# CONSERVATION LAWS AND INVARIANT SOLUTIONS IN THE FANNO MODEL FOR TURBULENT COMPRESSIBLE FLOW 

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#### Abstract

Asymptotic reductions of the Fanno model for one-dimensional turbulent compressible flow of a perfect gas in a long tube are investigated. Conservation laws are derived using the multiplier method for a nonlinear wave equation and a nonlinear diffusion equation for the mean velocity and a nonlinear diffusion equation for the mean pressure. Two conserved quantities for the mean velocity are obtained from the conservation laws and boundary conditions. An invariant solution is derived for the mean velocity using the Lie point symmetries associated with the conserved vector which generated the conserved quantity for the boundary value problem.


Keywords- Turbulent flow, conservation laws, multiplier method, associated Lie point symmetries.

## 1. INTRODUCTION

When the flow of a perfect gas in a tube is turbulent and the tube is long enough for wall drag to be important a more realistic model than that of inviscid flow is the Fanno flow model [1, 2]. The Fanno model contains a drag term in the momentum balance equation while the mass and energy balance equations are the same as for inviscid laminar flow. The Fanno model has applications to the air-jet spinning of polymer filaments [3], inlets to pressure transducers [4] and to high speed trains travelling through a long tunnel [5]. It may also be applicable to the analysis of air blasts in long tunnel networks in a mining environment $[6,7]$. The Fanno model for one-dimensional flow has been investigated mathematically by Ockenden et al [1]. These authors analysed flow of initially small amplitude in a semi-infinite tube as well as gas driven by a piston. They introduced a range of time scales and derived similarity solutions for boundary value problems for the mean turbulent velocity
and pressure averaged over the tube. Hastings et al [2] proved rigorously the existence, uniqueness and asymptotic behaviour of the waves. We will refer to the mean turbulent velocity and pressure of the gas simply as the gas velocity and pressure.

In this paper conservation laws for the partial differential equations obtained by Ockendon et al [1] in the asymptotic reductions of the Fanno model are derived. The gas velocity satisfies a nonlinear diffusion equation subject to homogeneous boundary conditions and a conserved quantity. A new method of solution due to Kara and Mahomed [8] is applied. A linear combination of the Lie point symmetries which are associated with the conservation law for the partial differential equation which generates the conserved quantity is used to derive the solution instead of a linear combination of the Lie point symmetries of the partial differential equation.

## 2. ASYMPTOTIC REDUCTIONS OF THE FANNO MODEL

The gas is initially at rest in a semi-infinite tube $x \geq 0$. We will consider two physical problems formulated by Ockendon et al [1]. These authors derived asymptotic reductions of the Fanno by introducing appropriate scales in $t, x$, gas velocity $u(t, x)$ and gas pressure $p(t, x)$. In this paper the different scales are denoted simply by $t, x, u$ and $p$.

### 2.1. Pressure increase at entrance

At $t=0$ the pressure at the entrance to the tube, $x=0$, is suddenly increased by a small amount and a shock moves into the undisturbed gas in the tube. In the region close to the shock the following boundary value problem for the nonlinear wave equation for $u(t, x)$ was derived:

$$
\begin{align*}
& \frac{\partial^{2} u}{\partial t^{2}}-\frac{\partial^{2} u}{\partial x^{2}}=-2 u \frac{\partial u}{\partial t}  \tag{2.1}\\
& x=t: \quad u=\frac{2}{t}, \quad u \sim \frac{3 t}{x^{2}} \quad \text { as } \quad x \rightarrow 0 . \tag{2.2}
\end{align*}
$$

In the region further from the shock and nearer the entrance to the tube the momentum and energy equations take the form

$$
\begin{equation*}
\frac{\partial p}{\partial x}=-u^{2}, \quad \frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}=0 \tag{2.3}
\end{equation*}
$$

The velocity $u(t, x)$ can be eliminated to give a nonlinear diffusion equation for the
pressure $p(t, x)$. Alternatively the pressure can be eliminated to give a nonlinear diffusion equation for the velocity $u(t, x)$. The following two boundary value problems for $p(t, x)$ and $u(t, x)$ were derived:

$$
\begin{align*}
& \frac{\partial p}{\partial t}=\frac{1}{2}\left(-\frac{\partial p}{\partial x}\right)^{-1 / 2} \frac{\partial^{2} p}{\partial x^{2}},  \tag{2.4}\\
& p(t, 0)=1, \quad p \sim \frac{3 t^{2}}{x^{3}} \quad \text { as } \quad x \rightarrow \infty  \tag{2.5}\\
& \frac{\partial^{2} u}{\partial x^{2}}=2 u \frac{\partial u}{\partial t},  \tag{2.6}\\
& \frac{\partial u}{\partial x}(t, 0)=0, \quad u \sim \frac{3 t}{x^{2}} \quad \text { as } \quad x \rightarrow \infty . \tag{2.7}
\end{align*}
$$

### 2.2 Compressive wave generated by piston

Ockendon et al [1] also considered the compressive wave generated by a piston moved impulsively into the gas with constant velocity much less than the speed of sound. The following boundary value problem for $u(t, x)$ was obtained:

$$
\begin{align*}
& \frac{\partial^{2} u}{\partial x^{2}}=2 u \frac{\partial u}{\partial t},  \tag{2.8}\\
& u(t, 0)=1, \quad u \sim \frac{3 t}{x^{2}} \quad \text { as } \quad x \rightarrow \infty \tag{2.9}
\end{align*}
$$

## 3. CONSERVATION LAWS

Conservation laws for a partial differential equation do not depend on the boundary conditions. We will obtain conservation laws for the partial differential equations (2.1), (2.4) and (2.6). The multiplier method will be used [9-11]. The derivation will be outlined for the nonlinear diffusion equation (2.6) and the results will then be stated for equations (2.1) and (2.4).

A conservation law for (2.6) is of the form

$$
\begin{align*}
& D_{1} T^{1}+\left.D_{2} T^{2}\right|_{(2.5)}=0  \tag{3.1}\\
& D_{1}=D_{t}=\frac{\partial}{\partial t}+u_{t} \frac{\partial}{\partial u}+u_{t t} \frac{\partial}{\partial u_{t}}+u_{x t} \frac{\partial}{\partial u_{x}}+\cdots  \tag{3.2}\\
& D_{2}=D_{x}=\frac{\partial}{\partial x}+u_{x} \frac{\partial}{\partial u}+u_{t x} \frac{\partial}{\partial u_{t}}+u_{x x} \frac{\partial}{\partial u_{x}}+\cdots \tag{3.3}
\end{align*}
$$

and subscripts denote partial derivatives. The vector $T=\left(T^{1}, T^{2}\right)$ is a conserved vector for the partial differential equation (2.6).

A multiplier $\Lambda$ of equation (2.6) has the property that

$$
\begin{equation*}
\Lambda\left(2 u u_{t}-u_{x x}\right)=D_{1} T^{1}+D_{2} T^{2} \tag{3.4}
\end{equation*}
$$

for all functions $u(t, x)$ and not only for solutions of (2.6). We will consider multipliers of the form $\Lambda=\Lambda(t, x, u)$. Multipliers of this form were found to be sufficient to derive significant conservation laws for two-dimensional and radial jets [12]. We will see that they are sufficient to derive the conserved vector which generates the invariant solution for the boundary value problem, (2.6) and (2.7). Multipliers which depend on the first order and higher order partial derivatives of $u$ could also be considered but the calculations rapidly become more complicated. Computer assisted calculations could then be performed which may lead to further conservation laws.

The right hand side of (3.4) is a divergence expression. The determining equation for the multiplier $\Lambda$ is

$$
\begin{equation*}
E_{u}\left[\Lambda\left(2 u u_{t}-u_{x x}\right)\right]=0 \tag{3.5}
\end{equation*}
$$

where $E_{u}$ is the standard Euler operator which annihilates divergence expressions:

$$
\begin{equation*}
E_{u}=\frac{\partial}{\partial u}-D_{x} \frac{\partial}{\partial u_{x}}-D_{y} \frac{\partial}{\partial u_{y}}+D_{x}^{2} \frac{p}{\partial u_{x x}}+D_{x} D_{y} \frac{\partial}{\partial u_{x y}}+D_{y}^{2} \frac{\partial}{\partial u_{y y}}-\cdots \tag{3.6}
\end{equation*}
$$

The expansion of (3.5) yields

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial u} u_{x x}+2 u \frac{\partial \Lambda}{\partial t}+\frac{\partial^{2} \Lambda}{\partial x^{2}}+2 \frac{\partial^{2} \Lambda}{\partial x \partial u} u_{x}+\frac{\partial^{2} \Lambda}{\partial u^{2}} u_{x}^{2}+\frac{\partial \Lambda}{\partial u} u_{x x}=0 \tag{3.7}
\end{equation*}
$$

Since (3.7) is satisfied for all functions $u(t, x)$ it can be separated by equating the coefficients of the partial derivatives of $u(t, x)$. The coefficients of $u_{x x}$ give

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial u}=0 \tag{3.8}
\end{equation*}
$$

and therefore $\Lambda=\Lambda(t, x)$. Equation (3.7) reduces to

$$
\begin{equation*}
2 u \frac{\partial \Lambda}{\partial t}+\frac{\partial^{2} \Lambda}{\partial x^{2}}=0 \tag{3.9}
\end{equation*}
$$

Separating (3.9) according to powers of $u$ gives

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial t}=0, \quad \frac{\partial^{2} \Lambda}{\partial x^{2}}=0 \tag{3.10}
\end{equation*}
$$

Thus $\Lambda=\Lambda(x)$ where

$$
\begin{equation*}
\Lambda(x)=c_{1}+c_{2} x \tag{3.1}
\end{equation*}
$$

and $c_{1}$ and $c_{2}$ are constants.
From (3.4)and (3.11) and by performing elementary manipulations,

$$
\begin{equation*}
\left(c_{1}+c_{2} x\right)\left(2 u u_{t}-u_{x x}\right)=D_{t}\left[c_{1} u^{2}+c_{2} x u^{2}\right]+D_{x}\left[c_{1}\left(-u_{x}\right)+c_{2}\left(u-x u_{x}\right)\right] \tag{3.12}
\end{equation*}
$$

for all functions $u(t, x)$. Thus when $u(t, x)$ is a solution of (2.5)

$$
\begin{equation*}
D_{t}\left[c_{1} u^{2}+c_{2} x u^{2}\right]+D_{x}\left[c_{1}\left(-u_{x}\right)+c_{2}\left(u-x u_{x}\right)\right]=0 \tag{3.13}
\end{equation*}
$$

Any conserved vector of the partial differential equation (2.6) with multiplier of the form $\Lambda(t, x, u)$ is therefore a linear combination of the two conserved vectors

$$
\begin{align*}
& T^{1}=u^{2},  \tag{3.14}\\
& T^{2}=-u_{x},  \tag{3.15}\\
& T^{1}=x u^{2}, \\
& T^{2}=u-x u_{x}
\end{align*}
$$

The conserved vectors (3.14) and (3.15) therefore form a basis of conserved vectors for the partial differential equation (2.5) with multipliers of the form $\Lambda(t, x, u)$. The conserved vectors were readily constructed by elementary manipulations once the multipliers had been derived. They can also be derived systematically using (3.4) with (3.11) as the determining equation.

Conservation laws for the nonlinear wave equation (2.1) with multiplier of the form $\Lambda(t, x, u)$ and of the nonlinear diffusion equation (2.4) with multiplier of the form $\Lambda(t, x, p)$ were investigated in the same way. The results are displayed in Table 3.1. In each case the first conserved vector is the elementary conserved vector which may readily be derived by writing the partial differential equation in conserved form. For equation (2.4) for $p(t, x)$ only one conserved vector with multiplier $\Lambda(t, x, p)$ was found.

Table 3.1 Conserved vectors

| Partial differential <br> equation | Multiplier <br> $\Lambda$ | Conserved vectors |
| :--- | :--- | :--- |
| $u_{x x}=2 u u_{t}$ | $\Lambda(t, x, u)$ | $T^{1}=u^{2}$ <br> $T^{2}=-u_{x}$ <br> $T^{1}=x u^{2}$ <br> $T^{2}=u-x u_{x}$ |
| $u_{t t}-u_{x x}=-2 u u_{t}$ | $\Lambda(t, x, u)$ | $T^{1}=u^{2}+u_{t}$ <br> $T^{2}=-u_{x}$ <br> $T_{1}=x\left(u^{2}+u_{t}\right)$ <br> $T^{2}=u-x u_{x}$ |
| $p_{t}=\frac{1}{2}\left(-p_{x}\right)^{-1 / 2} p_{x x}$ | $\Lambda(t, x, p)$ | $T^{1}=p$ <br> $T^{2}=\left(-p_{x}\right)^{1 / 2}$ |
| $\Lambda=c_{1}$ |  |  |

## 4. CONSERVED QUANTITIES

Conserved quantities for a boundary value problem are derived from the partial differential equation and the boundary conditions. The boundary conditions determine which conservation law to apply.

We will investigate conserved quantities for the two boundary value problems for the nonlinear diffusion equation for $u$. Since $u=u(t, x)$ the conserved vectors (3.14) and (3.15) for equation (2.6) can be expressed as functions of $t$ and $x$ and therefore

$$
\begin{equation*}
D_{t} T^{1}+D_{x} T^{2}=\frac{\partial}{\partial t} T^{1}(t, x)+\frac{\partial}{\partial x} T^{2}(t, x) . \tag{4.1}
\end{equation*}
$$

For a conserved vector the left hand side of (4.1) vanishes and hence

$$
\begin{equation*}
\frac{\partial}{\partial t} T^{1}(t, x)+\frac{\partial}{\partial x} T^{2}(t, x)=0 \tag{4.2}
\end{equation*}
$$

Consider first the boundary value problem (2.6) and (2.7). Substituting the elementary conserved vector (3.14) into (4.2) gives

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(u^{2}\right)+\frac{\partial}{\partial x}\left(-u_{x}\right)=0 \tag{4.3}
\end{equation*}
$$

and integrating with respect to $x$ along the tube from $x=0$ to $x=\infty$ we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{\infty} u^{2}(t, x) \mathrm{d} x=u_{x}(t, \infty)-u_{x}(t, 0) \tag{4.4}
\end{equation*}
$$

But from the boundary conditions (2.7), $u_{x}(t, 0)=0$ and $u_{x}(t, \infty)=0$, therefore

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{\infty} u^{2}(t, x) \mathrm{d} x=0 \tag{4.5}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\int_{0}^{\infty} u^{2}(t, x) \mathrm{d} x=\text { constant independent of } t \tag{4.6}
\end{equation*}
$$

The constant in (4.6) determines the strength of the flow along the tube due to the sudden pressure change at the entrance $x=0$. It is obtained by substituting the first equation in (2.3) into (4.6) and using the boundary conditions (2.5) for $p(t, x)$. This yields the conserved quantity

$$
\begin{equation*}
\int_{0}^{\infty} u^{2}(t, x) \mathrm{d} x=1 \tag{4.7}
\end{equation*}
$$

Consider next the piston problem, (2.8) and (2.9). Substituting the second conserved vector (3.15) into (4.2) and integrating with respect to $x$ along the tube from $x=0$ to $x=\infty$ we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{\infty} x u^{2}(t, x) \mathrm{d} x+\left[u(t, x)-x u_{x}(t, x)\right]_{0}^{\infty}=0 \tag{4.8}
\end{equation*}
$$

We impose the boundary condition (2.9) and assume that $x u_{x} \rightarrow 0$ as $x \rightarrow 0$. This gives

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{\infty} x u^{2}(t, x) \mathrm{d} x=1 \tag{4.9}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\int_{0}^{\infty} x u^{2}(t, x) \mathrm{d} x-t=\text { constant independent of } t . \tag{4.10}
\end{equation*}
$$

We see clearly that the boundary conditions determine which conservation law to apply.

## 5. INVARIANT SOLUTIONS

We know from the derivation of similarity solutions using a scaling transformation that all the parameters in the solution cannot be determined from the boundary conditions when the boundary conditions are homogeneous and a further condition is required [13]. The boundary condition (2.7) may be expressed as $u_{x}(t, 0)=0$ and $u(t, \infty)=0$ and is therefore homogeneous while the other boundary conditions, (2.2), (2.5) and (2.9) are non-homogeneous. We will derive the solution of the boundary value problem, (2.6) and (2.7). The additional condition required to complete the solution is the conserved quantity (4.7). It describes in terms of $u(x, t)$ the strength of the flow along the tube which cannot be prescribed by a boundary condition.

The conserved quantity (4.7) was derived from the elementary conserved vector (3.14). We will derive the solution for $u(x, t)$ using a linear combination of the Lie point symmetries associated with the conserved vector (3.14). This new method of deriving invariant solutions of problems with conserved quantities is due to Kara and Mahomed [8] and is more direct than the standard procedure of using a linear combination of all the Lie point symmetries of the partial differential equation [14].

The Lie point symmetry

$$
\begin{equation*}
X=\xi^{1}(t, x, u) \frac{\partial}{\partial t}+\xi^{2}(t, x, u) \frac{\partial}{\partial x}+\eta(t, x, u) \frac{\partial}{\partial u} \tag{5.1}
\end{equation*}
$$

of the partial differential equation (2.6) is said to be associated with the conserved vector $T=\left(T^{1}, T^{2}\right)$ for (2.6) if $[8,15]$

$$
\begin{equation*}
X\left(T^{i}\right)+T^{i} D_{k}\left(\xi^{k}\right)-T^{k} D_{k}\left(\xi^{i}\right)=0, \quad i=1,2 \tag{5.2}
\end{equation*}
$$

where $k$ is summed from 1 to 2 and the total derivatives, $D_{1}$ and $D_{2}$, are defined by (3.2) and (3.3). In (5.2), $X$ is prolongated as required when $T^{i}$ depends on derivatives of $u$.

Equation (5.2) is the determining equation for the Lie point symmetries $X$ associated with $T=\left(T^{1}, T^{2}\right)$. It consists of two components

$$
\begin{align*}
& X\left(T^{1}\right)+T^{1} D_{2}\left(\xi^{2}\right)-T^{2} D_{2}\left(\xi^{1}\right)=0,  \tag{5.3}\\
& X\left(T^{2}\right)+T^{2} D_{1}\left(\xi^{1}\right)-T^{1} D_{1}\left(\xi^{2}\right)=0 . \tag{5.4}
\end{align*}
$$

Substituting the elementary conserved vector (3.14) into (5.3) and (5.4) gives

$$
\begin{equation*}
2 u \eta+u^{2} \frac{\partial \xi^{2}}{\partial x}+u_{x}\left(u^{2} \frac{\partial \xi^{2}}{\partial u}+\frac{\partial \xi^{1}}{\partial x}\right)+u_{x}^{2} \frac{\partial \xi^{1}}{\partial u}=0 \tag{5.5}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\partial \eta}{\partial x}+u \frac{\partial \xi^{1}}{\partial t}+u^{2} \frac{\partial \xi^{2}}{\partial t}+u_{x}\left(\frac{\partial \eta}{\partial u}-\frac{\partial \xi^{2}}{\partial x}\right)-u_{x}^{2} \frac{\partial \xi^{2}}{\partial u}+u_{t}\left(u^{2} \frac{\partial \xi^{2}}{\partial u}-\frac{\partial \xi^{1}}{\partial x}\right)=0 \tag{5.6}
\end{equation*}
$$

Equations (5.5) and (5.6) are separated according to the derivatives of $u$. It is found that

$$
\begin{equation*}
X=\left(c_{1}+3 c_{3} t\right) \frac{\partial}{\partial t}+\left(c_{2}+2 c_{3} t\right) \frac{\partial}{\partial x}-c_{3} u \frac{\partial}{\partial u} \tag{5.7}
\end{equation*}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are constants. The generator (5.7) is a linear combination of the three Lie point symmetries associated with elementary conserved vector (3.14).

Now, $u=\Phi(t, x)$ is an invariant solution generated by the symmetries associated with the conserved vector (3.14) provided

$$
\begin{equation*}
\left.X(u-\Phi(t, x))\right|_{u=\Phi}=0 \tag{5.8}
\end{equation*}
$$

that is, provided

$$
\begin{equation*}
\left(c_{1}+3 c_{3} t\right) \frac{\partial \Phi}{\partial t}+\left(c_{2}+2 c_{3} x\right) \frac{\partial \Phi}{\partial x}=-c_{3} \Phi \tag{5.9}
\end{equation*}
$$

The general solution of (5.9) is

$$
\begin{equation*}
u(t, x)=\left(c_{1}+3 c_{3} t\right)^{-1 / 3} F(\xi), \quad \xi=\frac{c_{2}+2 c_{3} x}{\left(c_{1}+3 c_{3} t\right)^{2 / 3}} \tag{5.10}
\end{equation*}
$$

Substitution of (5.10) into (2.6) yields the ordinary differential equation

$$
\begin{equation*}
\frac{\mathrm{d}^{2} F}{\mathrm{~d} \xi^{2}}+\frac{1}{2 c_{3}} \frac{\mathrm{~d}}{\mathrm{~d} \xi}\left(\xi F^{2}\right)=0 \tag{5.11}
\end{equation*}
$$

and the conserved quantity (4.7) becomes

$$
\begin{equation*}
\frac{1}{2 c_{3}} \int_{0}^{\infty} F^{2}(\xi) \mathrm{d} \xi=1 \tag{5.12}
\end{equation*}
$$

The conserved quantity is independent of $t$ without a condition being placed on $c_{1}, c_{2}, c_{3}$ because $X$ is associated with the elementary conserved vector (3.14).

We choose $c_{2}=0$ to make $\xi=0$ when $x=0$. To satisfy the second boundary condition in (2.7) we choose $c_{1}=0$. There is only one non-zero constant $c_{3}$. Since $X$ contains $c_{3}$ as a constant factor, $u(t, x)$ will not depend
on $c_{3}$. (This can be verified by direct calculation keeping $c_{3}$ unspecified.) We can therefore choose $c_{3}$ conveniently. We choose $c_{3}=9 / 8$ in order to simplify $\xi$. Hence

$$
\begin{equation*}
u(t, x)=\frac{2}{3} \frac{F(\xi)}{t^{1 / 3}}, \quad \xi=\frac{x}{t^{2 / 3}} \tag{5.13}
\end{equation*}
$$

where $F(\xi)$ satisfies the differential equation

$$
\begin{equation*}
\frac{\mathrm{d}^{2} F}{\mathrm{~d} \xi^{2}}+\frac{4}{9} \frac{\mathrm{~d}}{\mathrm{~d} \xi}\left(\xi F^{2}\right)=0 \tag{5.14}
\end{equation*}
$$

subject to the boundary conditions

$$
\begin{equation*}
\frac{\mathrm{d} F}{\mathrm{~d} \xi}(0)=0, \quad F(\xi) \sim \frac{9}{2 \xi^{2}} \quad \text { as } \quad \xi \rightarrow \infty \tag{5.15}
\end{equation*}
$$

and to the initial condition

$$
\begin{equation*}
\int_{0}^{\infty} F^{2}(\xi) \mathrm{d} \xi=\frac{9}{4} \tag{5.16}
\end{equation*}
$$

The solution of (5.14) subject to the boundary conditions (5.15) is

$$
\begin{equation*}
F(\xi)=\frac{9}{2\left(\xi^{2}+k^{2}\right)} \tag{5.17}
\end{equation*}
$$

where $k$ is a constant which cannot be obtained from at the boundary conditions. It is obtained from the conserved quantity (5.16). Substituting (5.17) into (5.16) and using

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\mathrm{d} w}{\left(1+w^{2}\right)^{2}}=\frac{\pi}{4} \tag{5.18}
\end{equation*}
$$

gives

$$
\begin{equation*}
k=\left(\frac{9}{4} \pi\right)^{1 / 3} \tag{5.19}
\end{equation*}
$$

Finally from (5.13),

$$
\begin{equation*}
u(t, x)=\frac{3 t}{\left[x^{2}+\left(\frac{9 \pi}{4}\right)^{2 / 3} t^{4 / 3}\right]} \tag{5.20}
\end{equation*}
$$

Table 5.1 Lie point symmetries

| Partial differential | Lie point Symmetries |
| :--- | :--- |
| $\frac{\partial^{2} u}{\partial t^{2}}-\frac{\partial^{2} u}{\partial x^{2}}=-2 u \frac{\partial u}{\partial t}$ | $X_{1}=\frac{\partial}{\partial t}$ |
| $X_{2}=\frac{\partial}{\partial x}$ |  |
|  | $X_{3}=t \frac{\partial}{\partial t}+x \frac{\partial}{\partial x}-u \frac{\partial}{\partial u}$ |
| $\frac{\partial p}{\partial t}=\frac{1}{2}\left(-\frac{\partial p}{\partial x}\right)^{-1 / 2} \frac{\partial^{2} p}{\partial x^{2}}$ | $X_{1}=\frac{\partial}{\partial t} \quad X_{4}=3 t \frac{\partial}{\partial t}+2 x \frac{\partial}{\partial x}$ |
| $X_{2}=\frac{\partial}{\partial x} \quad X_{5}=t \frac{\partial}{\partial t}+2 p \frac{\partial}{\partial p}$ |  |
| $X_{3}=\frac{\partial}{\partial p}$ |  |
| $\frac{\partial^{2} u}{\partial x^{2}}=2 u \frac{\partial u}{\partial t}$ | $X_{1}=\frac{\partial}{\partial t} \quad X_{3}=2 t \frac{\partial}{\partial t}+x \frac{\partial}{\partial x}$ |
| $X_{2}=\frac{\partial}{\partial x}$ | $X_{4}=x \frac{\partial}{\partial x}-2 u \frac{\partial}{\partial u}$ |

From (5.7), the solution is generated by

$$
\begin{equation*}
X=3 t \frac{\partial}{\partial t}+2 x \frac{\partial}{\partial x}-u \frac{\partial}{\partial u} . \tag{5.21}
\end{equation*}
$$

The scaling symmetry (5.21) is a linear combination

$$
\begin{equation*}
X=a_{1} X_{1}+a_{2} X_{2}+a_{3} X_{3}+a_{4} X_{4} \tag{5.22}
\end{equation*}
$$

of the Lie point symmetries of (2.6), which are listed in Table 5.1, with $a_{1}=0$, $a_{2}=0, a_{3}=3 / 2, a_{4}=1 / 2$. In the standard method, starting from (5.22), the ratio $a_{3} / a_{4}=3$ is obtained by insisting that (4.7) is time independent.

The pressure $p(r, t)$ can now be obtained by integrating the first equation in $(2.3)$ and imposing the first boundary condition in (2.5). The results obtained for $u(r, t)$ and $p(r, t)$ are the same as derived by Ockendon et al [1]. These authors first derived the similarity solution for $p(r, t)$ by solving (2.4) subject to (2.5) and then obtained $u(r, t)$ using the first equation in(2.3).

The Lie point symmetries of equations (2.1) and (2.4) as well as (2.6) are listed in Table 5.1. These Lie point symmetries were derived directly using the determining equation of the partial differential equation and not using condition (5.2). The similarity transformations and solutions of Ockendon et al [1] may be re-derived using (5.8) and a linear combination of the Lie point symmetries of the partial differential equations.

It is only the boundary value problem, (2.6) with the homogeneous boundary conditions (2.7), that requires a conserved quantity to complete the solution and to which the method described in this paper to find the invariant solution can be applied. The other three boundary value problems formulated in Section 2 do not have a conserved quantity and do not need one to complete the solution because the boundary conditions are nonhomogenous.

## 6. DISCUSSION

Conservation laws for a partial differential equation depend only on the differential equation and are independent of the boundary conditions. The conservation laws derived here may therefore be useful in other problems described by the same partial differential equations but different boundary conditions.

The conservation laws for the nonlinear wave and diffusion equations were derived using multipliers which are independent of partial derivatives. It remains to be investigated if further conservation laws and conserved vectors can be obtained if multipliers which depend on the first and higher order partial derivatieves are considered.

The boundary value problem for the gas velocity due to a small pressure increase at the entrance to the tube, solved in Section 5, belongs to a class of boundary value problems with homogeneous boundary conditions which require a conserved quantity to complete their mathematical formulation. Important problems in this class include jet flows in fluid mechanics $[12,14]$.

In problems with a conserved quantity the derivation of the invariant solution using a linear combination of the Lie point symmetries associated with the conserved vector is more direct and shorter than the standard method of using a linear combination of all the Lie point symmetries of the partial differential equation. The derivation of the associated Lie point symmetries is less laborious because the order of the conserved vector is one less than that of the partial differential equation. For the invariant solution generated by the associated symmetries the conserved quantity is identically satisfied while in the standard approach a condition on the expansion constants is obtained for the quantity to be conserved. When the same
partial differential equation applies to more than one problem, the problems being distinguished by different boundary conditions, the conserved quantities will be different for each problem. The standard method then has advantages because the same Lie point symmetries can be used for all problems. Condition (5.2) can then be applied to obtain the ratio of constants in the linear combination of Lie point symmetries for each problem $[16,17]$.

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