Mathematical and Computational Applications, Vol. 15, No. 4, pp. 658-664, 2010.

AN ANALYSIS OF THE SYMMETRIES AND CONSERVATION LAWS OF THE CLASS OF ZAKHAROV-KUZNETSOV EQUATIONS

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Abstract- In this paper, we study and classify the conservation laws of the Zakharov-Kuznetsov equations. It is shown that these can be obtained by studying the interplay between symmetry generators and 'multipliers'. This is, particularly, useful for the higher-order multipliers. As a final note, we include Drinfeld-Sokolov-Wilson system to demonstrate the usefulness of the approach to systems of pdes.

Keywords- Conservation laws, symmetry generators.

1. INTRODUCTION AND BACKGROUND

The class of Zakharov-Kuznetsov equations with power law nonlinearity

$$u_t + au^n u_x + b(u_{xxx} + u_{xyy}) = 0, \tag{1}$$

has recently been a subject of extensive study in *plasma physics*, for e.g., [2, 3, 4, 5]. There are some interesting detailed accounts given in these references. However, none, it seems, categorizes analytic, exact or invariant solutions or studies the underlying conservation laws that are related to or independent of the symmetry properties of the equation. In this paper, an attempt at an analysis of both these aspects of the equation are done.

We include Drinfeld-Sokolov-Wilson system to demonstrate the usefulness of the approach to systems of pdes too.

The use of symmetry properties of a given system of partial differential equations to construct or generate new conservation laws from known conservation laws has been investigated [7, 8].

In this paper, we apply the recently established notion [1] that the symmetry invariance properties of the multipliers lead to a large class of conserved flows that would not be provided by variational techniques or the standard methods especially the higher-order multipliers.

Consider an rth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{2}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{3}$$

where the summation convention is used whenever appropriate.

A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{4}$$

along the solutions of (2).

It can be shown that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{5}$$

holds identically (i.e., off the solution space) for some current Φ^i 'modulo a curl'. When the PDE system is variational, multipliers are variational symmetries. There is a determining system for finding multipliers (and hence conservation laws) for any given PDE system. Then, the conserved density

is determined by a homotopy formula like
$\int_0^1 u\Lambda(t, x, \lambda u, \lambda u_x, \lambda u_{xx}, \ldots) \mathrm{d}\lambda$, where $\Lambda = \frac{\delta}{\delta u}\Phi^t$ and $\frac{\delta}{\delta u}$ is the Euler operator (see [6] for details).

Our method resorts mainly to the following theorem [1].

Theorem 0.1 *If Φ^i is a conserved current with multiplier Q_μ then $\Phi_X^i := pr\hat{X}\Phi^i$ is also a conserved current and has multiplier $Q_\mu^X := Q'_\mu(P) + \hat{R}^*(Q_\mu)$ where $\hat{R}^*$ is the adjoint of the operator $\hat{R}$. In the case of a point symmetry, this becomes $\Phi_X^i = prX\Phi^i + 2\Phi^{[i}D_j\xi^{j]}$ modulo curls and $Q_\mu^X = prXQ_\mu + Q_\mu D_i\xi^i + R^*(Q_\mu)$ where $R = \hat{R} + \xi^i D_i$ ie $prXG^\mu = R(G^\mu)$.*

2. RESULTS

The symmetry and conservation laws structure splits into two cases (i) $n \neq 1$ and (ii) $n = 1$.

(i) It can be shown that the point symmetry generators of (1) for this case is a four-dimensional Lie algebra spanned with basis are time and space translations $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$ and scaling $X_4 = \frac{1}{3}x\partial_x + \frac{1}{3}y\partial_y + t\partial_t - \frac{2}{3n}u\partial_u$.

In this section we construct multipliers that have the form determined by the ray invariance condition in Theorem 1.1, viz., $XQ = (\lambda + R)Q$, where R is determined by the action of X on the PDE and 'some' divergence term.

As a first case, we consider X_2 for which $R = 0$. The invariants of the equation $XQ = \lambda Q$ are given by the system

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}u_t}{0} = \frac{\mathrm{d}u_x}{0} = \frac{\mathrm{d}u_{xx}}{0} = \frac{\mathrm{d}u_{xt}}{0} = \ldots = \frac{\mathrm{d}u_{xxx}}{0} = \ldots = \frac{\mathrm{d}Q}{\lambda Q}$$

so that, for e.g.,

$$Q = e^{\lambda x} f(t, u, \theta, \epsilon, \mu, \kappa, \nu, \eta), \tag{6}$$

where $\theta = u_x$, $\epsilon = u_t$, $\mu = u_{xx}$, $\kappa = u_{xt}$, $\nu = u_{tt}$ and $\eta = u_{xxx}$. Since the Euler operator annihilates a total divergence, i.e., $\frac{\delta}{\delta u} D_i \Phi^i = 0$, we require

$$\frac{\delta}{\delta u}(Q(1)) = 0 \tag{7}$$

wherein we impose the form of Q to be as in (6). The greater the order of the assumed derivative of Q, the more cumbersome the expansion of the left hand side of (7). We have extensively employed the use of software to expand (7),

separate the resultant by monomials and solve the overdetermined system of PDEs - this would otherwise be impossible and the interesting forms of Q and, hence, the conserved flows would be lost (to some extent, the finer details can be seen for the KdV equation in [6]). In summary, we obtain the multipliers

$$Q_1 = u_{yy} + u_{xx} + \frac{a}{b(n+1)} u^{n+1}, \qquad Q_2 = u, \qquad Q_3 = f(y)$$

where $f(y)$ is an arbitrary function of y and each giving rise to corresponding conserved flows. We study the association of the conserved flows with symmetry by studying the action of the symmetries on the multipliers Q_i. Firstly, notice that there are no first-order (in derivatives) multipliers but we do have a second-order multiplier Q_1. This action is enumerated below as

$$X_i(Q_1) = 0, \qquad i = 1, 2, 3$$

$$X_4(Q_1) = -\frac{2}{3}(1 + \frac{1}{n})Q_1, \tag{8}$$

$$
\begin{aligned}
&X_i(Q_2) = 0, \qquad i = 1,2,3 \\
&X_4(Q_2) = -\frac{2}{3n} Q_2,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&X_i(Q_3) = 0, \qquad i = 1,2 \\
&X_3(Q_3) = f'(y) = g(y), \\
&X_4(Q_3) = \frac{1}{3} y f'(y) = h(y).
\end{aligned} \tag{10}
$$

Thus, Q_1 and Q_2 are strictly invariant under X_i for $i = 1,2,3$ but ray invariant under X_4 with $\lambda = -\frac{2}{3}(1+\frac{1}{n})$ and $-\frac{2}{3n}$, respectively. Q_3 is strictly invariant under X_1 and X_2, ray invariant under X_3 but not invariant with respect to X_4. The strict invariant condition is synonymous with the association of the symmetry X_j with the resultant conserved vector from the multiplier Q_k.

As an example, we note that the conserved flow corresponding to Q_1 for $n = \frac{1}{2}$ is

$$
\begin{aligned}
\Phi^x = {} & \frac{1}{6b\sqrt{u}} \big(-2abu_y{}^2 + 4abu(2u_{yy} + 3u_{xx}) + b\sqrt{u}\,(3u_t u_x \\
& + b\left((u_{yy} + u_{xx})(u_{yy} + 3u_{xx}) - u_y(u_{yyy} + u_{xxy})\right)) \\
& + u^{3/2}(12a^2 + b(-3u_{xt} + b(u_{yyyy} + u_{xxyy})))\big), \\
\Phi^x = {} & \frac{1}{6b\sqrt{u}} \big(2abu_y{}^2 + 4abuu_{xy} + \sqrt{u}\,(3u_t u_y + b\,(-u_x(u_{yyy} + u_{xxy}) \\
& + 2(u_{xy}(u_{yy} + u_{xx}) + u_y(u_{xyy} + u_{xxx}))) \\
& + u^{3/2}(3u_{yt} + b(u_{xyyy} + u_{xxxy})))\big), \\
\Phi^t = {} & \frac{u(8a\sqrt{u} + 3b(u_{yy} + u_{xx}))}{6b}
\end{aligned} \tag{11}
$$

(ii) For $n = 1$, we have an additional symmetry $X_5 = at\partial_x + \mathrm{d}_u$ and the calculations for the multipliers yield an additional one $Q_4 = -atu + x$. The action of the X_i's ($i = 1, \ldots, 5$) on Q_4 are as follows,

$$
\begin{aligned}
&X_1(Q_4) = -aQ_2, \\
&X_2(Q_4) = 1, \\
&X_3(Q_4) = 0, \\
&X_4(Q_4) = \tfrac{1}{3} Q_4, \\
&X_5(Q_4) = 0.
\end{aligned} \tag{12}
$$

For this case, therefore, Q_4 is strictly invariant under X_3 and X_5 so that the corresponding conserved vector is associated with X_3 and X_5. Also, as $X_2(Q_4) = f(y) = Q_3$ for f =constant which implies that the multiplier is obtainable by the symmetry action of X_2 on Q_4. Similarly, since $X_1(Q_4) = -aQ_2$, the action of X_1 on Q_4 yields Q_2 so that Q_2 as an independent multiplier can be dispensed with. That is, the conserved vector from Q_2 would not be in the basis set of conservation laws (see [8]).

The components of the conserved vector corresponding to the second-order multiplier Q_1 is

$$\begin{aligned}
\Phi^x &= -\tfrac{1}{6bu}(24abu_x{}^2 + au^3(3a + 4bu_{xy} + 6bu_{xx}) \\
&+2bu(3u_tu_x + u_{yy}(6a + 2bu_{xy} + 3bu_{xx}) \\
&-b(u_{yyy}u_x + 3u_xu_{xyy} - 2u_{xy}u_{xx} - 3u_{xx}{}^2 - 2u_xu_{xxy} + u_y(u_{xyy} \\
&+u_{xxx}))) + 2bu^2(-3u_{xt} + b(2u_{xyyy} + 3u_{xxyy} - u_{xxxy}))), \\
\Phi^y &= -\tfrac{1}{3u}(12au_yu_x + au^3u_{xx} \\
&+u(3u_tu_y - 6au_{xy} + b(u_{yy}u_{xx} + u_{xx}{}^2 + 3u_yu_{xxy} + 3u_yu_{xxx} - u_x(u_{xyy} \\
&+u_{xxx}))) - u^2(3u_{yt} + b(2u_{xxyy} + 3u_{xxxy} - u_{xxxx}))), \\
\Phi^t &= -\tfrac{u(au^2+3b(u_{yy}+u_{xx}))}{3b},
\end{aligned}$$

Q_2 is

$$\begin{aligned}
\Phi^x &= \tfrac{1}{6}(2au^3 - b(u_y{}^2 + 3u_x{}^2) + 2bu(u_{yy} + 3u_{xx})), \\
\Phi^y &= -\tfrac{1}{3}b(u_yu_x - 2uu_{xy}), \\
\Phi^t &= \tfrac{1}{2}u^2,
\end{aligned}$$

Q_3 is

$$\begin{aligned}
\Phi^x &= \tfrac{1}{6}(2b(uf'' - f'u_y) + f(3au^2 + 2b(u_{yy} + 3u_{xx}))), \\
\Phi^y &= -\tfrac{1}{3}b(f'u_x - 2fu_{xy}), \\
\Phi^t &= fu,
\end{aligned}$$

and Q_5 is

$$\begin{aligned}
\Phi^x &= \tfrac{1}{6}(3axu^2 - 2a^2tu^3 - 2abtu(u_{yy} + 3u_{xx}) \\
&+b(atu_y{}^2 + 2xu_{yy} - 6u_x + 3atu_x{}^2 + 6xu_{xx})), \\
\Phi^y &= \tfrac{1}{3}b(u_y(-1 + atu_x) + 2(x - atu)u_{xy}), \\
\Phi^t &= xu - \tfrac{1}{2}atu^2.
\end{aligned}$$

Notes. - systems example.

The Drinfeld-Sokolov-Wilson system

$$\begin{aligned}
&u_t + 2vv_x = 0 \\
&v_t - av_{xxx} + 3bu_xv + 3kuv_x = 0
\end{aligned}$$

admits a three-dimensional Lie point symmetry algebra spanned by

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = -2u\partial_u - 2v\partial_v + 3t\partial_t + x\partial_x$$

with commutator table

	X_1	X_2	X_3
X_1	0	0	$3X_1$
X_2	0	0	X_2
X_3	$-3X_1$	$-X_2$	0

Its zero order multipliers were shown to be $(1,0)$, $(0,1)$ and $(\frac{3}{2}bu, v)$ ([9])- the first two are of minimal interest as the corresponding conserved flow yield one equation in the system. Further detailed calculations, as done previously, shows, in fact, that a second-order multiplier exists, viz.,

$$(Q^1, Q^2) = (a(b-k)u_{xx} + 3(k^2 + kb - 2b^2)u^2 - (k+2b)v^2, 2(av_{xx} - kuv - 2buv)).$$

The corresponding conserved density ($b \neq k$) is

$$\Phi^t = \frac{1}{b-k}[-\frac{1}{2}(b-k)u_x^2 - av_x^2 + b(k-2b)u^3 - 2buv^2 + k^2u^3 - kuv^2].$$

The action of the X_i's ($i = 1, 2, 3$) on the Q^j's (j=1,2) are

$$X_1(Q^j) = X_2(Q^j) = 0,$$
$$X_3(Q^1) = -4Q^1, \quad X_3(Q^2) = -4Q^2.$$

Thus, the conserved density Φ^t and conserved flux are associated with X_1 and X_2 and not with X_3 as the multiplier (Q^1, Q^2) is ray invariant, as opposed to strictly invariant, under X_3.

Further investigation can be done for various combinations of b and k like $k = 2b$.

3. CONCLUSION

We have shown that pdes or systems of pdes may have multiplier that are higher-order (than two in derivatives) and lead to new and nontrivial conservation laws. A number of possible relationships between the multipliers and Lie point symmetry generators exist. These have consequences, inter alia, on the basis of conservation laws of the pde/systems of pdes.

Acknowledgements- The author thanks the referees for the useful comments and suggestions.

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