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ANALYTICAL ASPECT OF FOURTH-ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract- In this work, the homotopy analysis method (HAM) is applied to solve the fourth-order parabolic partial differential equations. This equation practically arises in the transverse vibration problems. The proposed iterative scheme finds the solution without any discritization, linearization, or restrictive assumptions. Some applications are given to verify the reliability and efficiency of the method. The convergence control parameter $\hbar$ in the HAM solutions has provided a convenient way of controlling the convergence region of series solutions. It is also shown that the solutions that are obtained by Adomian decomposition method (ADM) and variational iteration method (VIM) are special cases of the solutions obtained by HAM.

Keywords- Fourth-order parabolic partial differential equations, Homotopy analysis method, Convergence control parameter, transverse vibrations.

1. INTRODUCTION

Liao [1] employed the basic idea of the homotopy in topology to propose method for nonlinear problems, namely, homotopy analysis method (HAM) [2-3]. This method has many advantages over the classical methods; mainly, it is independent of any small or large quantities. So, the HAM can be applied no matter if governing equations and boundary/initial conditions contain small or large quantities or not. The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter; the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter. Various techniques for seeking analytical solutions to the nonlinear partial differential equations are proposed, Adomian Decomposition Method (ADM) [4-6], variation iteration method (VIM) [7] and homotopy perturbation method (HPM) [8]. The HAM is a general analytic approach to get series solution of various types of nonlinear equations, including algebraic equation, ordinary and partial differential equations. In this paper, the fourth-order parabolic partial differential equations with variable coefficients will be approached analytically. Evans et al [9] has studied fourth-order parabolic equation with constant coefficients via AGE method. Evans [10] studied the second order parabolic equations via finite difference method. Wazwaz [4-5] and Biazar et al [7] studied fourth-order parabolic partial differential equations with variable coefficients by ADM and VIM.
Let us assume the following nonlinear differential equation in form

$$N[u(\tau)] = 0 \tag{1}$$

where N is a nonlinear operator, τ is an independent variable and $u(\tau)$ is the solution of equation. We define the function, $\phi(\tau, p)$ as follows:

$$\lim_{p \to 0} \phi(\tau, p) = u_0(\tau) \tag{2}$$

where, $p \in [0,1]$ and $u_0(\tau)$ is the initial guess which satisfies the initial or boundary conditions

$$\lim_{p \to 1} \phi(\tau, p) = u(\tau) \tag{3}$$

and by using the generalized homotopy method, Liao's so-called zero-order deformation (1) will be:

$$(1-p)L[\phi(\tau, p) - u_0(\tau)] = p\hbar H(\tau) N[\phi(\tau, p)] \tag{4}$$

parameter $\hbar$, the auxiliary function $H(\tau)$, the initial guess $u_0(\tau)$ and the auxiliary linear operator L. This freedom plays an important role in establishing the keystone of validity and flexibility of HAM as shown in this paper. Thus, when p increases from 0 to 1 the solution $\phi(\tau, p)$ changes between the initial guess $u_0(\tau)$ and the solution $u(\tau)$. The Taylor series expansion of $\phi(\tau, p)$ with respect to p is

$$\phi(\tau, p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) p^m \tag{5}$$

and

$$u^{(m)}(\tau) = \left. \frac{\partial^m \phi(\tau; p)}{\partial p^m} \right|_{p=0} \tag{6}$$

Where $u^{(m)}(\tau)$ for brevity is called the mth order of deformation derivation which reads:

$$u^{(m)}(\tau) = \frac{u^{(m)}}{m!} = \frac{1}{m!} \left. \frac{\partial \phi(\tau; p)}{\partial p^m} \right|_{p=0} \tag{7}$$

It's clear that if the auxiliary parameter $\hbar = -1$ auxiliary function $H(\tau) = 1$, then (4) will become:

$$(1-p)L[\phi(\tau, p) - u_0(\tau)] + p\ N[\phi(\tau, p)] = 0 \tag{8}$$

This statement is commonly used in HPM procedure. Indeed, in HPM we solve the nonlinear differential equation by separating any Taylor expansion term. Now, we define the vector of

$$\vec{u}_m = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4 ...., \vec{u}_m\} \tag{9}$$

According to the definition in (7), the governing equation and the corresponding initial conditions can be deduced from zero-order deformation equation (1). Differentiating (1) *m* times with respect to the embedding parameter *p* and setting *p=0* and finally dividing by *m!*, we will have the so-called mth order deformation equation in the form:

$$L[u_m(\Omega) - \chi_m \, u_{m-1}(\Omega)] = \hbar \; R[\vec{u}_{m-1}] \tag{10}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(\Omega; p)]}{\partial \, p^{m-1}}\bigg|_{p=0} \tag{11}$$

and $\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$

By applying inverse linear operator to both sides of equation, (10), we can easily solve the equation and compute the solutions by computational software MATHEMATICA 6. Numerical results reveal that the HAM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy. For the sake of comparison, we take the same examples as used in [4-5,7].

2. EXEMPLIFICATION

Consider the following case of one-dimensional equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u(x,t)}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, \; t > 0 \tag{12}$$

subject to the initial and boundary conditions:

$$u(x,0) = 0, \; \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \; u\left(\frac{1}{2}, t\right) = \left(1 + \frac{0.5^5}{120}\right) S\,int, \; \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6}\frac{1}{2^3} S\,int$$

$$u(1,t) = \frac{121}{120} S\,int, \; u\left(\frac{1}{2}, t\right) = \frac{1}{6} S\,int \tag{13}$$

To solve this example by HAM, we take

$$u_0(x,t)=\left(1+\frac{x^5}{120}\right)t \tag{14}$$

$$N(\Phi)=\frac{\partial^2\Phi(x,t;p)}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4\Phi(x,t;p)}{\partial x^4} \tag{15}$$

and from (10), we have

$$R(N(\vec{u}_{m-1}))=\frac{\partial^2 u_{m-1}(x,t)}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_{m-1}(x,t)}{\partial x^4} \tag{16}$$

By applying (16) with (10), we have

$$u_1=\hbar\frac{t^3}{6}\left(1+\frac{x^5}{120}\right) \tag{17}$$

$$u_2=\left[\hbar(1+\hbar)\frac{t^3}{6}+\frac{1}{120}\hbar^2 t^5\right]\left(1+\frac{x^5}{120}\right) \tag{18}$$

$$u_3=\left[\hbar(1+\hbar)^2\frac{t^3}{6}+\frac{1}{120}(1+\hbar)\hbar^2 t^5+\frac{t^7}{5040}\right]\left(1+\frac{x^5}{120}\right) \tag{19}$$

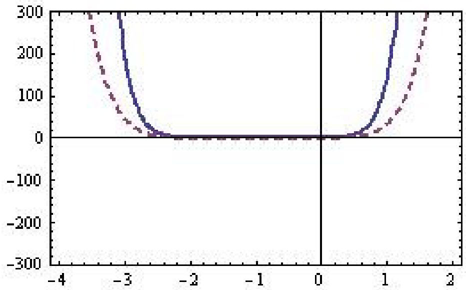

Fig. 1. The $\hbar$ -curve of the 10th and 8th (Thick, Dashed) order approximation for Example 1 at $x=0.9, t=0.9$

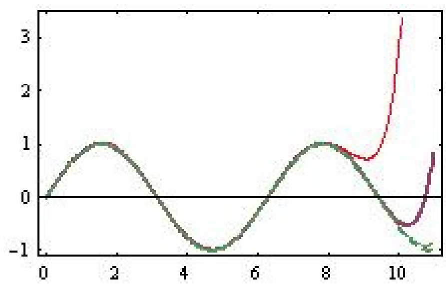

Fig. 2. Comparison between HAM ($\hbar=-1$ (ADM,VIM), $\hbar=-0.96,-0.88$ and exact), solution at $x=0.9$ for Example 1 after 10-iterations (Red, Thick, Automatic, Dashed)

The rest of the components of the HAM solution can be obtained. The fourth term approximate solution is given by

$$u=\left[t+\hbar\left(3+3\hbar+\hbar^2\right)\frac{t^3}{6}+\frac{1}{120}(3+2\hbar)\hbar^2 t^5+\frac{\hbar^3 t^7}{5040}\right]\left(1+\frac{x^5}{120}\right) \tag{20}$$

When $\hbar=-1$, the solution obtained by [4,7].

Consider the following case of one-dimensional equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\left(\frac{x}{Sinx}-1\right)\frac{\partial^4 u(x,t)}{\partial x^4}=0,\quad 0<x<1,\ t>0 \tag{21}$$

subject to the initial and boundary conditions:

$$u(x,0)=x-Sinx,\ \frac{\partial u}{\partial t}(x,0)=-x+Sinx,\ u(0,t)=0,\ \frac{\partial^2 u}{\partial x^2}(0,t)=0\ .$$
$$u(1,t)=e^{-t}(1-Sin1),\ \frac{\partial^2 u}{\partial x^2}(1,t)=e^{-t}Sin1 \tag{22}$$

To solve this problem by HAM, we take

$$u_0(x,t)=(1-t)(x-Sinx) \tag{23}$$

$$N(\Phi)=\frac{\partial^2\Phi(x,t;p)}{\partial t^2}+\left(\frac{x}{Sinx}-1\right)\frac{\partial^4\Phi(x,t;p)}{\partial x^4} \tag{24}$$

and from (10), we have

$$R(N(\vec{u}_{m-1}))=\frac{\partial^2 u_{m-1}(x,t)}{\partial t^2}+\left(\frac{x}{Sinx}-1\right)\frac{\partial^4 u_{m-1}(x,t)}{\partial x^4} \tag{25}$$

By applying (10), we have

$$u_1=\left(-\hbar\ \frac{t^2}{2}+\hbar\ \frac{t^3}{6}\right)(x-Sinx) \tag{26}$$

$$u_2=\left[\ -\hbar(1+\hbar)\ \frac{t^2}{2}+\hbar\ (1+\hbar)\ \frac{t^3}{6}+\hbar^2\ \frac{t^4}{24}-\frac{1}{120}\ \hbar^2\ t^5\ \right](x-Sinx) \tag{27}$$

$$u_3=\begin{bmatrix} -\hbar(1+\hbar)^2\frac{t^2}{2}+\hbar\ (1+\hbar)^2\ \frac{t^3}{6}+\hbar^2(1+\hbar)\ \frac{t^4}{24} \\ -\frac{1}{60}\ \hbar^2(1+\hbar)\,t^5-\ \frac{\hbar^3 t^6}{720} \end{bmatrix}(x-Sinx) \tag{28}$$

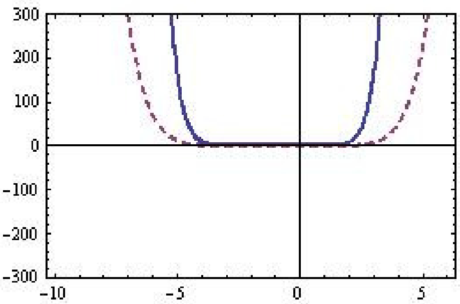

Fig. 3. The $\hbar$ -curve of the 10th and 8th (Thick, Dashed) order approximation for Example 2. at $x = 0.9, t = 0.9$

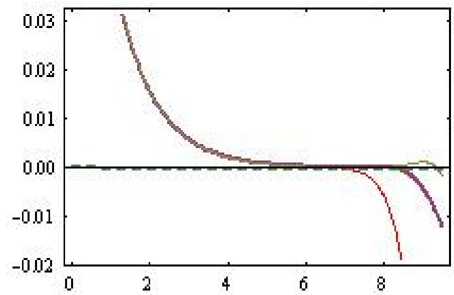

Fig. 4. Comparison between HAM ($\hbar = -1$ (ADM, VIM) $\hbar = -1.11, -1.05$ and exact), solutions at $x = 0.9$ for Example 2 after 10-iterations (Red, Thick, Automatic, Dashed)

The rest of the components of the HAM solution can be obtained. The fourth term approximate solution is given by

$$u = \begin{bmatrix} 1\text{-}t - \hbar\left(3+3\hbar+\hbar^2\right)\dfrac{t^2}{2} + \hbar\left(3+3\hbar+\hbar^2\right)\dfrac{t^3}{6} \\ +\hbar^2\left(3+2\hbar\right)\dfrac{t^4}{24} - \dfrac{1}{720}\hbar^3 t^6 + \dfrac{\hbar^3 t^7}{5040} \end{bmatrix}\left(x - Sinx\right) \tag{29}$$

When $\hbar = -1$, the solution obtained by [4,7].
Consider the following case of two-dimensional equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u(x,y,t)}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u(x,y,t)}{\partial y^4} = 0,\ \frac{1}{2} < x,\ y < 1, t > 0 \tag{30}$$

subject to the initial and boundary conditions:

$$u(x,y,0) = 0,\ \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!},\ u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{0.5^6}{6!} + \frac{y^6}{6!}\right)S\,int\ ,$$

$$u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)S\,int,\ \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right) = \frac{0.5^4}{24}S\,int\,,\ \frac{\partial^2 u}{\partial y^2}\left(x,\frac{1}{2},t\right) = \frac{0.5^4}{24}S\,int \tag{31}$$

$$\frac{\partial^2 u}{\partial x^2}(1,y,t) = \frac{1}{24}S\,int\,,\ \frac{\partial^2 u}{\partial y^2}(x,1,t) = \frac{1}{24}S\,int$$

To solve this example by HAM, we take

$$u_0(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t \tag{32}$$

$$N(\Phi) = \frac{\partial^2 \Phi(x,y,t;p)}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 \Phi(x,y,t;p)}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 \Phi(x,y,t;p)}{\partial y^4} \tag{33}$$

and from (10), we have

$$R(N(\vec{u}_{m-1})) = \frac{\partial^2 u_{m-1}(x,y,t)}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u_{m-1}(x,y,t)}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u_{m-1}(x,y,t)}{\partial y^4} \tag{34}$$

By applying (10), we have

$$u_1 = \hbar \frac{t^3}{6}\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \tag{35}$$

$$u_2 = \left[\hbar (1+\hbar) \frac{t^3}{6} + \frac{1}{120} \hbar^2 t^5\right]\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \tag{36}$$

$$u_3 = \left[\hbar(1+\hbar)^2 \frac{t^3}{6} + \hbar^2 (1+\hbar) \frac{t^5}{120} + \frac{\hbar^3 t^7}{5040}\right]\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \tag{37}$$

The rest of the components of the HAM solution can be obtained. The fourth term approximate solution is given by

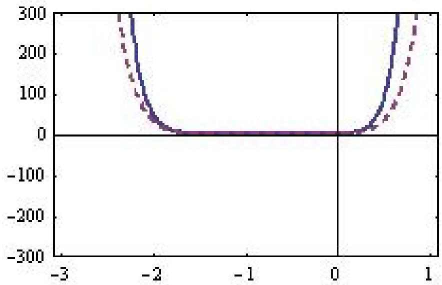

Fig. 5. The $\hbar$ -curve of the 10th and 8th (Thick, Dashed) order approximation for Example 3. at $x = 0.8, t = 0.9$

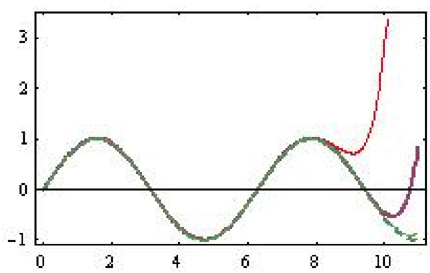

Fig. 6. Comparison between HAM ($\hbar = -1$ (ADM, VIM) $\hbar = -0.96, -0.86$ and exact), solutions $x = 0.9, t = 0.8$ for Example 3 after 10-iterations (Red, Thick, Automatic, Dashed)

The rest of the components of the HAM solution can be obtained. The fourth term approximate solution is given by

$$u=\left[t+\hbar\left(3+3\hbar+\hbar^{2}\right)\frac{t^{3}}{6}+\hbar^{2}\left(3+2\hbar\right)\frac{t^{5}}{120}+\frac{\hbar^{3}t^{7}}{5040}\right]\left(2+\frac{x^{6}}{6!}+\frac{y^{6}}{6!}\right) \tag{38}$$

When $\hbar=-1$, the solution obtained by [5,7].

3. FINAL REMARKS

In this paper, the HAM used to obtain the analytical solutions of fourth-order parabolic partial differential equations with variable coefficients. The computations associated with the examples in this work were performed by using MATHEMATICA. HAM provides us with a convenient way to control the convergence of approximation series by adapting $\hbar$ which is a fundamental qualitative difference in analysis between HAM and other methods. The proposed method is successfully implemented. There are two important points to make here. First, the method was used in a direct way without using linearization, perturbation or restrictive assumption. Second, the HAM avoids the cumbersome of the computational methods while still maintaining the higher level of accuracy. The comparison between the HAM, ADM and VIM made and it is found that HAM is more effective than ADM and VIM, at least for those particular examples. The suggested algorithm is quite efficient and is practically well suited for use in these problems, which are arising in transverse of a vibrating beam [11].

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