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LIE INFINITESIMAL CONSERVED QUANTITIES FOR ITÔ STOCHASTIC ODEs

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Abstract- A methodology for constructing conserved quantities with Lie symmetry infinitesimals in an Itô integral context is pursued. The basis of this construction relies on Lie bracket relations on both the instantaneous drift and diffusion of an Itô stochastic ordinary differential equation (SODE).

Keywords- Lie infinitesimal, stochastic ordinary differential equation, conserved quantity.

1. INTRODUCTION

A conserved quantity in the context of an Itô integral implies an entity which is constant on all sample paths for all time indices. The instantaneous drift and diffusion are zero. Trivially this says that these conserved quantities are all Martingales. That is, their expected value in the future or present is their eventuated values in the past.

Methods for constructing conserved quantities of SODEs by use of Lie transformations were analyzed for Stratonovich integral based SODEs by Misawa [1] and Albeverio and Fei [2]. The conserved quantity construction of Misawa [1] and Albeverio and Fei [2], precludes the necessity for a Lagrangian or a Hamiltonian formulation. The philosophy followed, highlighted the interplay between the infinitesimals of the symmetry operator H and the conserved quantity itself.

The Itô integral construction of the conserved quantities was later pursued by Ünal [3]. In this contribution Ünal [3] uses both the (Fokker-Planck) FP equation and its associated SODE to obtain the conserved quantity.

After having reconciled the determining equations obtained in Wafo and Mahomed [4] and Ünal [3] via Fredericks and Mahomed [5], we can focus on the conserved quantity analysis of [3]. We show that the symmetries of the FP equations are projectable by using the methodology of Mahomed and Momoniat [6]. This projectable nature of the temporal infinitesimal was an *ansatz* that Gaeta and Quintero [7] enforced on both the FP equation and its associated SODE. The work of Ünal [3] shows that in the SODE context, the temporal infinitesimal need not be a function of time only. This implies that the Lie algebra generated by the SODE can have non-projectable symmetries which will not belong to the Lie algebra generated by the FP equation.

In constructing the conserved quantity for Itô integral based SODEs, [3] tries to combine the determining equations associated with SODEs, which allows for the said infinitesimal to be non-projectable, with the determining equations based on the associated FP equation. However, we prove that the symmetries of the FP equation have to be projectable. Thus we have that only projectable symmetries will satisfy both the FP equation and its associated SODEs, which is what was utilized by [7].

In this paper, we first revisit the conserved quantity results of Ünal [3] and juxtapose it with the new findings of our deliberations. This scrutiny will be followed by an attempt to construct a conserved quantity based upon the methodology of [2] for Stratonovich integral SODEs.

2. CONSERVED QUANTITIES FOR ITÔ INTEGRALS REVISITED

We use the approach of [6] to firstly show that the symmetry operators of the FP equation

$$u_t + A_{ij}\, u_{x_i x_j} + B_i\, u_{x_i} + Cu = 0, \tag{1}$$

are projectable, where repeated indices imply summation. This is easily seen if we write the Lie operator in characteristic form via the Lie characteristic function $Q = \eta - \tau\, u_t - \xi_i\, u_{x_i}$, where u_{x_i} indicates the partial derivative of the dependent variable with respect to the ith spatial variable x_i. Then the symmetry condition for (1) yields the determining equation which is first singled out for the mixed derivatives in time and spatial variables t and x_i, respectively. These result in $Q = \alpha(t)u_{(t)} + \beta(t, x, u, u_x)$, which after insertion in the remaining determining equation; followed by separation with respect to the spatial derivatives gives $Q = \alpha(t)u_{(t)} + \alpha_j(t, x)\, u_{x_j} + \gamma(t, x, u)$. The determining equations belonging to the FP equation can be rewritten in terms of the instantaneous drift and diffusion coefficients of the Itô SODE $\mathbf{f}$ and $\mathbf{G}$, respectively. The original equations are

$$\frac{\partial(\tau A_{ik})}{\partial t} + \left(\xi_r \frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0 \tag{2}$$

$$\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} + \dots$$
$$-2\left(A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right) = 0 \tag{3}$$

$$\left(\frac{\partial}{\partial t} + f_i\frac{\partial}{\partial x_i} - A_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right)\left(\alpha_2(t,\mathbf{x}) + \frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{4}$$

where

$$A_{ij} = -\frac{1}{2}\sum_{k=1}^{M} G_i^k G_j^k, \tag{5}$$

$$B_i = f_i - 2\frac{\partial A_{ik}}{\partial x_k} \tag{6}$$

and

$$C = \left(\frac{\partial f_i}{\partial x_i}\right) - \frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}. \tag{7}$$

The dependent variable infinitesimal Φ of the FP equation has the relation

$$\Phi = \alpha_1(t,\,\mathbf{x}) + u\,\alpha_2(t,\,\mathbf{x}) \tag{8}$$

which is associated with the FP symmetry operator as

$$H_{FP} = \tau(t)\frac{\partial}{\partial t} + \xi_j(t,\,\mathbf{x})\frac{\partial}{\partial x_j} + \Phi(t,\,\mathbf{x},\,u)\frac{\partial}{\partial u}. \tag{9}$$

Equation (2) can be written as

$$\sum_{l=1}^{M} G_i^l Y^l(\xi_k) + \sum_{l=1}^{M} G_k^l Y^l(\xi_i) = H\left(\sum_{l=1}^{M} G_i^l G_k^l\right) + \sum_{l=1}^{M} G_i^l G_k^l \Gamma(\tau). \tag{10}$$

Since τ is a projectable in this context, i.e. a function of time only, we have that $\Gamma(\tau) = \dot{\tau}$. Further simplification gives

$$Y^l(\xi_k) = H\left(G_k^l\right) + \frac{1}{2}\,G_k^l\,\Gamma(\tau),\ \text{for } l = 1, M \text{ and } k = 1, N. \tag{11}$$

Equations (3) and (4) can likewise be written as

$$\Gamma(\xi_k) = \left(\Gamma(\tau) + H\right) f_k + \sum_{l=1}^{M} G_k^l Y^l\left(\alpha_2(t,\,\mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) \tag{12}$$

and

$$\Gamma\left(\alpha_2(t,\,\mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{13}$$

respectively. The projectable symmetries of the Itô SODE satisfy the determining equations

$$Y^l(\xi_k) = \left(\frac{1}{2}\,\Gamma(\tau) + H\right) G^l_k, \tag{14}$$

$$\Gamma(\xi_k) = \left(\Gamma(\tau) + H\right) f_k, \tag{15}$$

$$Y^l(\tau) = 0 \tag{16}$$

and

$$\Gamma(\tau) = Constant, \tag{17}$$

for $l = 1, M$ and $k = 1, N$; where the instantaneous drift and diffusion operators are respectively

$$\Gamma = \frac{\partial}{\partial t} + f_i\,\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{l\,=\,1}^{M} G^l_i\,G^l_m\,\frac{\partial^2}{\partial x_i \partial x_m} \tag{18}$$

and

$$Y^l = G^l_i\,\frac{\partial}{\partial x_i}, \tag{19}$$

in which the indices i and m run from one to N. Since these projectable symmetries form a subalgebra of the algebra belonging to the FP equation, we have that the determining equations associated with the FP equation become

$$Y^l\left(\alpha_2(t,\,\mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) = 0 \tag{20}$$

and

$$\Gamma\left(\alpha_2(t,\,\mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{21}$$

for all $l = 1, M$. Thus for projectable symmetries of the Itô integral based SODEs we have that $\alpha_2(t,\,\mathbf{x}) + \sum_{r=1}^{N} \partial\xi_r/\partial r$ is a conserved quantity as both its instantaneous drift and diffusion are zero. This is different from what was derived in [3], where extra terms involving the spatial derivative of the temporal infinitesimal survive, as a consequence of the preclusion of the fact that the temporal infinitesimal has to be projectable in the FP equation context.

3. AN ALTERNATIVE FORMULATION

An alternative formulation for deriving conserved quantities from Lie symmetries is adapted from [2] who derived conserved quantities from the Lie infinitesimals for Stratonovich based SODEs. This allows us to use both the projectable and non-projectable Lie symmetries of the Itô SODEs.

We first need a relation between the instantaneous drift and diffusion operators and the the symmetry operator. The use of Lie brackets achieves this. The determining equations (14) and (15) based on the SODEs can be written in terms of Lie brackets as

$$[\Gamma, H]\,(f_k) = \Gamma(\tau)\,\Gamma(f_k) \tag{22}$$

and

$$\left[Y^l, H\right](G^l_k) = \frac{1}{2}\,\Gamma(\tau)Y^l(G^l_k)\ l = 1, M, \tag{23}$$

where $[\Gamma, H] = \Gamma(H) - H(\Gamma)$, and where condition (16) dictates that

$$Y^l(H) = \sum_{k=1}^{N} Y^l(\xi_k)\,\partial/\partial x_k$$

for all $l = 1, M$. However the drift and diffusion coefficients of the SODE are arbitrary, so we have

$$[\Gamma, H] = \Gamma(\tau)\,\Gamma \tag{24}$$

$$\left[Y^l, H\right] = \frac{1}{2}\,\Gamma(\tau)Y^l,\ l = 1, M. \tag{25}$$

We next define $\mathcal{I} \equiv \{I\,(t, \mathbf{x})|\, dI = 0,$ wherever (14) and (15) are satisfied$\}$. If $I \in \mathcal{I}$, i.e. satisfies $\Gamma(I) = 0$ and $Y^l(I) = 0$, then $H(I) \in \mathcal{I}$, where H satisfies (24) and (25). Proof. From (24), we have that

$$\begin{aligned}
\Gamma(H(I)) &= (\Gamma(H))\,(I) + H\,(\Gamma(I)) && (26)\\
&= [\Gamma, H]\,(I) + H(\Gamma(I)) && (27)\\
&= \Gamma(\tau)\Gamma(I) && (28)\\
&= 0. && (29)
\end{aligned}$$

By (25) we also deduce

$$\begin{aligned}
Y^l(H(I)) &= \left(Y^l(H)\right)(I) + H\left(Y^l(I)\right) && (30)\\
&= \left[Y^l, H\right](I) + H(Y^l(I)) && (31)\\
&= \frac{1}{2}\Gamma(\tau)Y^l(I) && (32)\\
&= 0. && (33)
\end{aligned}$$

Let $\mathcal{L}$ denote the set of all H satisfying (24) and (25). Having established $\mathcal{L}$ it can be shown that it is a complex Lie algebra.

3.1. Conserved Quantities for First Order SODEs

We propose that for first order SODEs,

$$I = \sum_{j=1}^{N} \xi_j + \Gamma(\tau) + H(\phi) \tag{34}$$

is a conserved quantity, where ϕ (not yet specified) is at least twice continuous with respect to spacial and temporal variables. This implies

$$\begin{aligned} \Gamma(I) &= \Gamma(\xi_j) + \Gamma(H(\phi)) + \Gamma(\Gamma(\tau)) && (35)\\ &= \sum_{j=1}^{N} (\Gamma(\tau) + H) f_j + \Gamma(H\phi) && (36) \end{aligned}$$

which we arrive at by using relations (15) and (17) for first order SODEs. Utilizing (24) gives

$$\Gamma(I) = \sum_{j=1}^{N} (\Gamma(\tau) + H) f_j + \Gamma(\tau)\Gamma(\phi) + H\Gamma(\phi), \tag{37}$$

which simply means that

$$(H + \Gamma(\tau)) \left(\sum_{j=1}^{N} f_j + \Gamma(\phi) \right) = 0. \tag{38}$$

The function ϕ is chosen such that

$$\sum_{j=1}^{N} f_j + \Gamma(\phi) = 0. \tag{39}$$

Next we have to show that $Y^l I$ is zero. We have

$$\begin{aligned} Y^l I &= Y^l \left(\sum_{j=1}^{N} \xi_j \right) + Y^l(H\phi) + Y^l(\Gamma(\tau)) && (40)\\ &= \sum_{j=1}^{N} \left(\frac{1}{2}\Gamma(\tau) + H \right) G_j^l + \frac{1}{2}\Gamma(\tau) Y^l(\phi) + H Y^l(\phi) && (41)\\ &= \left(H + \frac{1}{2}\Gamma(\tau) \right) \left(\sum_{j=1}^{N} G_j^l + Y^l(\phi) \right). && (42) \end{aligned}$$

This is due to the fact that we proved $Y^l(\Gamma(\tau)) = 0$. The main calculations above were arrived at in a similar manner to what we did before only now using (25).

In summary, this forces ϕ to be chosen such that

$$\sum_{j=1}^{N} f_j + \Gamma(\phi) = 0, \tag{43}$$

and

$$\sum_{j=1}^{N} G_j^l + Y^l(\phi) = 0. \tag{44}$$

3.2. Conserved Quantities based on the FP equation

Although we are limited to only projectable symmetries under the FP equation context, we can still derive interesting results. By considering only the projectable symmetries of the associated SODEs, we showed that the FP determining equations simplify to

$$\Gamma\left(\alpha_2(t, \mathbf{x}) + \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}\right) = 0 \tag{45}$$

and

$$Y^l\left(\alpha_2(t, \mathbf{x}) + \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}\right) = 0. \tag{46}$$

Focusing now only on (45), we expand in the following manner

$$\begin{aligned}
\Gamma(\alpha_2) &= -\Gamma(\sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}) & (47)\\
&= \sum_{j=1}^{N}\left[-\frac{\partial}{\partial x_j}\left(\Gamma(\xi_j)\right) + \frac{\partial f_k}{\partial x_j}\left(\frac{\partial \xi_j}{\partial x_k}\right) - \frac{\partial A_{rs}}{\partial x_j}\left(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\right)\right] & (48)\\
&= \sum_{j=1}^{N}\left[-\xi_i \frac{\partial^2 f_j}{\partial x_i\, x_j} - \frac{\partial \xi_i}{\partial x_j}\frac{\partial f_j}{\partial x_i} + \frac{\partial f_k}{\partial x_j}\left(\frac{\partial \xi_j}{\partial x_k}\right) - \frac{\partial A_{rs}}{\partial x_j}\left(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\right)\right] & (49)
\end{aligned}$$

which we deduce by using (14) and the fact that the temporal infinitesimal is projectable, i.e. a function of time only. Thus we have the relation

$$\Gamma(\alpha_2) = \sum_{j=1}^{N}\left[-\xi_i \frac{\partial^2 f_j}{\partial x_i\, x_j} - \frac{\partial A_{rs}}{\partial x_j}\left(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\right)\right]. \tag{50}$$

In a similar fashion we modify (46) as

$$Y^l\Big(\alpha_2(t,\,\mathbf{x})\Big) = Y^l\Big(\sum_{j=1}^{N}\frac{\partial \xi_j}{\partial x_j}\Big) \tag{51}$$

$$= \sum_{j=1}^{N}\Big[-\frac{\partial}{\partial x_j}\Big(Y^l(\xi_j)\Big) + \frac{\partial G_k^l}{\partial x_j}\Big(\frac{\partial \xi_j}{\partial x_k}\Big) \tag{52}$$

$$= \sum_{j=1}^{N}\Big[-\xi_i\frac{\partial^2 G_j^l}{\partial x_i\,x_j} - \frac{\partial \xi_i}{\partial x_j}\frac{\partial G_j^l}{\partial x_i} + \frac{\partial G_k^l}{\partial x_j}\Big(\frac{\partial \xi_j}{\partial x_k}\Big)\Big] \tag{53}$$

which we obtain by using (15) and the fact that the temporal infinitesimal is projectable. Thus we have

$$Y^l\Big(\alpha_2(t,\,\mathbf{x})\Big) = \sum_{j=1}^{N}\Big[-\xi_i\frac{\partial^2 G_j^l}{\partial x_i\,x_j}\Big]. \tag{54}$$

If we can find an $\alpha_2(t,\,\mathbf{x})$ such that (50) and (54) are satisfied, then we can use the projectable symmetries of the SODEs to generate conserved quantities. Thus it is also possible to generate the conserved quantities from the determining equations of the associated Fokker-Plank equation, but only for the case where $\tau(t)$, $\xi(t,\mathbf{x})$ and $\Phi(t,\mathbf{x},u)$, which is what [7] used as an *ansatz* for both the SODE and the FP equation.

4. EXAMPLE

In the work of [3], it was stated that the temporal infinitesimal of the form

$$H = \tau\frac{\partial}{\partial t}$$

gives rise to a conserved quantity. This is not necessarily the case. The temporal infinitesimal also has to satisfy the condition.

$$e^{\epsilon\,\Gamma(\tau)} = \Gamma\left(e^{\epsilon\,H}(t)\right), \tag{55}$$

which was derived in [5]. In the concluding example in [3] we have the following SODE

$$d\mathbf{X}(t) = \mathbf{f}\;dt + \mathbf{G}\;dW(t), \tag{56}$$

where $\mathbf{f}$ is the vector

$$\begin{pmatrix} -\frac{1}{2}X_1(t) \\ -\frac{1}{2}X_2(t) \end{pmatrix} \tag{57}$$

and $\mathbf{G}$ the vector

$$\begin{pmatrix} -X_2(t) \\ X_1(t) \end{pmatrix}. \tag{58}$$

In previous articles [5] and [3], the following symmetry infinitesimals were found

$$\tau(t,\mathbf{X}(t)) = C_0F_0\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right), \tag{59}$$

$$\xi_1(t,\mathbf{X}(t)) = C_1F_1\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)X_1(t) + C_2F_2\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)X_2(t) \tag{60}$$

and

$$\xi_2(t,\mathbf{X}(t)) = C_1F_1\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)X_2(t) - C_2F_2\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)X_1(t). \tag{61}$$

Since the temporal infinitesimal is not projectable, it does not belong to a subalgebra of the FP equation. However, it is a conserved quantity. Thus only the spatial infinitesimals are symmetries of the FP equation itself. The temporal infinitesimal in this case, is a conserved quantity because the drift and diffusion coefficients are not functions of time and the instantaneous drift of the temporal infinitesimal is zero, i.e. $\Gamma(\tau) = 0$; thus the instantaneous drift and diffusion are zero. In this instance it also satisfies the condition (55). We now construct conserved quantities using both our alternate method and the FP associated method from [7] presented above.

4.1. Alternative Method

Considering equation (44), we have

$$Y(\phi) = x_2 - x_1 \tag{62}$$

which implies that

$$-x_2\frac{\partial\phi}{\partial x_1} + x_1\frac{\partial\phi}{\partial x_2} = -x_1 + x_2, \tag{63}$$

which easily solves as

$$\phi = F_3\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)F_4(t) - (X_1(t)+X_2(t)). \tag{64}$$

Invoking relation (43) gives

$$F_3\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)\dot{F}_4(t) = 0 \tag{65}$$

since $\Gamma(F_3\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right)) = 0$. This forces the following simplification

$$\phi = F_3\left(\frac{X_2^2(t)+X_1^2(t)}{2}\right) - (X_1(t)+X_2(t)). \tag{66}$$

The conserved quantity is constructed by utilizing the non-projectable temporal infinitesimal in this instance. By invoking the projectable symmetries only, we now implement the FP associated conserved quantity construction.

4.2. FP associated conserved quantity construction

The conserved quantity is of the form

$$I = \alpha_2 + \sum_{j=1}^{2} \frac{\partial \xi_j}{\partial x_j}. \tag{67}$$

Equation (54) becomes

$$Y(\alpha_2) = 0, \tag{68}$$

since $\mathbf{G}$ has linear components. Thus we have

$$\alpha_2 = F_5\,(u)\,F_6(t), \tag{69}$$

where

$$u = \frac{X_2^2(t) + X_1^2(t)}{2}. \tag{70}$$

By invoking equation (50) we have

$$\Gamma(\alpha_2) = -\frac{\partial A_{22}}{\partial x_1}\frac{\partial^2 \xi_1}{\partial^2 x_2} - \frac{\partial A_{11}}{\partial x_2}\frac{\partial^2 \xi_2}{\partial^2 x_1} \tag{71}$$

where

$$\begin{aligned} -\frac{\partial A_{11}}{\partial x_2}\frac{\partial^2 \xi_2}{\partial^2 x_1} &= -2x_1 \big(C_1 F_1''(u) x_1 x_2^2 + C_1 F_1'(u) x_1 \\ &\quad + C_2 F_2''(u) x_2^3 + 2C_2 F_2'(u) x_2 + C_2 F_2'(u) x_2\big)\,, \end{aligned} \tag{72}$$

and

$$\begin{aligned} -\frac{\partial A_{22}}{\partial x_1}\frac{\partial^2 \xi_1}{\partial^2 x_2} &= -2\,x_2\,\big(C_1\,F_1''(u)\,x_2\,x_1^2 + C_1\,F_1'(u)\,x_2 \\ &\quad - C_2\,F_2''(u)\,x_1^3 - 2C_2\,F_2'(u)\,x_1 - C_2\,F_2'(u)\,x_1\big)\,. \end{aligned} \tag{73}$$

Comparing coefficients of various combinations of the spatial variables which are independent of u, we find

$$F_1''(u) = 0 \tag{74}$$

which implies

$$F_1(u) = \frac{C_1\,u^2}{2} + C_2 u + C_3 \tag{75}$$

and

$$F_2''(u) = 0. \tag{76}$$

These result in a quadratic form

$$F_2(u) = \frac{C_4\, u^2}{2} + C_5\, u + C_6. \tag{77}$$

Thus we ultimately have

$$F_5(u)\dot{F}_6(t) = -2C_1 \left(C_2\, u^2 + C_3\, u\right), \tag{78}$$

which we can solve as

$$F_5(u) = C_2\, u^2 + C_3\, u \tag{79}$$

and

$$F_6(t) = -2\, C_1\, t + C_8. \tag{80}$$

Eventually we can write our unknown variable α_2 as

$$\alpha_2 = (C_8 - 2\, C_1\, t) \left(C_2\, u^2 + C_3\, u\right), \tag{81}$$

which implies that our conserved quantity is

$$I = (C_8 - 2\, C_1\, t) \left(C_2\, u^2 + C_3\, u\right) + 2\, C_1 \left(C_2\, u^2 + C_3\, u\right) + 2\, C_2\, F_2(u), \tag{82}$$

since

$$\frac{\partial \xi_1}{\partial x_1} = C_1\, F_1'(u)\, x_1^2 + C_2\, F_2(u) + C_2\, F_2'(u)\, x_1\, x_2 \tag{83}$$

and

$$\frac{\partial \xi_1}{\partial x_1} = C_1\, F_1'(u)\, x_2^2 + C_2\, F_2(u) - C_2\, F_2'(u)\, x_1\, x_2. \tag{84}$$

Remark. The two methods yield two unrelated conserved quantities. Neither of the two have been found in the past. It is also interesting to note that the last method further dictates the form of the arbitrary functions F_1 and F_2, which generate the two spatial infinitesimals.

5. CONCLUDING REMARKS

We have derived new methods of constructing conserved quantities. We have noted that the part of the conservation analysis of Ünal [3] has to satisfy an extra condition (55) to ensure that the Lie point invariance holds. The two novel ways of constructing conserved quantities are based on two independent approaches: one based on the projectable symmetries of the SODEs and thus a sub-algebra of the FP equation and the other method takes advantage of both the projectable and non-projectable symmetries of the SODE alone. Both methods precludes the necessity for a Hamiltonian or a Lagrangian formulation.

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