots, n, \tag{2}$$

where the summation convention is used whenever appropriate.

A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{3}$$

along the solutions of (1).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots, \tag{4}$$

where ξ^i, $\eta^{\alpha} \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \\ &\ \vdots \end{aligned} \tag{5}$$

and W^{α} is the Lie characteristic function defined by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{6}$$

Here, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u.

The Euler-Lagrange equations, if they exist, associated with (1) are the system $\delta L/\delta u^{\alpha} = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^{\alpha}$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^{\alpha}}, \quad \alpha = 1, \ldots, m. \tag{7}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_{\Omega} L(x, u, u_{(1)}, \ldots, u_{(r)}) \mathrm{d}x \tag{8}$$

defined over Ω. If we include point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries X are given by

$$XL + LD_i\xi^i = D_i f_i. \tag{9}$$

Corresponding to each X, a conserved flow is obtained via Noether's Theorem.

For partial Lagrangians (see [7]), L, the Noether type generators, X, are determined by

$$XL + LD_i\xi^i = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i f_i \tag{10}$$

and the conserved vector from the expression as in Noether's theorem (see [9]).

2. APPLICATIONS

2.1. The Shallow Water Wave Equation

The shallow water wave equation (SWW), models basic water waves that reasonably approximates the behavior of real ocean waves, viz.,

$$u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0, \tag{11}$$

where α and β are arbitrary constants. From the equation (11), we separate the cases, (1) $\alpha \neq \beta$ and (2) $\alpha = \beta$.

Case (1) $\alpha \neq \beta$, will be referred to as shallow water wave-1 (SSW-1), and corresponding to the case (2) $\alpha = \beta$, in (11), α is replaced by β, and referred to as the shallow water wave-2 (SSW-2), viz.,

$$u_{xxxt} + \beta u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0. \tag{12}$$

2.1.1. Shallow Water Wave-1 (SSW-1)

Here, we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t - \frac{1}{2}\beta u_t u_x^2, \tag{13}$$

for which

$$\frac{\delta L}{\delta u} = (2\beta - \alpha)u_{tx}u_x. \tag{14}$$

Substituting into (10) and separating by monomials, we obtain two cases that emerge, (a) $\alpha = 2\beta$ and (b) $\alpha \neq 2\beta$.

Subcase (a): $\alpha = 2\beta$ leads to the following generators and conserved vectors.

(i) $X = \partial_t, \quad W = -u_t$

The conserved flow is given by

$$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$$

and

$$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}.$$

The divergence becomes

$$D_t(T^1) + D_x(T^2) = \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}. \tag{15}$$

We observe that extra terms emerge. By some adjustments, these terms can be absorbed into the conservation law. That is,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\ &= \tfrac{1}{2}D_t(u_t u_{xxx}) - \tfrac{1}{2}D_x(u_{xx}u_{tt}) \end{aligned} \tag{16}$$

Taking these terms across and including them into the conserved flows, we get

$$D_t(T^1 - \tfrac{1}{2}u_t u_{xxx}) + D_x(T^2 + \tfrac{1}{2}u_{xx}u_{tt}) = 0 \tag{17}$$

The modified conserved quantities are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$, modulo the equation. Then,

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx}, \\ &= \tfrac{1}{2}u_x^2 \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt}, \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 \end{aligned} \tag{18}$$

We have a similar situation below.

(ii) $X = \partial_x, \quad W = -u_x$

The conserved flow is given by

$$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$$

and

$$T^2 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2 \beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$$

so that a redefinition leads to

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2, \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 \\ \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2\beta + u_x u_{xxt} \end{aligned} \tag{19}$$

Subcase (b): $\alpha \neq 2\beta$. The symmetry generators and conserved vectors are

$$\text{(i)}\;\; X = \partial_u, \quad B^1 = \frac{1}{2}u_x^2(2\beta - \alpha), \quad B^2 = 0, \quad W = 1.$$

The conserved flow is given by

$$T^1 = \frac{1}{2}u_x - \frac{1}{2}\beta u_x^2 - \frac{1}{2}u_{xxx} + \frac{1}{2}u_x^2(2\beta - \alpha)$$

and

$$T^2 = u_x + \frac{1}{2}u_t - u_t u_x \beta - u_{xxt}$$

for the total divergence is

$$D_t(T^1) + D_x(T^2) = -\tfrac{1}{2}u_{xxxt} \tag{20}$$

From the equation (20), u_{xxxt} has two derivative consequences,

$$\begin{aligned} u_{xxxt} &= D_t(u_{xxx}), \\ &= D_x(u_{xxt}), \end{aligned} \tag{21}$$

which leads to two possible forms of the same conserved quantity, viz.,

$$\begin{aligned} \tilde{T}_1^1 &= T^1 + \tfrac{1}{2}u_{xxx}, \\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 + \tfrac{1}{2}u_x^2(2\beta - \alpha) \\ \\ \tilde{T}_1^2 &= T^2, \\ &= u_x + \tfrac{1}{2}u_t - u_t u_x\beta - u_{xxt} \end{aligned} \tag{22}$$

or

$$\begin{aligned} \tilde{T}_2^1 &= T^1, \\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha) \\ \tilde{T}_2^2 &= T^2 + \tfrac{1}{2}u_{xxt}, \\ &= u_x + \tfrac{1}{2}u_t - u_t u_x\beta - \tfrac{1}{2}u_{xxt} \end{aligned} \tag{23}$$

(ii) $X = \partial_x, \quad B^1 = \frac{1}{3}u_x^3(2\beta - \alpha), \quad B^2 = 0, \quad W = -u_x$

We get

$$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 - \frac{1}{3}u_x^3(2\beta - \alpha) - \frac{1}{2}u_x u_{xxx}$$

and

$$T^2 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2\beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}, \tag{24}$$

so that

$$\begin{aligned} \tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2, \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 - \tfrac{1}{3}u_x^3(2\beta - \alpha) \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2\beta + u_x u_{xxt} \end{aligned} \tag{25}$$

2.1.2. Shallow Water Wave-2 (SSW-2)

For equation (12), we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t - \frac{1}{2}\beta u_t u_x^2, \tag{26}$$

so that

$$\frac{\delta L}{\delta u} = \beta u_{tx}u_x. \tag{27}$$

The separation of monomials after substitution in (10) gives rise to a splitting $\beta \neq 0$ or $\beta = 0$. If $\beta \neq 0$, we have a trivial solution, and if $\beta = 0$, then equation (12) changes to

$$u_{xxxt} - u_{tx} - u_{xx} = 0 \tag{28}$$

and the partial Lagrangian (26) becomes a standard Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t, \tag{29}$$

and the conserved quantities are as follows:

(i) $X = \partial_t, \quad W = -u_t$

The conserved quantities

$$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$$

and

$$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt},$$

lead to a redefinition

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx}, \\ &= \tfrac{1}{2}u_x^2 \\ \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt}, \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2. \end{aligned} \tag{30}$$

(ii) $X = \partial_x, \quad W = -u_x$

Similarly, we obtain

$$T^1 = -\frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$$

and

$$T^2 = -\frac{1}{2}u_x^2 + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$$

so that

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2, \\ &= -\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{xx}^2 \\ \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= -\tfrac{1}{2}u_x^2 + u_x u_{xxt} \end{aligned} \tag{31}$$

2.2. Camassa-Holms, Hunter-Saxton, Inviscid Burgers and KdV family of equations

We now consider the family of equations

$$\alpha(v_t + 3vv_x) - \beta(v_{txx} + 2v_x v_{xx} + vv_{xxx}) - \gamma v_{xxx} = 0. \tag{32}$$

Even though it represents a class of nonlinear evolution equations, it displays variational/Hamiltonian properties and would then be subject to, amongst other things, Noether's theorem [10]. This is well documented in the case of the KdV equation [5]. Also, it displays interesting soliton or soliton like solutions. Equation (32) represents a version of the KdV equation ($\alpha = 1, \beta = 0$), the Camassa-Holm equation ($\alpha = 1, \beta = 1$), the Hunter-Saxton equation ($\alpha = 0, \beta = 1$) and the inviscid Burgers equation $u_t + 3uu_x = 0$ [3, 4, 8]. We modify this equation by letting $v = u_x$ to obtain

$$\alpha(u_{tx} + 3u_x u_{xx}) - \beta(u_{txxx} + 2u_{xx}u_{xxx} + u_x u_{xxxx}) - \gamma u_{xxxx} = 0. \tag{33}$$

Equation (33) displays variational properties with respect to the Lagrangian

$$L = -\frac{\alpha}{2}(u_x u_t + u_x^3) - \frac{\beta}{2}(u_x u_{xx}^2 + u_{tx}u_{xx}) - \frac{\gamma}{2}u_{xx}^2. \tag{34}$$

The symmetries and corresponding conserved vectors are

(i) $X = \partial_t, \quad W = -u_t$

The conserved quantities are

$$T^1 = -\frac{\alpha}{2}u_x^3 - \frac{\beta}{2}u_x u_{xx}^2 - \frac{\gamma}{2}u_{xx}^2 - \frac{\beta}{2}u_t u_{xxx}$$

and

$$\begin{aligned} T^2 = {} & \frac{\alpha}{2}u_t^2 + \frac{3\alpha}{2}u_t u_x^2 - \frac{\beta}{2}u_t u_{xx}^2 - \beta u_t u_{txx} - \beta u_t u_x u_{xxx} \\ & -\gamma u_t u_{xxx} + \frac{\beta}{2}u_{tt}u_{xx} + \beta u_x u_{xx}^2 + \frac{\beta}{2}u_{tx}u_{xx} + \gamma u_{xx}^2 \end{aligned}$$

The total divergence is

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= 2\gamma u_{xx}u_{xxx} - \gamma u_{tx}u_{xxx} - \gamma u_{xx}u_{txx} + \tfrac{1}{2}\beta u_{tx}u_{xxx} + \beta u_{xx}^3 \\ &+ \tfrac{1}{2}\beta u_{xx}u_{txx} - \beta u_{tx}u_{txx} + 2\beta u_x u_{xx}u_{xxx} - bu_x u_{xx}u_{txx} \\ &- \tfrac{\beta}{2}u_t u_{txxx} + \tfrac{\beta}{2}u_{xx}u_{ttx} - \tfrac{\beta}{2}u_{tx}u_{txx} - \beta u_x u_{tx}u_{xxx} \end{aligned} \tag{35}$$

As before, extra terms that require further analysis emerge. By making an adjustment to these terms, they can be absorbed into the conservation law if we note that

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_x(\gamma u_{xx}^2) - D_x(\gamma u_{tx}u_{xx}) + D_x(\tfrac{\beta}{2}u_{tx}u_{xx}) \\ &- D_x(\tfrac{\beta}{2}u_t u_{txx}) + D_x(\beta u_x u_{xx}^2) - D_x(u_x u_{tx}u_{xx}) \\ &- D_x(\tfrac{\beta}{2}u_{tx}^2) + D_t(\tfrac{\beta}{2}u_{tx}u_{xx}). \end{aligned} \tag{36}$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law. The modified conserved quantity are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$ along the equation, viz.,

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{\beta}{2}u_{tx}u_{xx}, \\ \tilde{T^2} &= T^2 - \gamma u_{xx}^2 + \gamma u_{tx}u_{xx} - \tfrac{\beta}{2}u_{tx}u_{xx} \\ &+ \tfrac{\beta}{2}u_t u_{txx} - \beta u_x u_{xx}^2 - u_x u_{tx}u_{xx} + \tfrac{\beta}{2}u_{tx}^2 \end{aligned} \tag{37}$$

(ii) $X = \partial_x, \quad W = -u_x$

With

$$T^1 = \frac{\alpha}{2}u_x^2 - \frac{\beta}{2}u_x u_{xxx} + \frac{\beta}{2}u_{xx}^2$$

and

$$T^2 = -\frac{\beta}{2}u_{tx}u_{xx} + \frac{\gamma}{2}u_{xx}^2 + \alpha u_x^3 - \beta u_x u_{txx} - \beta u_x^2 u_{xxx} - \gamma u_x u_{xxx} + \beta u_{tx}u_{xx}$$

we get

$$D_t(T^1) + D_x(T^2) \quad = \quad -\tfrac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx}), \tag{38}$$

so that, since $-\frac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx})$ has derivative consequences, $-\frac{1}{2}\beta(D_x(u_x u_{txx}) - D_t(u_{xx}^2))$, so that a redefinition leads to

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}\beta u_{xx}^2, \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}\beta u_x u_{xx}, \end{aligned} \tag{39}$$

(iii) $X = n(t)\partial_u, \quad W = n(t)$

Here, we get

$$T^1 = -\frac{\alpha}{2}n(t)u_x + \frac{\beta}{2}n(t)u_{xxx}$$

and

$$\begin{aligned} T^2 = &-\frac{\alpha}{2}n(t)u_x u_t - \frac{\alpha}{2}n(t)u_x^3 - \frac{\beta}{2}n(t)u_x u_{xx}^2 - \frac{\beta}{2}n(t)u_{tx}u_{xx} \\ &-\frac{\gamma}{2}n(t)u_{xx}^2 - \frac{\beta}{2}n_t(t)u_{xx} + \frac{\alpha}{2}n_t(t)u, \end{aligned}$$

so that

$$D_t(T^1) + D_x(T^2) \quad = \quad -\tfrac{1}{2}n(t)\beta u_{txxx}, \tag{40}$$

and

$$\begin{aligned} \tilde{T}_2^1 &= T^1, \\ \tilde{T}_2^2 &= T^2 + \tfrac{1}{2}n(t)\beta u_{txx}. \end{aligned} \tag{41}$$

3.DISCUSSION AND CONCLUSION

We used the Noether identity to find symmetry generators and then conservation laws for some high order equations containing mixed derivatives in the highest term. All the conserved vectors produce extra terms that become essential parts of the constructed conserved vector for the equation in question.

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